INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF OPTICS B: QUANTUM AND SEMICLASSICAL OPTICS

J. Opt. B: Quantum Semiclass. Opt. **7** (2005) S408–S412 doi:10.1088/1464-4266/7/10/030

Characterization of tomographically faithful states in terms of their Wigner function

G M D'Ariano[1] and M F Sacchi

QUIT, INFM and Dipartimento di Fisica 'A Volta', Università di Pavia, via A Bassi 6, I-27100 Pavia, Italy

Received 2 March 2005, accepted for publication 9 August 2005
Published 26 September 2005
Online at stacks.iop.org/JOptB/7/S408

Abstract

A bipartite quantum state is *tomographically faithful* when it can be used as an input of a quantum operation acting on one of the two quantum systems, such that the joint output state carries complete information about the operation itself. Tomographically faithful states are a necessary ingredient for the tomography of quantum operations and for complete quantum calibration of measuring apparatuses. In this paper we provide a complete classification of such states for continuous variables in terms of the Wigner function of the state. For two-mode Gaussian states faithfulness simply resorts to correlation between the modes.

Keywords: process tomography, quantum operations, Wigner function

1. Introduction

Quantum operations describe any kind of physical process affecting quantum states, including unitary evolutions of closed systems and non-unitary transformations of open quantum systems, such as systems interacting with a reservoir, or subjected to noise or measurements of any kind. The problem of determining the quantum operation experimentally occurs in different scenarios, typically for quantum calibration of controlled transformations [1] and of measuring apparatuses [2].

In a *naive* process tomography one varies the input state over a suitably complete set in order to recover enough information about the quantum operation. The tensor structure of the bipartite quantum system, however, allows us to use a *single* fixed bipartite state that scans the complete set of single-system states in a quantum parallel fashion [1]. The bipartite states that can be used in this way in order to carry complete information of the process are called *tomographically faithful* [3].

The problem of evaluating the faithfulness of a state can be expressed in terms of an invertibility condition of a map associated to the state. In many situations it is not simple to check such a condition. In this paper we address the *continuous variables* case (i.e. quantum harmonic oscillators), and solve the problem of the complete classification of faithfulness in terms of the Wigner function of the state. The use of Wigner functions has proved very useful as a generalized phase-space technique to express density operators in terms of c-number functions, thus leading to considerable simplification of the evaluation of quantum dynamics and of expectation values [4, 5].

In this paper we present a general result that provides a necessary and sufficient condition for the faithfulness in terms of the Wigner function. Such a condition, as we will show, makes use of highly irregular functions, such as the customary P-functions in quantum optics. We then specialize our results to the case of Gaussian states of two modes of the electromagnetic field. The class of Gaussian states constitutes a fortunate framework both for theoreticians and experimentalists, since on the one hand all calculations can be done analytically, whereas, on the other hand, these states are easily generated in a laboratory, using lasers, linear optics, and parametric amplifiers. We will show that for Gaussian states the condition of faithfulness is simply the existence of correlations between the two modes.

The paper is organized as follows. In section 2 we briefly recover the general result about the faithfulness of a quantum state, and recall the problem of inversion of a special operator associated to the state. We then restate the problem in terms of the Wigner function of the state, and write a necessary and

[1] Author to whom any correspondence should be addressed.

sufficient condition. The section presents some examples of faithful (both entangled and separable) and unfaithful states. In section 3 we simplify the result of section 2 for the case of two-mode Gaussian states. The condition of faithfulness then simply restates itself as the existence of correlations between the two modes. We conclude the paper in section 4, with a summary of the results and some remarks about the statistical errors that affect the reconstruction of a quantum operation, and the connection with quantum images.

2. Faithfulness in terms of the Wigner function

In mathematical terms, a quantum operation $\mathcal{E}$ is described by a completely positive map [6]. This can be written in the Kraus form

$$\mathcal{E}(\rho) = \sum_n K_n \rho K_n^\dagger, \tag{1}$$

where the K_n are operators on the Hilbert space $\mathcal{H}$ of the quantum system. For simplicity we will consider quantum operations with the same input and output space $\mathcal{H}$, and that are trace-preserving—the so-called *channels*—corresponding to the completeness relation $\sum_n K_n^\dagger K_n = I$. The concept of a *tomographically faithful* state [3] relies on using a bipartite state R on $\mathcal{H} \otimes \mathcal{H}$, such that the output state

$$R_{\mathcal{E}} = (\mathcal{E} \otimes I) R \tag{2}$$

is in one-to-one correspondence with the quantum operation $\mathcal{E}$. In [3] it was proved that a state R is faithful iff the following operator on $\mathcal{H} \otimes \mathcal{H}$

$$\check{R} = (E R)^{\tau_2} E = (R^{\tau_2} E)^{\tau_1} \tag{3}$$

is invertible. In equation (3), $E = \sum_{i,j} |ij\rangle\langle ji|$ denotes the swap operator, and O^{τ_l} denotes the partial transposition of the operator O on the lth Hilbert space, $l = 1, 2$.

Using the notation of [7] for bipartite vectors

$$|A\rangle\!\rangle \equiv \sum_{n,m} \langle n|A|m\rangle \, |n\rangle \otimes |m\rangle, \tag{4}$$

one can generally write a bipartite state in the form

$$R = \sum_{i,j} |A_i\rangle\!\rangle\langle\!\langle B_j|. \tag{5}$$

From the identity

$$A \otimes B|C\rangle\!\rangle = |ACB^\tau\rangle\!\rangle \tag{6}$$

it follows that

$$\check{R} = \sum_{i,j} A_i^\tau \otimes B_j^\dagger, \tag{7}$$

where the transpose τ is defined on the basis chosen in the decomposition of equation (4). Similarly, for a state R written as

$$R = \sum_{i,j} A_i \otimes B_j, \tag{8}$$

using equation (3), one easily writes $\check{R}$ as follows:

$$\check{R} = \sum_{i,j} |B_j\rangle\!\rangle\langle\!\langle A_i^*|, \tag{9}$$

where O^* denotes the complex conjugation on the fixed basis. Notice that the evaluation of the operator $\check{R}$ does not need the diagonalization of R. Moreover, all the previous sums can be suitably replaced with integrals.

A state R is faithful iff the associated operator $\check{R}$ is invertible. We are interested in finding the conditions of faithfulness in terms of the Wigner function of the state. For simplicity, we consider bipartite states that correspond to two-mode states. However, our results are easily generalized to the case of two-party multimode states.

We recall the Cahill–Glauber formulae [8] between the single-mode density matrix ρ and Wigner function $W(\alpha, \alpha^*)$:

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \mathrm{Tr}[\rho D(2\alpha)(-1)^{a^\dagger a}], \tag{10}$$

$$\rho = 2 \int_{\mathbb{C}} \mathrm{d}^2\alpha \, W(\alpha, \alpha^*) D(2\alpha)(-1)^{a^\dagger a}, \tag{11}$$

where α^* denotes the complex conjugate of α, $\mathrm{d}^2\alpha \equiv \mathrm{d}\,\mathrm{Re}\,(\alpha)\,\mathrm{d}\,\mathrm{Im}\,(\alpha)$, and $D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}$ represents the displacement operator for the mode a, with $[a, a^\dagger] = 1$.

For a two-mode bipartite state R we write the Wigner function as a function of two complex variables α and β by direct generalization of equation (11) as follows:

$$R = 4 \int_{\mathbb{C}} \mathrm{d}^2\alpha \int_{\mathbb{C}} \mathrm{d}^2\beta \, W(\alpha, \alpha^*, \beta, \beta^*) D(2\alpha)(-1)^{a^\dagger a} \otimes D(2\beta)(-1)^{b^\dagger b}, \tag{12}$$

where a is a shorthand notation for the annihilation operator otherwise denoted $a \otimes I$, as well as b for $I \otimes b$. In the following we will write the Wigner function in short form as $W(\alpha, \beta) \equiv W(\alpha, \alpha^*, \beta, \beta^*)$, omitting the dependence on the complex conjugated variables. According to equations (8) and (9), the condition of faithfulness of the state in (12) corresponds to the condition of invertibility of the operator

$$\begin{aligned}\check{R} &= 4 \int_{\mathbb{C}} \mathrm{d}^2\alpha \int_{\mathbb{C}} \mathrm{d}^2\beta \, W(\alpha, \beta) \, |D(2\beta)(-1)^{a^\dagger a}\rangle\!\rangle \\ &\quad \times \langle\!\langle D(2\alpha^*)(-1)^{a^\dagger a}| \\ &= 4(I \otimes (-1)^{b^\dagger b}) \int_{\mathbb{C}} \mathrm{d}^2\alpha \int_{\mathbb{C}} \mathrm{d}^2\beta \, W(\alpha, \beta) \, |D(2\beta)\rangle\!\rangle \\ &\quad \times \langle\!\langle D(2\alpha^*)|(I \otimes (-1)^{b^\dagger b}),\end{aligned} \tag{13}$$

where in the second line we used identity (6). Since the set $\{|D(\alpha)\rangle\!\rangle\}$ is an orthonormal basis (in the Dirac sense) for $\mathcal{H} \otimes \mathcal{H}$, namely $\int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi} |D(\alpha)\rangle\!\rangle\langle\!\langle D(\alpha)| = I \otimes I$, the condition of faithfulness in terms of the Wigner function is then the following.

Necessary and sufficient condition for faithfulness. A bipartite state with Wigner function $W(\alpha, \beta)$ is faithful iff one can find a function $f(\beta, \gamma)$ such that

$$\int_{\mathbb{C}} \mathrm{d}^2\beta \, W(\alpha, \beta) \, f(\beta, \gamma) = \delta^{(2)}(\alpha - \gamma), \tag{14}$$

where $\delta^{(2)}(\sigma) = \int_{\mathbb{C}} \frac{\mathrm{d}^2\lambda}{\pi^2} \mathrm{e}^{\lambda\sigma^* - \lambda^*\sigma}$ denotes the Dirac delta over the complex plane.

Equation (14) should be read in a distributional sense. When such a condition is satisfied one has

$$\check{R}^{-1} = \frac{4}{\pi}(I \otimes (-1)^{b^\dagger b}) \int_{\mathbb{C}} \mathrm{d}^2\delta \int_{\mathbb{C}} \mathrm{d}^2\gamma \, f(\delta, \gamma) \, |D(2\gamma^*)\rangle\!\rangle \times \langle\!\langle D(2\delta)|(I \otimes (-1)^{b^\dagger b}). \tag{15}$$

2.1. Example 1: twin-beam state

Consider the twin-beam state that can be easily generated by nondegenerate optical parametric amplifiers

$$R = (1-\lambda^2)|\lambda^{a^\dagger a}\rangle\!\rangle\langle\!\langle\lambda^{a^\dagger a}|, \qquad 0 \leqslant \lambda < 1, \tag{16}$$

where the parameter λ is simply related to the total number of photons $\bar{n} = 2\lambda^2/(1-\lambda^2)$. The corresponding Wigner function is given by

$$W_R(\alpha,\beta) = \frac{4(1-\lambda^2)}{\pi^2}\,\mathrm{Tr}[\lambda^{a^\dagger a}D(2\alpha)\lambda^{a^\dagger a}D(-2\beta^*)]. \tag{17}$$

By normal ordering equation (17) and a lengthy calculation, one obtains

$$W_R(\alpha,\beta) = \frac{4}{\pi^2}\exp\left[-\frac{2(1+\lambda^2)}{1-\lambda^2}(|\alpha|^2+|\beta|^2) + \frac{4\lambda}{1-\lambda^2}(\alpha\beta+\alpha^*\beta^*)\right]. \tag{18}$$

Using the solution (41) of identity (14) derived in the appendix, the function $f(\beta,\gamma)$ can formally be written as

$$f(\beta,\gamma) = \frac{4\lambda^2}{\pi(1-\lambda^2)^2}\exp\left(\frac{1+3\lambda^2}{1-\lambda^2}|\beta|^2\right) \times \exp\left(2\frac{1-8\lambda^2+\lambda^4}{(1-\lambda^2)^2}|\gamma|^2\right)\int_{\mathbb{C}} \mathrm{d}^2\xi\, \mathrm{e}^{|\xi|^2} \times \exp\left(\xi\left(\frac{4\lambda}{1-\lambda^2}\gamma - \beta^*\right) - \xi^*\left(\frac{4\lambda}{1-\lambda^2}\gamma^* - \beta\right)\right), \tag{19}$$

which should be treated as a distribution, in the sense that the integral in ξ has to be performed after the integration on β of equation (14).

Notice that in this simple example, the faithfulness is more easily checked using equation (7) and writing immediately

$$\check{R}^{-1} = \frac{1}{1-\lambda^2}\left(\frac{1}{\lambda}\right)^{a^\dagger a} \otimes \left(\frac{1}{\lambda}\right)^{b^\dagger b}. \tag{20}$$

2.2. Example 2: classically correlated coherent states

A mixture of correlated coherent states can be easily generated by splitting the thermal radiation in a 50/50 beam splitter. For such a kind of state we can write

$$R = \int_{\mathbb{C}} \frac{\mathrm{d}^2\gamma}{\pi\sigma^2}\exp\left(-\frac{|\gamma|^2}{\sigma^2}\right)|\gamma\rangle\langle\gamma|^{\otimes 2}, \tag{21}$$

where the variance σ is related to the total number of photons by $\bar{n} = \sigma/2$. The corresponding Wigner function is given by

$$W_R(\alpha,\beta) = \frac{4}{\pi^2(1+2\sigma^2)}\exp\left[-\frac{2}{1+2\sigma^2}(|\alpha|^2+|\beta|^2) + \frac{4\sigma^2}{1+2\sigma^2}(\alpha\beta^*+\alpha^*\beta)\right]. \tag{22}$$

Using again the solution (41) of identity (14) derived in the appendix, the function $f(\beta,\gamma)$ can formally be written as

$$f(\beta,\gamma) = \frac{4\sigma^4}{\pi(1+2\sigma^2)}\exp\left(\frac{1-2\sigma^2}{1+2\sigma^2}|\beta|^2\right) \times \exp\left(2\frac{1+2\sigma^2-8\sigma^4}{(1+2\sigma^2)^2}|\gamma|^2\right)\int_{\mathbb{C}} \mathrm{d}^2\xi\, \mathrm{e}^{|\xi|^2} \times \exp\left(\xi\left(\beta - \frac{4\sigma^2}{1+2\sigma^2}\gamma\right) - \xi^*\left(\beta^* - \frac{4\sigma^2}{1+2\sigma^2}\gamma^*\right)\right), \tag{23}$$

and thus the state (21) is an example of a separable faithful state.

2.3. Example 3: product states

Consider a product state

$$R = \rho \otimes \sigma. \tag{24}$$

The Wigner function is given by the product of the independent Wigner functions for ρ and σ:

$$W_R(\alpha,\beta) = W_\rho(\alpha,\alpha^*)W_\sigma(\beta,\beta^*). \tag{25}$$

Of course the state R is not faithful, and in fact the condition (14) can never be satisfied.

2.4. Example 4: classical correlation between orthogonal states

Consider the state

$$R = (1-\lambda)\sum_{n=0}^{\infty}\lambda^n|n\rangle\langle n|^{\otimes 2}, \tag{26}$$

where $|n\rangle$ denotes the Fock state. From the relation [8]

$$\langle n|D(\alpha)|n\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)L_n(|\alpha|^2), \tag{27}$$

and the identity [9]

$$\sum_{n=0}^{\infty}\lambda^n L_n(x)L_n(y) = \frac{1}{1-\lambda}\exp\left(-\frac{\lambda}{1-\lambda}(x+y)\right)I_0\left(2\frac{\sqrt{xy\lambda}}{1-\lambda}\right), \tag{28}$$

where $L_n(x)$ and $I_0(x)$ denote the nth-order Laguerre polynomials and the 0-order modified Bessel function, one obtains the Wigner function

$$W_R(\alpha,\beta) = \frac{4}{\pi^2}\exp\left(-2\frac{1+\lambda}{1-\lambda}(|\alpha|^2+|\beta|^2)\right) \times I_0\left(\frac{8\sqrt{\lambda}}{1-\lambda}|\alpha\beta|\right). \tag{29}$$

Condition (14) can never be satisfied, since there is no dependence of the Wigner function on the phase of β. In fact, $\check{R} \equiv R$ is clearly not invertible, whence the state R is not faithful.

3. Simplification for Gaussian states

Unfortunately, it is often difficult to inspect condition (14), since it holds more generally in a distribution sense. For Gaussian bipartite states, however, it is possible to derive a more practical condition in terms of the correlation matrix.

According to the derivation in the appendix, the term of the Wigner function of a Gaussian bipartite state that is relevant for the condition (14) is the factor of the form

$$g(\alpha, \beta) = \exp[(A\alpha\beta + B\alpha\beta^*) + \text{h.c.}]. \tag{30}$$

In fact, as shown in the appendix, the condition (14) can be satisfied iff

$$|A|^2 - |B|^2 \neq 0. \tag{31}$$

In order to clarify the meaning of condition (31), it is useful to consider the state R in terms of the characteristic function $\Gamma(\alpha, \beta) = \text{Tr}[RD(\alpha) \otimes D(\beta)]$, which corresponds to the Fourier transform of the Wigner function, and hence will be Gaussian as well. One has

$$R = \int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi} \int_{\mathbb{C}} \frac{\mathrm{d}^2\beta}{\pi} \Gamma(\alpha, \beta) D^\dagger(\alpha) \otimes D^\dagger(\beta). \tag{32}$$

The operator $\check{R}$ then can be written

$$\check{R} = \int_{\mathbb{C}} \frac{\mathrm{d}^2\alpha}{\pi} \int_{\mathbb{C}} \frac{\mathrm{d}^2\beta}{\pi} \Gamma(\alpha, \beta) |D^\dagger(\beta)\rangle\rangle\langle\langle D^\dagger(\alpha^*)|. \tag{33}$$

Similarly to equation (14), $\check{R}$ is invertible iff one can find a function $f(\beta, \gamma)$ such that

$$\int_{\mathbb{C}} \mathrm{d}^2\beta\, \Gamma(\alpha, \beta) f(\beta, \gamma) = \delta^{(2)}(\alpha - \gamma), \tag{34}$$

and $\check{R}^{-1}$ writes

$$\check{R}^{-1} = \int_{\mathbb{C}} \mathrm{d}^2\delta \int_{\mathbb{C}} \mathrm{d}^2\gamma\, f(\delta, \gamma) |D^\dagger(\gamma^*)\rangle\rangle\langle\langle D^\dagger(\delta)|. \tag{35}$$

The same consideration before equation (30) applies here. The condition (34) can be satisfied iff $|A|^2 - |B|^2 \neq 0$, where A and B are the coefficients in front of the variables $\alpha\beta$ and $\alpha\beta^*$ of the characteristic function. Using the identities

$$\begin{aligned}
A &= \partial^2_{\alpha\beta}\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
&\quad - \partial_\alpha\Gamma(\alpha, \beta)|_{\alpha=\beta=0}\, \partial_\beta\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
A^* &= \partial^2_{\alpha^*\beta^*}\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
&\quad - \partial_{\alpha^*}\Gamma(\alpha, \beta)|_{\alpha=\beta=0}\, \partial_{\beta^*}\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
B &= \partial^2_{\alpha\beta^*}\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
&\quad - \partial_\alpha\Gamma(\alpha, \beta)|_{\alpha=\beta=0}\, \partial_{\beta^*}\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
B^* &= \partial^2_{\alpha^*\beta}\Gamma(\alpha, \beta)|_{\alpha=\beta=0} \\
&\quad - \partial_{\alpha^*}\Gamma(\alpha, \beta)|_{\alpha=\beta=0}\, \partial_\beta\Gamma(\alpha, \beta)|_{\alpha=\beta=0},
\end{aligned} \tag{36}$$

the condition of faithfulness can be restated in terms of the correlation matrix as follows.

Necessary and sufficient condition for the faithfulness of Gaussian states. A bipartite Gaussian state is faithful iff the following condition on the correlations is satisfied

$$\chi(R) \doteq \langle\Delta a^\dagger b^\dagger\rangle\langle\Delta ab\rangle + \langle\Delta a^\dagger b\rangle\langle\Delta ab^\dagger\rangle \neq 0, \tag{37}$$

where for any two operators P and Q

$$\langle\Delta PQ\rangle \doteq \langle PQ\rangle - \langle P\rangle\langle Q\rangle. \tag{38}$$

In terms of the quadratures $X_c = (c + c^\dagger)/2$ and $Y_c = (c - c^\dagger)/(2\text{i})$ of the modes $c = a, b$, the correlation $\chi(R)$ can be rewritten as

$$\chi(R) = \tfrac{1}{2}\left(\langle\Delta X_a X_b\rangle^2 + \langle\Delta Y_a Y_b\rangle^2 + \langle\Delta X_a Y_b\rangle^2 + \langle\Delta Y_a X_b\rangle^2\right). \tag{39}$$

In the examples 1–4 given in the previous section, one has $\chi(R) = \frac{\lambda^2}{(1-\lambda^2)^2}, \sigma^4, 0$, and 0, respectively (notice, however, that the state in example 4 is not Gaussian). Using equation (39), the condition $\chi(R) \neq 0$ shows that bipartite Gaussian states are always faithful, except when they are product states ($\chi(R)$ is the sum of nonnegative terms which all vanish when there is no correlation between the two modes). A rigorous relation between statistical errors that affect tomographic reconstructions and the strength of correlations is in order, but is beyond the scope of this paper.

4. Conclusions

Tomographically faithful states are a necessary ingredient for the tomography of quantum operations and for complete quantum calibration of measuring apparatuses. In this paper we have provided a complete classification of two-mode faithful states in terms of the Wigner function of the state. This classification has been derived from the general faithfulness condition resorting to the invertibility of a special operator associated to the state. Some examples of faithful states have been presented, both entangled and separable, along with examples of not faithful states. For two-mode Gaussian states we have shown that faithfulness is simply equivalent to nonvanishing correlations between the modes.

We conclude by noticing that the actual statistical efficiency of a faithful state in the tomography of a quantum operation in infinite dimensions is connected to the increase of the singular values of the unbounded operator $\check{R}^{-1}$. Such unboundedness is responsible for the increasingly large statistical errors in the Fock representation of the quantum operation, accounting for the finite experimental data sample used to infer information on a infinite set of matrix elements of the quantum operation. As a rule of thumb, the statistical efficiency increases for greater correlation $\chi(R)$.

Finally, it is worth mentioning that the framework of *quantum images* [10] bears a strict analogy with that of the quantum tomography of a channel using an input bipartite state, with the role of the channel here played by the density contour of the image analysed by one of the twin beams from parametric down-conversion of vacuum. Clearly, when the state is faithful one has quantum imaging on the other beam, and our result is consistent with the recent demonstration [10] that entanglement is not necessary for quantum imaging. In particular, the thermal state split by a beam splitter in equation (21) is suitable for quantum imaging.

Acknowledgments

This work has been sponsored by INFM through the project PRA-2002-CLON, and by the EC and MIUR through the cosponsored ATESIT project IST-2000-29681 and Cofinanziamento 2003.

Appendix

We show that for a function of the form

$$g(\alpha,\beta)=h(\alpha)k(\beta)\mathrm{e}^{A\alpha\beta+A^*\alpha^*\beta^*}\mathrm{e}^{B\alpha\beta^*+B^*\alpha^*\beta}, \qquad (40)$$

with $|A|\neq|B|$, and both h and k (generally not analytical) invertible functions, the following function

$$\begin{aligned}f(\beta,\gamma)=&\frac{(|A|^2-|B|^2)^2}{\pi^3}k^{(-1)}(\beta)\,h^{(-1)}(\gamma)\\&\times\mathrm{e}^{-(|A|^2+|B|^2)|\gamma|^2-(AB\gamma^2+A^*B^*\gamma^{*2})}\mathrm{e}^{-|\beta|^2}\\&\times\int\mathrm{d}^2\xi\;\mathrm{e}^{(|A|^2+|B|^2)|\xi|^2+(A^*B^*\xi^2+AB\xi^{*2})}\mathrm{e}^{(|A|^2-|B|^2)(\xi\gamma-\xi^*\gamma^*)}\\&\times\mathrm{e}^{\beta(A\xi^*+B^*\xi)-\beta^*(A^*\xi+B\xi^*)},\end{aligned} \qquad (41)$$

satisfies the identity

$$\int_{\mathbb{C}}\mathrm{d}^2\beta\;g(\alpha,\beta)f(\beta,\gamma)=\delta^{(2)}(\alpha-\gamma). \qquad (42)$$

The function $f(\beta,\gamma)$ should be treated as a distribution, in the sense that the integral in ξ has to be performed after the integration on β of equation (42).

One has

$$\begin{aligned}\int_{\mathbb{C}}\mathrm{d}^2\beta\;g(\alpha,\beta)f(\beta,\gamma)=&\frac{(|A|^2-|B|^2)^2}{\pi^3}h(\alpha)\,h^{(-1)}(\gamma)\\&\times\mathrm{e}^{-(|A|^2+|B|^2)|\gamma|^2-(AB\gamma^2+A^*B^*\gamma^{*2})}\\&\times\int\mathrm{d}^2\xi\;\mathrm{e}^{(|A|^2+|B|^2)|\xi|^2+(A^*B^*\xi^2+AB\xi^{*2})}\mathrm{e}^{(|A|^2-|B|^2)(\xi\gamma-\xi^*\gamma^*)}\\&\times\int\mathrm{d}^2\beta\;\mathrm{e}^{-|\beta|^2}\mathrm{e}^{\beta(A\xi^*+B^*\xi+A\alpha+B^*\alpha^*)-\beta^*(A^*\xi+B\xi^*-A^*\alpha^*-B\alpha)}\\=&\frac{(|A|^2-|B|^2)^2}{\pi^2}h(\alpha)\,h^{(-1)}(\gamma)\\&\times\mathrm{e}^{(|A|^2+|B|^2)(|\alpha|^2-|\gamma|^2)+AB(\alpha^2-\gamma^2)+A^*B^*(\alpha^{*2}-\gamma^{*2})}\\&\times\int\mathrm{d}^2\xi\;\mathrm{e}^{(|A|^2-|B|^2)[\xi(\gamma-\alpha)-\xi^*(\gamma^*-\alpha^*)]}\\=&\,\delta^{(2)}(\alpha-\gamma),\end{aligned} \qquad (43)$$

where the integral in $\mathrm{d}^2\beta$ has been performed by using the identity

$$\int_{\mathbb{C}}\mathrm{d}^2\beta\;\exp\left(-\frac{|\beta|^2}{\sigma^2}\right)\mathrm{e}^{\beta\alpha^*-\beta^*\gamma}=\pi\sigma^2\mathrm{e}^{-\sigma^2\alpha^*\gamma}. \qquad (44)$$

References

[1] D'Ariano G M and Lo Presti P 2001 *Phys. Rev. Lett.* **86** 4195
[2] D'Ariano G M, Lo Presti P and Maccone L 2004 *Phys. Rev. Lett.* **93** 250407
[3] D'Ariano G M and Lo Presti P 2003 *Phys. Rev. Lett.* **91** 047902
[4] Kim Y S and Zachary W W (ed) 1986 *The Physics of Phase Space* (Berlin: Springer)
[5] Gardiner C W 1991 *Quantum Noise* (Berlin: Springer)
[6] Kraus K 1983 *States, Effects and Operations* (*Springer Lecture Notes in Physics* vol 190) (Berlin: Springer)
[7] D'Ariano G M, Lo Presti P and Sacchi M F 2000 *Phys. Lett.* **272** 32
[8] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
[9] Gradstein I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)
[10] Magatti D, Ferri F, Gatti A, Bache M, Brambilla E and Lugiato L A 2005 *Phys. Rev. Lett.* **94** 183602