

Secret Quantum Communication of a Reference Frame

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We propose quantum-cryptographic protocols to secretly communicate a reference frame—unspeakable information in the sense it cannot be encoded into a string of bits. Two distant parties can secretly align their Cartesian axes by exchanging N spin- $\frac{1}{2}$ particles, achieving the optimal accuracy $1/N$. A possible eavesdropper cannot gain any information without being detected.

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Any communication about the properties of a physical object requires not only the transmission of a string of bits, but also the existence of notions shared between sender and receiver. For example, to communicate the length of an object, both parties must agree on a unit of length. In many cases, the natural units of length, time, and mass [1] provide a common standard that allows any two parties to communicate by simply exchanging bits, even though they have never met. However, such standard references do not exist for all properties, as some cannot be communicated as strings of bits [2,3] and are hence referred to as unspeakable information [4,5]. For example, due to the fundamental isotropy of space, a string of bits can encode only the relative orientation of two directions, but not the absolute orientation of a single direction in space. In such cases, the only possibility for the sender to establish a shared notion with the receiver is to send a physical object—such as a gyroscope—that provides the common standard. This situation arises whenever sender and receiver try to establish a shared reference frame (SRF), e.g., by aligning their Cartesian axes, or by synchronizing their clocks.

Using quantum cryptography, one can securely send speakable information (i.e., strings of bits) through a public channel. One might then ask whether the same is true for unspeakable information. The answer is not trivial since to convey unspeakable info a physical carrier must be sent, and it can be intercepted by an eavesdropper. In this Letter we present quantum-cryptographic protocols to encode spatial directions and reference frames so that no eavesdropper can obtain information on them. This is useful also for quantum cryptography, since private SRFs are an important resource to realize both classical and quantum secret communication [6]. Here, we consider communication through the transmission of N spin- $\frac{1}{2}$ particles. We start by showing a simple, experimentally feasible protocol that uses N shared secret bits and N separable states to communicate a secret direction with accuracy $1/\sqrt{N}$. Then, we present the optimal protocol, which exploits entanglement and secretly transmits a whole Cartesian frame. Such a protocol employs $3\log N$ secret bits to achieve accuracy $1/N$. Remarkably, this amount of secret

bits exactly coincides with the number of secret bits that can be exchanged once a private SRF is established [6]. Such a tight balance provides a new understanding of reference frames as a communication resource. Note that the above protocols do not need the two parties to possess the shared secret bits beforehand: They can obtain them using quantum key distribution, which does not necessarily require a prior shared reference [7,8]. This suggests that protocols that do not use prior classical randomness can be implemented. We conclude the Letter by providing two such protocols that require only quantum and classical communication over a public channel.

Separable protocol using secret random bits.—The intuitive idea of this protocol is that two parties (say Alice and Bob) can transform a secret shared random string into a secret shared direction. Imagine that Alice wants to communicate to Bob the secret direction of her z axis. For each 0 and 1 in the secret string, she sends a spin- $\frac{1}{2}$ particle pointing, respectively, up and down (according to her own z axis). Then, Bob just needs to rotate the z axis of his Stern-Gerlach apparatus until its measurement results (0 for “up” and 1 for “down”) match the bits of the shared secret string. When this happens, he knows that he has aligned his z axis with Alice’s. A possible eavesdropper Eve will not be able to gain any information on the direction of the z axis, since she does not know the secret string. In fact, from her point of view a random sequence of spin up and spin down is equivalent to the maximally chaotic state, where the spins are randomly oriented in any direction she might choose to sample. Moreover, if Eve is tampering with the communication, Bob will find it out as any direction he chooses for his Stern-Gerlach apparatus will never yield the secret string as outcome bits.

A more practical implementation of the above protocol can be obtained if Bob performs all measurements aligning his Stern-Gerlach apparatus alternately along his own x , y , and z axes. The probability that Alice’s spin up states (identified by zeros in the shared secret key) will have outcome up in Bob’s z -oriented Stern-Gerlach apparatus is $p(\theta) = \cos^2\theta/2$, where θ is the unknown angle between Alice and Bob’s z axes. This is also the probability that Alice’s spin down states (identified by ones) will have

outcome down at Bob's apparatus. Therefore, Bob can estimate such probability from his outcomes on N spins as $\bar{p}_\theta = [N(\text{up}, 0) + N(\text{down}, 1)]/N$, where $N(i, j)$ is the number of spins that gave outcome i when the corresponding shared secret bit was j . Once the angle θ has been estimated from \bar{p}_θ , Bob orients his Stern-Gerlach apparatus along his x and y axes, and repeats the procedure to recover the angles between these and Alice's z axis. The eavesdropper Eve can be detected by Bob. In fact, her action would result in a depolarization of the transmitted spins and the three angles Bob recovers would be inconsistent. He can discover it if the sum of their squared cosines differs from one by more than a purely statistical error would allow. The accuracy of Bob's procedure can be evaluated as follows. The rms error in the estimate of $p(\theta)$ from the data is $\Delta p(\theta) = \sqrt{\bar{p}_\theta - \bar{p}_\theta^2}/\sqrt{N}$. Then, from error propagation theory, the error on each of the angles θ estimated from \bar{p}_θ is

$$\Delta\theta = \Delta\bar{p} \left/ \left| \frac{\partial p(\theta)}{\partial\theta} \right| \right. = \sqrt{\frac{\bar{p}_\theta(1 - \bar{p}_\theta)}{p(\theta)[1 - p(\theta)]}} \frac{1}{\sqrt{N}} \simeq \frac{1}{\sqrt{N}}, \quad (1)$$

since $\bar{p}_\theta \rightarrow p(\theta)$ for large N . Thus, Bob's overall error on his estimate of the direction of Alice's z axis using $3N$ spins will be $3/\sqrt{N}$. Notice that, without using entangled states at the preparation stage, the $1/\sqrt{N}$ asymptotic scaling cannot be beaten [9].

To send a spatial direction, the exchanged qubits must possess directional information, as is the case of spin- $\frac{1}{2}$ particles. Nevertheless, a partial directional information can be encoded also in the polarization of a photon. In fact, using single photons one can transmit a direction in the plane orthogonal to the wave vector. In this case Bob just needs to orient his polarizers in two directions (say the x and y axes), and the above procedure gives him Alice's secret direction with accuracy $2\sqrt{2}/N$.

Optimal protocol using secret random bits.—The previous protocol requires N qubits and N secret bits to communicate a secret spatial direction with an rms error $3\sqrt{3}/N$. Now we show that a suitable use of entanglement allows one to transmit a whole frame of Cartesian axes with rms error $1/N$, with only $3 \log N$ secret bits needed. To this purpose, we first recall the basic features of the optimal protocol to publicly transmit a Cartesian frame [3], then showing how to modify it in order to achieve unconditional security.

The optimal state for the reference frame communication requires the decomposition of the Hilbert space $\mathcal{H}^{\otimes N}$ of N spin- $\frac{1}{2}$ particles into irreducible representations (irreps) of the rotation group. This decomposition is given by

$$\mathcal{H}^{\otimes N} = \bigoplus_{j=0(\frac{1}{2})}^{N/2} \mathcal{H}_j \otimes \mathbb{C}^{m_j}, \quad (2)$$

where j is the quantum number of the total angular momentum, ranging from $0(\frac{1}{2})$ to $N/2$ for N even (odd), \mathcal{H}_j is a $(2j+1)$ -dimensional space supporting an irrep of the rotation group, and \mathbb{C}^{m_j} is a multiplicity space, whose dimension m_j is equal to the number of equivalent irreps corresponding to the quantum number j . In this decomposition of the Hilbert space, the action of a collective rotation $U_g^{\otimes N}$, $g \in \text{SU}(2)$ becomes

$$U_g^{\otimes N} = \bigoplus_{j=0(\frac{1}{2})}^{N/2} U_g^j \otimes \mathbb{1}_{m_j}, \quad (3)$$

where $\{U_g^j\}$ is the irrep with angular momentum j , and $\mathbb{1}_d$ denotes the identity in a d -dimensional Hilbert space. From Eq. (3) it is clear that the multiplicity spaces are the rotationally invariant subsystems of the global Hilbert space $\mathcal{H}^{\otimes N}$.

For large N , the optimal states for the transmission of a reference frame are given by [3]

$$|A\rangle = \bigoplus_{j=0(\frac{1}{2})}^{(N/2)-1} \frac{A_j}{\sqrt{2j+1}} |E_j\rangle, \quad (4)$$

A_j being suitable coefficients [$A_j \approx \sqrt{4/N} \sin(\frac{2\pi j}{N})$ for $N \gg 1$], and $|E_j\rangle \in \mathcal{H}_j \otimes \mathbb{C}^{m_j}$ being the maximally entangled vector

$$|E_j\rangle = \sum_{m=-j}^j |jm\rangle |m\rangle, \quad (5)$$

where $\{|jm\rangle \in \mathcal{H}_j\}$ are the eigenstates of the total angular momentum J_z , and the vector $|m\rangle$ runs on the first $2j+1$ elements of a basis of the multiplicity space \mathbb{C}^{m_j} . Since the state $|A\rangle$ is prepared by Alice referring to her Cartesian frame, from Bob's point of view all spins are rotated by the unknown rotation $g \in \text{SU}(2)$ that connects his axes with Alice's. Accordingly, Bob receives the state $U_g^{\otimes N}|A\rangle$, and his aim is to perform the best possible measurement to infer g . For a state of the form (4) such measurement is given by the positive operator-valued measurement (POVM) $M(h)dh = U_h^{\otimes N}|B\rangle\langle B|U_h^{\dagger \otimes N}dh$, where dh is the invariant measure over the rotation group, and $|B\rangle$ is

$$|B\rangle = \bigoplus_j \sqrt{2j+1} |E_j\rangle. \quad (6)$$

The use of the state $|A\rangle$ and of the measurement $M(h)$ allows one to communicate optimally a Cartesian frame with an asymptotic rms error $1/N$ [3]. The optimality proof for this protocol is given in Ref. [10]. A transmission scheme using only the state $|A\rangle$ is not secret, since anybody can intercept the spins, perform the optimal measurement, and reprepare the state according to the outcome.

To construct a secret protocol, notice that the vector $|E_j\rangle$ in the state $|A\rangle$ can be replaced by any other maximally entangled vector $|W_j\rangle = (W_j \otimes \mathbb{1}_{m_j})|E_j\rangle$, where W_j is a

local unitary on \mathcal{H}_j . With the same substitution in the POVM (6), the outcome probabilities in the orientation-measurement are unchanged. Thus the estimation is still optimal. The idea then is to randomize the choice of the maximally entangled vector $|W_j\rangle$ in order to make it impossible for Eve to extract any kind of information about Alice's axes. To this purpose, consider the unitaries W_{j,p_j,q_j} defined

$$W_{j,p_j,q_j} = \sum_{m=-j}^j \exp\left(\frac{2\pi i m q_j}{2j+1}\right) |m \oplus p_j\rangle \langle m|, \quad (7)$$

\oplus here denoting addition modulo $2j+1$. These unitaries form a representation of the ‘‘shift and multiply’’ group $\mathbb{Z}^{2j+1} \times \mathbb{Z}^{2j+1}$, which is irreducible on \mathcal{H}_j . A completely secure communication can be obtained if Alice sends one of the states

$$|A_{\{p_j,q_j\}}\rangle = \bigoplus_{j=0(\frac{1}{2})}^{(N/2)-1} \frac{A_j}{\sqrt{2j+1}} (W_{j,p_j,q_j} \otimes \mathbb{1}_{m_j}) |E_j\rangle, \quad (8)$$

chosen according to a secret random sequence $\{p_j, q_j\}$ that she shares only with Bob. The number of possible sequences is $C = \sum_{j=0(\frac{1}{2})}^{N/2} (2j+1)^2 \approx \mathcal{O}(N^3)$. This means that Alice and Bob asymptotically need $3 \log N$ bits of shared randomness. From the point of view of Eve, the randomization procedure is equivalent to the preparation of the mixed state $\rho_E = \sum_{\{p_j,q_j\}} |A_{\{p_j,q_j\}}\rangle \langle A_{\{p_j,q_j\}}| / C$. Because of the irreducibility of the representations $\{W_{j,p_j,q_j}\}$ this averaged state can be easily calculated, obtaining

$$\rho_E = \bigoplus_{j=0(\frac{1}{2})}^{(N/2)-1} \frac{|A_j|^2}{(2j+1)^2} \mathbb{1}_{2j+1} \otimes \sum_{m=-j}^j |m\rangle \langle m|. \quad (9)$$

Using Eq. (3), it is immediate to see that Eve's state ρ_E is completely invariant under rotations, therefore no useful information can be extracted from it about the orientation of Alice's axes. On the other hand, since Bob knows which pure state $|A_{\{p_j,q_j\}}\rangle$ was sent, he can perform the optimal orientation measurement. He then recovers Alice's axes with the asymptotically optimal rms error $1/N$.

This result sheds a new light on the role of private shared reference frames (SRF) as a physical resource. In fact, we can relate the above protocol with the cryptographic protocol of Ref. [6], where a private SRF is used to communicate a secret string of bits. While in our case we have asymptotically

$$N \text{ qubits} + 3 \log N \text{ secret bits} \rightarrow \text{private SRF}, \quad (10)$$

in the case of Ref. [6] one has

$$N \text{ qubits} + \text{private SRF} \rightarrow 3 \log N \text{ secret bits}. \quad (11)$$

In other words, the comparison of the two results gives a tight balance between the number of secret random bits

needed to establish a private SRF and the secret classical capacity associated to it.

Is it really necessary to have a string of secret bits to establish a private SRF? Remarkably, this is not the case. In the following we sketch two alternative protocols, modeled on the Bennett-Brassard 1984 protocol [11] and the Ekert protocols [12], in which the private SRF is established by only using quantum and classical communication over an authenticated public channel: The main idea of both protocols is to exploit the fact that the information encoded in the multiplicity spaces \mathbb{C}^{m_j} of Eq. (2) is frame independent [7], since according to Eq. (3) the multiplicity spaces are invariant under rotations. Thus, even though Alice and Bob do not share a Cartesian frame, they can exchange qubits and test whether an eavesdropper is acting in the rotationally invariant subsystems of the Hilbert space. If the security level is too low, they can decide to abort the protocol (before any reference information is transmitted). This strategy prevents Eve from accessing the information encoded in the multiplicity spaces, namely, her POVM must have the form $P_i = \bigoplus_j P_{ij} \otimes \mathbb{1}_{m_j}$, for some suitable $P_{ij} \geq 0$, $\sum_i P_{ij} = \mathbb{1}_{2j+1}$. Since for states of the form (4) representation spaces and multiplicity spaces are maximally entangled, Eve's measurement gives no information about Alice's axes [again the state seen by Eve is the ρ_E of Eq. (9)].

Bennett-Brassard 1984 protocol-type protocol.—With probability $\frac{1}{2}$ Alice sends the state $U_h^{\otimes N} |A\rangle$, where h is a random rotation, otherwise she sends a test-state. The possible test states are

$$\begin{aligned} \tau_{j,m} &= \frac{\mathbb{1}_{2j+1}}{2j+1} \otimes |m\rangle \langle m| \\ \text{and} \quad \tilde{\tau}_{j,m} &= \frac{\mathbb{1}_{2j+1}}{2j+1} \otimes |\tilde{m}\rangle \langle \tilde{m}|, \end{aligned} \quad (12)$$

where $\{|m\rangle\}$ and $\{|\tilde{m}\rangle\}$ are two bases of \mathbb{C}^{m_j} related by a Fourier transform. An eavesdropper cannot tell whether the state $U_h^{\otimes N} |A\rangle$ or the test states were sent, as there is a large overlap between them, i.e., a fidelity $F = A_j^2 / (2j+1)^2$ (of order $1/N^3$ for $j \approx N/4$). On the other hand, Bob performs with probability $\frac{1}{2}$ the optimal measurement of orientation, otherwise he performs a test measurement. For the test measurement he randomly chooses one of the two von Neumann measurements $V_{j,m} = \mathbb{1}_{2j+1} \otimes |m\rangle \langle m|$, or $\tilde{V}_{j,m} = \mathbb{1}_{2j+1} \otimes |\tilde{m}\rangle \langle \tilde{m}|$. Then, as in the Bennett-Brassard 1984 protocol, using the authenticated public classical channel, Alice and Bob declare the ‘‘basis’’ they used: Alice announces whether she prepared an orientation state, a test state of the kind τ , or a test state of the kind $\tilde{\tau}$, and similarly Bob announces whether he measured the orientation, the observable V , or the observable \tilde{V} . They keep only the cases where Bob's measurement coincided with Alice's preparation, and discard the rest. Alice announces the values of j and m for the test states she sent, so that Bob can check whether or not the results of his

measurements match with her data. In this way, an eavesdropper in the rotationally invariant subsystems can be detected, and Bob can decide to abort the protocol, if the security level is not sufficiently high. Otherwise, Alice publicly communicates the random rotation h she performed on the optimal state $|A\rangle$. On each of the states Alice sent, Bob estimated the rotation gh with rms error $1/N$, where g is the unknown rotation connecting his axes with Alice's ones. Thus, knowing h , Bob can immediately infer g and align his axes. Notice that here N is the number of spins in each of the states that Alice sends. If she sends M states, the total number of transmitted spins is MN and Bob's accuracy will scale as $4/\sqrt{MN}$ (where the square root comes from the central limit theorem, and the factor 4 refers to the fact that only $\frac{1}{4}$ of the states are used in average for the orientation measurement). Compared with the previous protocol, the scaling in accuracy with the total number MN of spins is suboptimal, but the decrease in accuracy is needed because of the elimination of the prior shared random bits.

Ekert-type protocol.—Alice starts with two sets of N spin- $\frac{1}{2}$ particles, half of which she sends to Bob. The Hilbert spaces of both sets of spins, $\mathcal{H}_A^{\otimes N}$ and $\mathcal{H}_B^{\otimes N}$, can be decomposed as in Eq. (2), and she can prepare the entangled state

$$|\Phi\rangle_{AB} = \bigoplus_{j=0}^{(N/2)-1} \frac{A_j}{\sqrt{(2j+1)m_j}} \sum_{m,n} |jm\rangle_A |n\rangle_B |jm\rangle_B |n\rangle_A, \quad (13)$$

where A_j are the same coefficients as in the optimal state in Eq. (4). Note that for any value of the angular momentum the state $|\Phi\rangle_{AB}$ exhibits a maximal entanglement between the multiplicity spaces $\mathbb{C}_A^{m_j}$ and $\mathbb{C}_B^{m_j}$. For this reason, an eavesdropper in the rotationally invariant subsystems can be detected by Alice and Bob by testing Bell inequalities, as in the original Ekert protocol: to evade detection Eve must not act on the rotationally invariant subsystems of Bob. A SRF is established when Alice and Bob both perform the optimal orientation measurement, given by Eq. (6). In fact, the probability density that they measure the rotations h_A and h_B is $p(h_A, h_B) = \langle \Phi | M(h_A) \otimes U_g^{\otimes N \dagger} M(h_B) U_g^{\otimes N} | \Phi \rangle$, where g is the unknown rotation connecting their axes. A short calculation allows one to derive the following equality:

$$p(h_A, h_B) = \langle \Phi | M(h_A) \otimes U_g^{\otimes N \dagger} M(h_B) U_g^{\otimes N} | \Phi \rangle = \langle A | M(h_A h_B^{-1} g) | A \rangle, \quad (14)$$

which implies that $h_A h_B^{-1} g$ is distributed with the rms error $1/N$ of the optimal orientation measurement, even though the outcomes h_A and h_B are completely random, as the local states of Alice and Bob are rotationally invariant. Once Alice has communicated (with the authenticated public channel) the outcome h_A of her measurement, Bob

can use this information to retrieve the unknown rotation g with the precision $1/N$. Again, since Eve cannot touch the rotationally invariant subsystems, she can gain no information about Alice's axes. Also here, part of the exchanged spins is employed (in the Bell inequalities tests) to make up for the absence of the prior shared secret bits.

Conclusions.—In this Letter we have shown that quantum mechanics allows one to secretly communicate directions in space, either with or without the need of prior secret random bits. We have given a simple protocol that needs no entanglement and an entangled protocol that achieves the ultimate bounds in the precision of reference frame transmission. The unentangled protocol can be easily implemented experimentally using spin- $\frac{1}{2}$ particles or single photon polarization states. Two further protocols that do not need secret random bits have also been presented. The ideas of this Letter have been presented for spatial reference frames, however, they can be exploited to achieve the secret transmission of any possible reference frame, e.g., to obtain a secret clock synchronization.

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