

Transforming quantum operations: Quantum supermaps

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received 1 April 2008; accepted in final form 9 June 2008
published online 11 July 2008

PACS 03.67.-a – Quantum information
PACS 03.65.-w – Quantum mechanics
PACS 03.65.Ta – Foundations of quantum mechanics; measurement theory

Abstract – We introduce the concept of *quantum supermap*, describing the most general transformation that maps an input quantum operation into an output quantum operation. Since quantum operations include as special cases quantum states, effects, and measurements, quantum supermaps describe all possible transformations between elementary quantum objects (quantum systems as well as quantum devices). After giving the axiomatic definition of supermap, we prove a realization theorem, which shows that any supermap can be physically implemented as a simple quantum circuit. Applications to quantum programming, cloning, discrimination, estimation, information-disturbance trade-off, and tomography of channels are outlined.

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The input-output description of any quantum device is provided by the *quantum operation* of Kraus [1], which yields the most general probabilistic evolution of a quantum state. Precisely, the output state ρ_{out} is given by the quantum operation \mathcal{E} applied to the input state ρ_{in} as follows:

$$\rho_{out} = \frac{\mathcal{E}(\rho_{in})}{\text{Tr}[\mathcal{E}(\rho_{in})]}, \quad p(\mathcal{E}|\rho_{in}) := \text{Tr}[\mathcal{E}(\rho_{in})], \quad (1)$$

where $p(\mathcal{E}|\rho_{in})$ is the probability of \mathcal{E} occurring on state ρ_{in} , when \mathcal{E} is one of a set of alternative transformations, such as in a quantum measurement. Owing to its physical meaning, a quantum operation \mathcal{E} must be a linear, trace non-increasing, completely positive (CP) map (see, e.g. [2]). The most general form of such a map is known as Kraus form

$$\mathcal{E}(\rho) = \sum_j E_j \rho E_j^\dagger, \quad (2)$$

where the operators E_j satisfy the bound $\sum_j E_j^\dagger E_j \leq I$ so that $0 \leq p(\mathcal{E}|\rho_{in}) \equiv \text{Tr}[\sum_j E_j^\dagger E_j \rho_{in}] \leq 1$. Trace-preserving maps, i.e. those achieving the bound, are a particular kind of quantum operations: they occur deterministically and are referred to as *quantum channels*.

In general it is convenient to consider two different input and output Hilbert spaces \mathcal{H}_{in} and \mathcal{H}_{out} , respectively. In

this way, the concept of quantum operation can be used to treat also quantum states, effects, and measurements, which describe the properties of elementary quantum objects such as quantum systems and measuring devices. Indeed, states can be described as quantum operations with one-dimensional \mathcal{H}_{in} , i.e. with Kraus operators E_j given by ket-vectors $\sqrt{p_j}|\psi_j\rangle \in \mathcal{H}_{out}$, thus yielding the output state $\rho_{out} = \mathcal{E}(1) = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. A *quantum effect* $0 \leq P \leq I$ [3] corresponds instead to a quantum operation with one-dimensional \mathcal{H}_{out} , i.e. with Kraus operators given by bra-vectors $E_i = \langle v_i|$, yielding the probability $p(\mathcal{E}|\rho_{in}) \equiv \mathcal{E}(\rho_{in}) = \sum_i \langle v_i|\rho_{in}|v_i\rangle = \text{Tr}[P\rho_{in}]$ with $P = \sum_i |v_i\rangle\langle v_i|$. More generally, any quantum measurement can be viewed as a particular quantum operation, namely as a quantum-to-classical channel [4].

Channels, states, effects, and measurements are all special cases of quantum operations. What about then considering maps between quantum operations themselves? They would describe the most general kind of transformations between elementary quantum objects. For example a programmable channel [5] would be a map of this type, with a quantum state at the input and a channel at the output. Or else, a device that optimally clones a set of unknown unitary gates would be a map from channels to channels. We will call such a general class of quantum maps *quantum supermaps*, as they transform CP maps (sometimes referred to as superoperators) into CP maps.

In this paper we develop the basic tools to deal with quantum supermaps. The concept of quantum supermap is first introduced axiomatically, by fixing the minimal

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requirements that a map between quantum operations must fulfill. We then prove a realization theorem that provides any supermap with a physical implementation in terms of a simple quantum circuit with two open ports in which the input operation \mathcal{E} can be plugged. This result allows one to simplify the description of complex quantum circuits and to prove general theorems in quantum information theory. Moreover, the generality of the concept of supermap makes it fit for application in many different contexts, among which quantum programming, calibration, cloning, and estimation of devices.

To start with, we define the *deterministic supermaps* as those sending channels to channels. Conversely, a *probabilistic* supermap will send channels to arbitrary trace-non-increasing quantum operations. The minimal requirements that a deterministic supermap $\tilde{\mathcal{S}}$ must satisfy in order to be physical are the following: it must be i) linear and ii) completely positive. Linearity is required to be consistent with the probabilistic interpretation. Indeed, if the input is a random choice of quantum operations $\mathcal{E} = \sum_i p_i \mathcal{E}_i$, the output must be given by the same random choice of the transformed operations $\tilde{\mathcal{S}}(\mathcal{E}) = \sum_i p_i \tilde{\mathcal{S}}(\mathcal{E}_i)$, and, if the input is the quantum operation \mathcal{E} with probability p , the output must be the $\tilde{\mathcal{S}}(\mathcal{E})$ with probability p , implying $\tilde{\mathcal{S}}(p\mathcal{E}) = p\tilde{\mathcal{S}}(\mathcal{E})$. Clearly, these two conditions imply that $\tilde{\mathcal{S}}$ is a linear map on the linear space generated by quantum operations. Complete positivity is needed to ensure that the output of $\tilde{\mathcal{S}}$ is a legitimate quantum operation even when $\tilde{\mathcal{S}}$ is applied locally to a bipartite joint quantum operation, *i.e.* a quantum operation \mathcal{E} with bipartite input space $\mathcal{H}_{in} = \mathcal{H}_{in,A} \otimes \mathcal{H}_{in,B}$ and bipartite output space $\mathcal{H}_{out} = \mathcal{H}_{out,A} \otimes \mathcal{H}_{out,B}$. If $\tilde{\mathcal{S}}$ is a supermap transforming quantum operations with input (output) space $\mathcal{H}_{in,A}$ ($\mathcal{H}_{out,A}$), complete positivity corresponds to require that $\tilde{\mathcal{S}} \otimes \mathcal{J}_B(\mathcal{E})$ is a CP map for any bipartite quantum operation \mathcal{E} , \mathcal{J}_B denoting the identity supermap on the spaces labeled by B .

In order to deal with complete positivity it is convenient to use the Choi representation [6] of a CP map \mathcal{E} in terms of the positive operator E on $\mathcal{H}_{out} \otimes \mathcal{H}_{in}$

$$E := \mathcal{E} \otimes \mathcal{I}(|I\rangle\langle I|), \quad (3)$$

where $|I\rangle$ is the maximally entangled vector $|I\rangle = \sum_n |n\rangle|n\rangle \in \mathcal{H}_{in}^{\otimes 2}$, $\{|n\rangle\}$ an orthonormal basis, and \mathcal{I} is the identity operation. The correspondence $E \leftrightarrow \mathcal{E}$ is one-to-one, the inverse relation of eq. (3) being

$$\mathcal{E}(\rho) := \text{Tr}_{\mathcal{H}_{in}}[(I \otimes \rho^\top)E], \quad (4)$$

where $^\top$ denotes transposition in the basis $\{|n\rangle\}$. In terms of the Choi operator, the probability of occurrence of \mathcal{E} is given by $p(\mathcal{E}|\rho_{in}) = \text{Tr}[\rho_{in}^\top P]$, where P is the effect $P := \text{Tr}_{\mathcal{H}_{out}}[E]$. To have unit probability on any state, a quantum channel must have $P = I$, *i.e.* its Choi operator must satisfy the normalization

$$\text{Tr}_{\mathcal{H}_{out}}[E] = I_{\mathcal{H}_{in}}. \quad (5)$$

A supermap $\tilde{\mathcal{S}}$ maps quantum operations into quantum operations as $\mathcal{E}' = \tilde{\mathcal{S}}(\mathcal{E})$. In the Choi representation, the supermap $\tilde{\mathcal{S}}$ induces a linear map \mathcal{S} on Choi operators, as $E' = \mathcal{S}(E)$. Using eq. (4), we can get back $\tilde{\mathcal{S}}$ from \mathcal{S} as follows:

$$\mathcal{E}'(\rho) = \tilde{\mathcal{S}}(\mathcal{E})(\rho) = \text{Tr}_{\mathcal{K}_{out}}[(I \otimes \rho^\top)\mathcal{S}(E)]. \quad (6)$$

Of course complete positivity of \mathcal{E}' implies that the map \mathcal{S} is positive. On the other hand, it is easily seen that the bipartite structure of a joint operation \mathcal{E} over a composite system induces a bipartite structure of the Choi operator E . The local application of the supermap $\tilde{\mathcal{S}}$ —given by $\tilde{\mathcal{S}} \otimes \mathcal{J}(\mathcal{E})$ —then corresponds to the local application of \mathcal{S} —given by $\mathcal{S} \otimes \mathcal{J}(E)$ —whence $\tilde{\mathcal{S}}$ is CP, if and only if \mathcal{S} is CP.

Since the correspondence $\tilde{\mathcal{S}} \leftrightarrow \mathcal{S}$ is one-to-one, in the following we will focus our attention on \mathcal{S} . The supermap \mathcal{S} sends positive operators E on $\mathcal{H}_{out} \otimes \mathcal{H}_{in}$ to positive operators $\mathcal{S}(E)$ on generally different Hilbert spaces $\mathcal{K}_{out} \otimes \mathcal{K}_{in}$. Complete positivity of \mathcal{S} is equivalent to the existence of a Kraus form

$$\mathcal{S}(E) = \sum_i S_i E S_i^\dagger, \quad (7)$$

where $\{S_i\}$ are operators from $\mathcal{H}_{out} \otimes \mathcal{H}_{in}$ to $\mathcal{K}_{out} \otimes \mathcal{K}_{in}$.

The following Lemmas provide the characterization of deterministic supermaps:

Lemma 1. *Any linear operator C on $\mathcal{H}_{out} \otimes \mathcal{H}_{in}$ such that $\text{Tr}[CE] = 1$ for all Choi operators E of channels has the form $C = I \otimes \rho$, with ρ on \mathcal{H}_{in} satisfying $\text{Tr}[\rho] = 1$. For $C \geq 0$ one has $\rho \geq 0$.*

Proof. Consider a Choi operator E with effect $\text{Tr}_{\mathcal{H}_{out}}[E] = P \leq I$. Upon defining $D := \sigma \otimes (I - P)$ for some state σ on \mathcal{H}_{out} , we have that $E + D$ is the Choi operator of a channel, normalized as in eq. (5). Since by hypothesis $\text{Tr}[C(E + D)] = 1$, we have that

$$\begin{aligned} \text{Tr}[CE] &= 1 - \text{Tr}[CD] = 1 - \text{Tr}[C(\sigma \otimes I)] + \text{Tr}[C(\sigma \otimes P)] \\ &= \text{Tr}[C(\sigma \otimes P)], \end{aligned} \quad (8)$$

since $\sigma \otimes I$ is the Choi operator of a channel. Therefore,

$$\begin{aligned} \text{Tr}[CE] &= \text{Tr}[C(\sigma \otimes P)] = \text{Tr}[P \text{Tr}_{\mathcal{H}_{out}}[C(\sigma \otimes I)]] \\ &= \text{Tr}[\text{Tr}_{\mathcal{H}_{out}}[E] \text{Tr}_{\mathcal{H}_{out}}[C(\sigma \otimes I)]] \\ &= \text{Tr}[\rho P], \end{aligned} \quad (9)$$

where $\rho := \text{Tr}_{\mathcal{H}_{out}}[C(\sigma \otimes I)]$. Since ρ does not depend on E , the last equality can be rewritten as $\text{Tr}[CE] = \text{Tr}[(I \otimes \rho)E]$ for all positive E , whence $C = I \otimes \rho$, and in order to have $\text{Tr}[CE] = 1$ for all E , we must have $\text{Tr}[\rho] = 1$. Clearly $C \geq 0$ implies $\rho \geq 0$. ■

Lemma 2. *The supermap \mathcal{S} is deterministic iff there exists a channel \mathcal{N}_* from states on \mathcal{K}_{in} to states on \mathcal{H}_{in} such that, for any state ρ on \mathcal{K}_{in} , one has*

$$\mathcal{S}_*(I_{\mathcal{K}_{out}} \otimes \rho) = I_{\mathcal{H}_{out}} \otimes \mathcal{N}_*(\rho), \quad (10)$$

where \mathcal{S}_* is the dual map of \mathcal{S} defined in terms of the Kraus form in eq. (7) by

$$\mathcal{S}_*(O) := \sum_i S_i^\dagger O S_i. \quad (11)$$

Proof. One has $\text{Tr}[C\mathcal{S}(E)] = \text{Tr}[\mathcal{S}_*(C)E]$. Consider a positive operator $C = I \otimes \rho$ on $\mathcal{K}_{out} \otimes \mathcal{K}_{in}$, where ρ is a state on \mathcal{K}_{in} . We have that

$$1 = \text{Tr}[C\mathcal{S}(E)] = \text{Tr}[\mathcal{S}_*(C)E], \quad (12)$$

for all Choi operators E of channels. According to Lemma 7, this implies $\mathcal{S}_*(I \otimes \rho) = I \otimes \sigma$, where σ is a state for any state ρ . Since the maps $\rho \rightarrow I \otimes \rho$, \mathcal{S}_* and $I \otimes \sigma \rightarrow \sigma$ are all CP, we have $\sigma = \mathcal{N}_*(\rho)$, where \mathcal{N}_* is a CP trace preserving map from states on \mathcal{K}_{in} to states on \mathcal{H}_{in} . ■

Remarkably, the same mathematical structure of Lemma 2 characterizes *semi-causal quantum operations* [7], *i.e.* operations on bipartite systems that allow signaling from system A to system B but not viceversa. In our case, this structure originates from the causality of input-output relations. An equivalent condition for a supermap to be deterministic is given by the following:

Lemma 3. *The supermap \mathcal{S} is deterministic iff there exists an identity preserving completely positive map \mathcal{N} such that, for any operator E on $\mathcal{H}_{out} \otimes \mathcal{H}_{in}$, one has*

$$\text{Tr}_{\mathcal{K}_{out}}[\mathcal{S}(E)] = \mathcal{N}(\text{Tr}_{\mathcal{H}_{out}}[E]). \quad (13)$$

Proof. This lemma follows from the previous one by considering that

$$\begin{aligned} \text{Tr}[\rho \text{Tr}_{\mathcal{K}_{out}}[\mathcal{S}(E)]] &= \text{Tr}[(I \otimes \rho)\mathcal{S}(E)] = \\ \text{Tr}[\mathcal{S}_*(I \otimes \rho)E] &= \text{Tr}[(I \otimes \mathcal{N}_*(\rho))E] = \\ \text{Tr}[(I \otimes \rho)(\mathcal{I} \otimes \mathcal{N})(E)] &= \text{Tr}[\rho \mathcal{N}(\text{Tr}_{\mathcal{H}_{out}}[E])], \end{aligned} \quad (14)$$

for all states ρ on \mathcal{K}_{in} . The map \mathcal{N} is identity preserving because it represents \mathcal{N}_* in the Schrödinger picture. ■

Equation (13) shows that the effect $P' = \text{Tr}_{\mathcal{K}_{out}}[\mathcal{S}(E)]$ depends only on the effect $P = \text{Tr}_{\mathcal{H}_{out}}[E]$, *e.g.* not on $\text{Tr}_{\mathcal{H}_{in}}[E]$. Basically, this reflects the fact that, in the input/output bipartition of the Choi operator, the output must not influence the transformation of the input effect.

Now we show that deterministic supermaps, so far introduced on a purely axiomatic level, can be physically realized with simple quantum circuits. Upon writing a canonical Kraus form for the completely positive map \mathcal{N} as follows:

$$\mathcal{N}(P) = \sum_l N_l^\dagger P N_l, \quad (15)$$

and substituting the Kraus forms (7) and (15) into eq. (13), one obtains

$$\begin{aligned} \sum_n (\langle k_n | \otimes I) S_i E S_i^\dagger (I \otimes |k_n\rangle) = \\ \sum_m (\langle h_m | \otimes N_j^\dagger) E (N_j \otimes |h_m\rangle), \end{aligned} \quad (16)$$

where $\{|k_n\rangle\}$ and $\{|h_m\rangle\}$ are orthonormal basis for \mathcal{K}_{out} and \mathcal{H}_{out} , respectively, and identity operators must be considered as acting on the appropriate Hilbert spaces — \mathcal{K}_{in} on the top and \mathcal{H}_{in} on the bottom part of eq. (16). Equation (16) gives two equivalent Kraus forms for the same CP map, of which the second one is canonical (since $\{N_j\}$ is canonical and $\{|h_m\rangle\}$ are orthogonal). Therefore, there exists an isometry W connecting the two sets of Kraus operators as follows:

$$(\langle k_n | \otimes I) S_i = \sum_{mj} W_{ni,mj} (\langle h_m | \otimes N_j^\dagger), \quad (17)$$

with $W^\dagger W = I$. Explicitly

$$W_{ni,mj} := (\langle k_n | \otimes \langle a_i |) W (|h_m\rangle \otimes |b_j\rangle), \quad (18)$$

where $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ are orthonormal basis for two ancillary systems with Hilbert spaces \mathcal{A} and \mathcal{B} . From eq. (17) we then obtain

$$S_i = (I \otimes \langle a_i |) W (I \otimes Z), \quad (19)$$

where

$$Z = \sum_j |b_j\rangle \otimes N_j^\dagger. \quad (20)$$

Using eq. (7), we can now evaluate the output Choi operator as follows:

$$\mathcal{S}(E) = \text{Tr}_{\mathcal{A}}[W(I \otimes Z)E(I \otimes Z^\dagger)W^\dagger]. \quad (21)$$

Finally, using eq. (6) we get

$$\begin{aligned} \mathcal{E}'(\rho) &= \text{Tr}_{\mathcal{K}_{in}}[(I \otimes \rho^\top)\mathcal{S}(E)] = \\ \text{Tr}_{\mathcal{K}_{in} \otimes \mathcal{A}}[(I_{\mathcal{K}_{out} \otimes \mathcal{A}} \otimes \rho^\top)W(I \otimes Z)E(I \otimes Z^\dagger)W^\dagger] &= \\ \text{Tr}_{\mathcal{A}}[W(\mathcal{E} \otimes \mathcal{I}_{\mathcal{B}})(V\rho V^\dagger)W^\dagger], \end{aligned} \quad (22)$$

where $V = \sum_j |b_j\rangle \otimes N_j^*$ is the partial transposed of Z (see eq. (20)) on the second space. Since the map \mathcal{N} is identity preserving, V is an isometry, namely $V^\dagger V = I$. We have then proved the following realization theorem:

Theorem 1. *Every deterministic supermap $\tilde{\mathcal{S}}$ can be realized by a four-port quantum circuit where the input operation \mathcal{E} is inserted between two isometries V and W and a final ancilla is discarded as in fig. 1. The output operation $\mathcal{E}' = \tilde{\mathcal{S}}(\mathcal{E})$ is given by*

$$\tilde{\mathcal{S}}(\mathcal{E})(\rho) = \text{Tr}_{\mathcal{A}}[W(\mathcal{E} \otimes \mathcal{I}_{\mathcal{B}})(V\rho V^\dagger)W^\dagger]. \quad (23)$$

Since any isometry can be realized as a unitary interaction with an ancilla initialized in some reset state, the above Theorem entails a realization of supermaps in terms of unitary gates. However, we preferred stating it in terms of isometries in order to avoid the arbitrariness in the choice of the initial ancilla state.

We want now to emphasize that deterministic supermaps with $\mathcal{H}_{in} = \mathcal{K}_{in}$ do not preserve in general

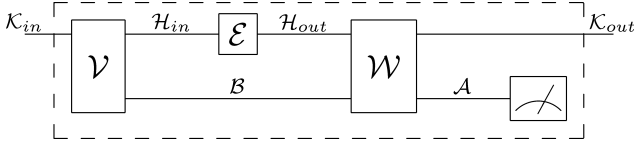


Fig. 1: Realization scheme for a supermap $\tilde{\mathcal{S}}$ sending the quantum operation \mathcal{E} to the quantum operation $\mathcal{E}' = \tilde{\mathcal{S}}(\mathcal{E})$, here represented by the dashed-boxed circuit. The input quantum operation \mathcal{E} sends states in \mathcal{H}_{in} to states in \mathcal{H}_{out} . The output quantum operation \mathcal{E}' sends states on \mathcal{K}_{in} to states on \mathcal{K}_{out} . The supermap is realized by two maps $\mathcal{V} = V \cdot V^\dagger$ and $\mathcal{W} = W \cdot W^\dagger$ made by isometries V and W located at the input and at the output ports of the quantum operation \mathcal{E} , respectively. The two isometries are possibly connected by an identity channel on the ancillary system with Hilbert space \mathcal{B} . At the output the ancilla with Hilbert space \mathcal{A} is either measured, each outcome post-selecting a *probabilistic supermap*, or simply discarded, thus realizing a *deterministic supermap*.

the probabilities of occurrence of arbitrary quantum operations: if \mathcal{E}' is not a channel $p(\mathcal{E}'|\rho_{in})$ can be different from $p(\mathcal{E}|\rho_{in})$. This is clear from fig. 1, since the input state ρ_{in} is generally changed by the isometry V . Indeed, one can have the extreme situation in which for every ρ_{in} the isometry V feeds into \mathcal{E} a fixed state on which \mathcal{E} occur with certainty, thus transforming the probabilistic quantum operation \mathcal{E} into a deterministic one \mathcal{E}' . For the above reason, we will call *probability preserving* those special deterministic supermaps with $\mathcal{H}_{in} = \mathcal{K}_{in}$ that also preserve occurrence probability for all states, namely which preserve the effect $P = \text{Tr}_{\mathcal{H}_{out}}[E]$. Since the input effect $P = \text{Tr}_{\mathcal{H}_{out}}[E]$ is transformed into the output effect $P' = \text{Tr}_{\mathcal{H}_{out}}[E']$ by the map \mathcal{N} , we may denote \mathcal{N} as the *effect-map* associated to the supermap \mathcal{S} , as in eq. (13). It is immediate to see that the supermap \mathcal{S} is probability preserving if and only if its effect-map \mathcal{N} is the identity map.

Up to now we have considered only deterministic supermaps. What about the probabilistic ones? By definition a probabilistic supermap \mathcal{S} turns quantum channels into arbitrary trace-nonincreasing quantum operations. In this case, it is not always possible to associate an effect-map \mathcal{N} to \mathcal{S} . However, a probabilistic supermap \mathcal{S}_1 is always completed to a deterministic one by some other supermaps $\mathcal{S}_2, \mathcal{S}_3, \dots$, which can occur in place of \mathcal{S}_1 , so that $\mathcal{S}_1 + \mathcal{S}_2 + \dots =: \mathcal{S}$ is deterministic. Each supermap \mathcal{S}_j is completely positive, hence it has a Kraus form with operators $\{S_k^{(j)}\}$, and all Kraus forms together provide a Kraus form for the deterministic supermap \mathcal{S} . Therefore, since according to eq. (19) each Kraus term $S_k^{(j)}$ is associated to an outcome of a von Neumann measurement $\{P_k^{(j)} = |a_k^{(j)}\rangle\langle a_k^{(j)}|\}$ over the ancilla with Hilbert space \mathcal{A} , any probabilistic supermap can be realized by a quantum circuit as in fig. 1, via postselection induced by a projective measurement $\{P^{(j)} = \sum_k P_k^{(j)}\}$ over the ancilla.

Theorem 2. *Every probabilistic supermap $\tilde{\mathcal{S}}$ can be realized by a four port scheme with measurement as in fig. 1, namely*

$$\tilde{\mathcal{S}}(\mathcal{E})(\rho) = \text{Tr}_{\mathcal{A}}[(P \otimes I)W(\mathcal{E} \otimes \mathcal{I}_{\mathcal{B}})(V\rho V^\dagger)W^\dagger] \quad (24)$$

with V and W isometries, and P orthogonal projector over a subspace of the ancillary space \mathcal{A} .

We want to stress the generality of the realization Theorem 2, which can be seen as the analog for probabilistic supermaps (here presented in finite dimensions) of Ozawa's realization theorem for quantum instruments [8]. Indeed eq. (24) describes any circuit in which an input device can be plugged, *e.g.* circuits with measurements performed in different stages, including the possibility of conditioning transformations on measurement outcomes. Therefore, whatever the input and the output of the quantum circuit might be (states, channels, or measurements), the following *delayed reading principle* will hold at a fundamental axiomatic level:

Corollary 1 (Delayed reading principle). *Every probabilistic quantum circuit is equivalent to a unitary circuit with a single orthogonal measurement at the output.*

Quantum supermaps can be applied to a tensor product of quantum operations, namely to a set of quantum operations that are not causally connected (the output of one map is not used as the input for another map). Assorted input sets of states, channels, and measurements can be considered as well, as long as they are not causally connected. Differently, if one wants to map an input set of two causally connected quantum operations, or possibly a memory channel [9], one needs to move to higher level of supermaps, namely *supermaps of supermaps*¹. Since the supermap is CP, one can introduce its Choi operator, and then consider the physically admissible mappings. In this way, one can build up a whole hierarchy of supermaps by considering the completely positive maps acting on the Choi operators of the lower level. An efficient diagrammatic approach to treat this problem is provided in ref. [10] by introducing the notion of *quantum comb*. The normalization condition for such higher-level supermaps has a recursive form, entailing the causal structure of input-output relations.

Before concluding, we outline list here some remarkable *ante litteram* examples of supermaps as well as some novel applications of this theoretical tool:

1. *Quantum compression of information and error correction.* The realization scheme of fig. 1 entails as a special case the coding/decoding scheme at the basis of quantum error correction and information compression. Schumacher's information compression [11] is a beautiful *ante litteram* example of supermap, which turns a noiseless communication channel on a smaller system into a

¹Two causally connected quantum operations are indeed a four-port circuit in which an input device can be plugged, *i.e.* referring to fig. 1 they are a supermap with one-dimensional ancilla space \mathcal{B} .

channel that reliably transfers states in a larger Hilbert space. Similarly, also error correction can be seen as a supermap, now turning a noisy channel on a larger Hilbert space into a noiseless channel acting on a smaller space. In both cases the supermap is given by the insertion of the input channel \mathcal{E} between two deterministic channels \mathcal{C} and \mathcal{D} (the *coding* and *decoding* maps, respectively), namely $\tilde{\mathcal{S}}(\mathcal{E}) = \mathcal{D}\mathcal{E}\mathcal{C}$, with the additional constraint that the ancilla \mathcal{B} in fig. 1 must be one-dimensional.

2. *Cloning of transformations.* An interesting application of quantum supermaps is the optimal cloning of transformations, instead of states. For example, an optimal $1 \rightarrow 2$ cloner of unitary transformations would be a four-port circuit that turns an unknown unitary channel $\mathcal{U} = U \cdot U^\dagger$ into a channel $\tilde{\mathcal{S}}(\mathcal{U})$ that maximizes the average channel fidelity with the bipartite channel $\mathcal{U} \otimes \mathcal{U}$. This device has been recently studied in ref. [12], and has optimal global fidelity $F = (d + \sqrt{d^2 - 1})/d^3$, surpassing the value achievable by any classical cloning scheme. The non-classical performances of the cloning circuit essentially depend on the possibility of entangling system \mathcal{H}_{in} with the ancilla \mathcal{B} in fig. 1 [12]. It is rather intriguing to investigate the possible cryptographic connections of the problem of optimally cloning unitary channels, which appear to be a harder task than cloning pure quantum states.

3. *Discrimination/estimation of channels and memory channels.* A probabilistic supermap $\tilde{\mathcal{S}}$ with one-dimensional \mathcal{K}_{in} and \mathcal{K}_{out} sends a quantum operation \mathcal{E} into a probability $p = \tilde{\mathcal{S}}(\mathcal{E})$. In this case the Kraus operators S_i are given by bra-vectors $\langle v_i |$ with $|v_i\rangle \in \mathcal{H}_{out} \otimes \mathcal{H}_{in}$, and eq. (7) yields $p = \sum_i \langle v_i | \mathcal{E} | v_i \rangle = \text{Tr}[EP]$ where $P := \sum_i |v_i\rangle\langle v_i|$. A set of such probabilistic supermaps $\{\tilde{\mathcal{S}}^{(j)}\}$ that sums up to a deterministic supermap $\tilde{\mathcal{S}} = \sum_j \tilde{\mathcal{S}}^{(j)}$ plays for channels of the same role that a POVM plays for states: for any channel \mathcal{E} , the supermap $\tilde{\mathcal{S}}^{(j)}$ gives a probability

$$p_j = \tilde{\mathcal{S}}^{(j)}(\mathcal{E}) = \text{Tr}[EP_j] \quad (25)$$

with $p_j \geq 0$, and $\sum_j p_j = 1$. The normalization of probabilities is ensured by the normalization condition of eq. (10), which here reads

$$\sum_j P_j = I_{\mathcal{H}_{out}} \otimes \sigma, \quad (26)$$

$\sigma = \mathcal{N}_*(1)$ being a quantum state on \mathcal{H}_{in} . This set of probabilistic supermaps, completely specified by the operators $\{P_j \geq 0\}$, describes the most general setup one can devise in order to test a given property of a quantum channel, and can be used to discriminate between two or more channels, or else to estimate a signal encoded into a parametric family of channels and quantum operations. Such a set of probabilistic supermaps is a particular case of *quantum circuit tester* introduced in refs. [10,13] to treat the discrimination of causally ordered sequences of channels and the discrimination of memory channels. We notice

that the particular case of probabilistic supermaps treated in this paragraph has been independently introduced in ref. [14] under the name *process POVM (PPOVM)*.

4. *Information-disturbance trade-off for quantum operations.* When the spaces \mathcal{K}_{in} and \mathcal{K}_{out} are non-trivial, a set of probabilistic supermaps $\{\tilde{\mathcal{S}}^{(j)}\}$ summing up to a deterministic one provides for channels the analog of an instrument. Differently from a tester, which has only classical output (the outcome j), the output of such a supermap is both a classical outcome and an output quantum operation. In this setting, supermaps provide the opportunity to address the completely new problem of information-disturbance trade-off for quantum channels. For example, we may try to estimate a completely unknown unitary U , producing at the same time a channel that is the most possibly similar to U . Similarly to the problem of cloning quantum channels, the information-disturbance trade-off rises the intriguing possibility of new cryptographic protocols based on channels instead of states.

5. *Quantum tomography of devices.* An interesting example of supermap is also that corresponding to tomography of quantum devices based their local application on bipartite states [15–17]. Tomography of quantum operations is based on the supermap that sends an input operation \mathcal{E} into an output state $\tilde{\mathcal{S}}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{I}(F)$ where F is a faithful state on $\mathcal{H}_{in}^{\otimes 2}$ [15], so that the output state is in one-to-one correspondence with the input operation. Note that, in order to have such a one-to-one correspondence, the map $\mathcal{S}(E) = \sum_i S_i E S_i^\dagger$ must be invertible, namely $\mathcal{S}(E) = 0$ if and only if $E = 0$. Tracing eq. (7) with an arbitrary operator O on $\mathcal{K}_{out} \otimes \mathcal{K}_{in}$, one can easily see that invertibility of \mathcal{S} is equivalent to the condition

$$\text{span}\{\mathcal{S}_*(O) | O \in \mathcal{B}(\mathcal{K}_{out} \otimes \mathcal{K}_{in})\} = \mathcal{B}(\mathcal{H}_{out} \otimes \mathcal{H}_{in}), \quad (27)$$

where span denotes the linear span of a set, and $\mathcal{B}(\mathcal{H})$ the set of all operators on \mathcal{H} . Since $\tilde{\mathcal{S}}$ sets an invertible correspondence between operations and states, one can perform an informationally complete measurement on the output state to completely characterize it. Note that, using probabilistic supermaps, we can also combine the deterministic map $E \mapsto \mathcal{S}(E)$ and the infocomplete measurement in a single object, introducing the notion of *informationally complete tester* (see also ref. [14]), which is a tester with the property that the mapping $E \rightarrow \{p(j|E) = \text{Tr}[P_j E]\}$ is invertible. In this case, the invertibility condition of eq. (27) becomes

$$\text{span}\{P_j\} = \mathcal{B}(\mathcal{H}_{out} \otimes \mathcal{H}_{in}). \quad (28)$$

As regards tomography of a POVM $\mathbf{P} = \{P_n\}$, this can be identified with the quantum-to-classical channel $\mathcal{E}_{\mathbf{P}}(\rho) = \sum_n \text{Tr}[P_n \rho] |n\rangle\langle n|$, and the above scheme applies as well.

6. *Programmable devices.* Programmable quantum channels [5] and measurements [18] are a remarkable example of supermaps. Impossibility of ideal programmability suggests to study optimal programmability.

One could consider either deterministic-approximate case [18–20], or the probabilistic-exact one [21–23]. In these applications an input state σ (the program) is turned into a channel \mathcal{E}_σ or into a measurement (POVM) $\mathbf{P}_\sigma = \{P_{\sigma,j}\}$. For channels the supermap is given by $\mathcal{E}_\sigma(\rho) = \tilde{\mathcal{S}}(\sigma)(\rho) = \text{Tr}_2[U(\rho \otimes \sigma)U^\dagger]$, where U is a unitary interaction. For programmable POVMs one has the set of probabilistic supermaps $\{\tilde{\mathcal{S}}^{(j)}\}$ such that $P_{\sigma,j} = \tilde{\mathcal{S}}^{(j)}(\sigma) = \text{Tr}_2[E_j(I \otimes \sigma)]$, where $\{E_j\}$ is a joint POVM. Equivalently, regarding the POVM as a channel from states to classical outcomes, one has the set of probabilistic supermaps $\{\tilde{\mathcal{S}}^{(j)}\}$ with one-dimensional \mathcal{H}_{in} and \mathcal{K}_{out} so that $p(j|\rho) = \tilde{\mathcal{S}}^{(j)}(\sigma)(\rho) = \text{Tr}[P_{\sigma,j}\rho] = \text{Tr}[E_j(\rho \otimes \sigma)]$, where $\{E_j\}$ is a joint POVM.

In conclusion, we have introduced the concept of *quantum supermap*, as a tool to describe all possible transformations between elementary quantum objects, *i.e.* states, channels, and measurements, with numerous applications to quantum information processing, cloning, discrimination, estimation, and information-disturbance trade-off for channels, tomography and calibration of devices, and quantum programming. A realization theorem has been presented, which shows that any abstract supermap can be physically implemented as a simple quantum circuit. The generality of the concept of supermap, describing any quantum evolution, allows one to use it as a tool to formulate and prove general theorems in quantum information theory and quantum mechanics, and to efficiently address a large number of novel applications.

We thank D. M. SCHLINGEMANN and D. KRETSCHMANN for many useful discussions about semi-causality and memory channels during the development of this work. We thank an anonymous referee for valuable suggestions in the revision of the original manuscript. This work has been funded by Ministero Italiano dell'Università e della Ricerca (MIUR) through PRIN 2005.

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