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GROUP THEORETICAL QUANTUM TOMOGRAPHY*

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A general method is presented for estimating the ensemble average of all operators of an arbitrary quantum system from a set of measurements of a *quorum* of observables. The *quorum*—i. e. a "complete" set of noncommuting observables for determining the quantum state of the system—is generated from a maximal commuting set of observables—the "seed observables"—under the action of a dynamical group of the quantum system. A method for deconvolving noise of any kind in the measurement is given in terms of the completely positive (CP) map pertaining the noise. This approach leads to a group theoretical classification of physically realizable quantum tomographic machines. These are made of two devices: 1) a measuring apparatus for the seed observables; 2) a transformation apparatus that achieves the dynamical group. Examples of applications are given in different physical contexts.

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1 Introduction

The name *quantum tomography* originated in quantum optics, where the set of quadrature probability distributions for varying phase was recognized [1] to be the Radon transform of the Wigner function, the Radon transform being the basic imaging tool in computerized medical tomography. Such analogy gave the name to a first qualitative technique for measuring the matrix elements of the radiation density operator [2]. Then, a first quantitative technique has been presented [3] (for a review, see Ref. [4]), which is now used in the lab [5]. The method has been then generalized to the estimation of an arbitrary observable of the field [6]. Finally, very recently, the route for a generalization to arbitrary quantum systems has been recognized [7, 8]. In this paper this route is pursued, and a general group theoretical approach is presented for estimating the ensemble average of all operators of an arbitrary quantum system from a set of measurements of a *quorum* of observables. The *quorum*—a concept introduced by

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U. Fano [9]—is a "complete" set of noncommuting observables for determining the quantum state of the system. In this paper this set will be generated from a maximal commuting set of observables—the "seed observables"—under the action of a dynamical group of the quantum system. This will lead to a complete group theoretical classification of physically realizable quantum tomographic machines, which are made of a single measuring apparatus for the seed observables and a transformation apparatus that achieves the dynamical group. We will also see that there is a way for deconvolving the noise in the measurement, using the CP map that describes the noise.

After presenting the general framework and the basic concepts in Section 2, in Section 3 the group theoretical classification of quantum tomographic machines is given. In Section 4 the group theoretical method is presented for deriving a general unbiased tomographic estimation rule for the ensemble average of all system operators. Some examples of applications in different physical contexts are given in Section 5. The max-likelihood estimation method is shortly discussed in Section 6. Finally, Section 7 closes the paper with conclusions and a list of open problems and further developments.

2 Quantum tomography for arbitrary quantum system

In the following, I will generalize the homodyne tomography technique to a method for an arbitrary quantum system $\mathcal{S}$ with Hilbert space $\mathcal{H}$. I will use the name *quantum tomography* to denote a technique for estimating the ensemble average $\langle A\rangle$ of all (generally complex unbounded) operators $A\in\mathcal{L}(\mathcal{H})$ of the quantum system $\mathcal{S}$ from a single set of measurement outcomes of a fixed set—socalled *quorum*—of observables. I call the set $\mathcal{Q}=\{Q_\lambda,\lambda\in\mathbf{\Lambda}\}$ of observables Q_λ (the index λ ranging in the manifold $\mathbf{\Lambda}$) a "*quorum*" for $\mathcal{S}$ if there is an *unbiased tomographic estimation rule* $\mathcal{E}$ for the *quorum*. An unbiased tomographic estimation rule $\mathcal{E}$: $\mathcal{L}(\mathcal{H})\times\mathcal{Q}\to\mathcal{L}(\mathcal{H})$ for the *quorum* $\mathcal{Q}$ is a family of operator valued functions $\mathcal{E}[A]$ over $\mathcal{Q}$ labeled by $A\in\mathcal{L}(\mathcal{H})$ with $[\mathcal{E}[A](q),q]=0$ for $q\in\mathcal{Q}$, such that the ensemble average $\langle A\rangle=\mathrm{Tr}[A\varrho]$ of A for arbitrary *unknown* state ϱ can be obtained by averaging over the *quorum* as follows

$$\langle A\rangle=\int_{\mathbf{\Lambda}}\mathrm{d}\mu(\lambda)\,\langle\mathcal{E}[A](Q_\lambda)\rangle\ , \tag{1}$$

where μ is a probability measure over $\mathbf{\Lambda}$ ($\int_{\mathbf{\Lambda}}\mathrm{d}\mu(\lambda)=1$, the integral is a sum for discrete $\mathbf{\Lambda}$). Because Eq. (1) must be true for arbitrary state ϱ, it is equivalent to

$$A=\int_{\mathbf{\Lambda}}\mathrm{d}\mu(\lambda)\,\mathcal{E}[A](Q_\lambda)\ , \tag{2}$$

where the integral convergence has to be considered in the ultra-weak sense. The function $\mathcal{E}[A]$ over $\mathcal{Q}$ is called *unbiased tomographic estimator* for A. From Eq. (1) it is clear that the unbiased tomographic estimation rule $\mathcal{E}$ must be linear on $\mathcal{L}(\mathcal{H})$.

The above definition corresponds to the following

Measuring procedure: in order to estimate the ensemble average $\langle A\rangle$ of any operator A one:

1) selects an observable Q_λ randomly in $\mathcal{Q}$ according to the probability measure μ;

2) measures Q_λ;
3) evaluates the function $\mathcal{E}[A](Q_\lambda)$;
4) averages the result over many measurements with varying $Q_\lambda \in \mathcal{Q}$.

Notice that the estimator rule is not unique, as there exist *null estimators* $\mathcal{N}$ over $\mathcal{Q}$ satisfying the identity

$$\int_{\mathbf{\Lambda}} \mathrm{d}\mu(\lambda)\, \mathcal{N}(Q_\lambda) = 0\ . \tag{3}$$

This sets an equivalence relation $\simeq$ between unbiased estimators: two estimators are equivalent iff they differ by a null estimator, namely $\mathcal{E}[A] \simeq \mathcal{E}'[A]$ iff $\mathcal{E}'[A] = \mathcal{E}[A] + \mathcal{N}$, with $\mathcal{N}$ null estimator.

3 Classification of quantum tomographic machines

A physically realizable *quorum* is the *group-dynamical quorum*. A group-dynamical *quorum* $\mathcal{Q}$—or $\mathbf{G}$-*quorum*—is achieved starting from a maximal commuting set of *seed observables* $\{H_\nu\}$ as the orbit of $\{H_\nu\}$ under the action $\mathrm{Ad}(g)H_\nu = gH_\nu g^{-1}$ of a group $\mathbf{G}$ of physical transformations, $g \in \mathbf{G}$. $\mathbf{G}$ has unitary representation over $\mathcal{H}$ and is called *dynamical group* for $\mathcal{S}$. Then, the manifold $\mathbf{\Lambda}$ of the $\mathbf{G}$-*quorum* is isomorphic to the homogeneous space $\mathbf{\Lambda} = \mathbf{G}/\mathbf{H}$, where $\mathbf{H}$ denotes the (generalized) stabilizer of the seed observables $\mathrm{Ad}(\mathbf{H})H_\nu \propto H_\nu$. For *composite systems* with Hilbert space $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$ a *quorum* is given by the Cartesian-product *quorum* $\mathcal{Q} = \times_{n=1}^N \mathcal{Q}_n$, and an estimation rule $\mathcal{E}$ is given by the tensor-product estimation rule $\mathcal{E} = \otimes_{n=1}^N \mathcal{E}_n$. This means that for tensor product operators $\otimes_{n=1}^N A_n$, one has

$$\mathcal{E}[\otimes_{n=1}^N A_n](Q_1, \ldots, Q_N) = \prod_{n=1}^N \mathcal{E}_n[A_n](Q_n)\ , \tag{4}$$

and the rule is extended to all operators in $\mathcal{L}(\mathcal{H})$ by linearity. There are, however, "entangled" estimator rules over $\otimes_{n=1}^N \mathcal{H}_n$ (see the following).

The tomographic estimation is also possible in the presence of controlled noise, i. e. when the dynamics of noise is known. In quantum mechanics any noise is described by a unit-preserving CP map $\Gamma\colon\ \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. In the following, the noise described by the CP map Γ will be referred to as Γ-noise. The Γ-noise can be *deconvolved* for the A-estimation if $\mathcal{E}[A](\mathcal{Q}) \in \mathcal{D}(\Gamma^{-1})$ and $[\Gamma^{-1}(\mathcal{E}[A](q)), q] = 0$ for $q \in \mathcal{Q}$. Then, the ensemble average of A can be estimated in the presence of Γ-noise, using the *deconvolved estimator* $\Gamma^{-1}(\mathcal{E}[A])$, and we will call the tomographic estimation *robust* to Γ-noise. Notice that the notions of *robustness* to noise and deconvolved estimation can be extended to the case when there is a new *quorum* $\mathcal{Q}_\Gamma$ isomorphic to $\mathcal{Q}$ and a one-to-one map $m_\Gamma\colon\ \mathcal{Q}_\Gamma \leftrightarrow \mathcal{Q}$ such that $[\Gamma^{-1}(\mathcal{E}[A](q)), m_\Gamma(q)] = 0$ for $q \in \mathcal{Q}$.

We are now in position to classify physical realizable tomographic machines in terms of dynamical groups. A *tomographic machine* $\mathcal{T} = [\{H_\nu\}, \mathbf{G}, R, \mathcal{E}]$ for the quantum system $\mathcal{S}$ with Hilbert space $\mathcal{H}$ is given by

1) a *measuring apparatus* for the *seed observables* $\{H_\nu\}$;
2) a *transformation apparatus* which achieves the dynamical group $\mathbf{G}$ of transformations with unitary representation R over $\mathcal{H}$;
3) a *unbiased tomographic estimation rule* $\mathcal{E}$ to find a set of unbiased estimators $\mathcal{E}[A]$ for $A \in \mathcal{L}(\mathcal{H})$.

In the following section I will present a general group theoretical method to derive an unbiased tomographic estimation rule for arbitrary system $\mathcal{S}$.

4 The estimation rule

We can easily obtain an unbiased tomographic estimation rule when a group $\mathbf{T}$ is available that has unitary irreducible representation (UIR) over the Hilbert space $\mathcal{H}$ of the quantum system $\mathcal{S}$. Here, for simplicity, I consider the case of unimodular group $\mathbf{T}$. Some of the following results, however, can be extended to groups having only left- or right-invariant measure. Let consider a UIR R of the group $\mathbf{T}$ over $\mathcal{H}$, with $R(g^{-1}) = R^\dagger(g)\ \forall g \in \mathbf{T}$. The case of projective UIR can be regarded as a central extension of $\mathbf{T}$. Consider the following operator on $\mathcal{H} \otimes \mathcal{H}$

$$E = \int \mathrm{d}g\, R^\dagger(g) \otimes R(g) \;, \tag{5}$$

where dg denotes a (Haar) invariant measure over $\mathbf{T}$ and the integral is extended to the whole $\mathbf{T}$ manifold. It is clear from the form of the operator E in Eq. (5) that the center $\mathbf{C}$ of the group is irrelevant: hence, in the following, I will consider the group $\mathbf{T}/\mathbf{C}$, and use the same symbol $\mathbf{T}$ to denote it. It is easy to verify that $E = E^\dagger$ is an *intertwining operator*, with the following action

$$E\, A \otimes B = B \otimes A\, E \;, \tag{6}$$

For square-integrable representation one also has

$$\mathrm{Tr}_1(E) = \mathrm{Tr}_2(E) = 1_{\mathcal{H}} \;, \tag{7}$$

with invariant measure dg normalized as follows

$$\int \mathrm{d}g\, |\langle u|R(g)|v\rangle|^2 = 1 \;, \tag{8}$$

for any two unit vectors $|u\rangle$ and $|v\rangle$ in the Hilbert space (for unimodular $\mathbf{T}$ and square-integrable UIR the integral in Eq. (8) is finite and is independent on the particular choice of vectors $|u\rangle$ and $|v\rangle$). One also has the useful identity on $\mathcal{H}$: $E|a\rangle\langle b| \otimes |c\rangle\langle \mathrm{d}| = |c\rangle\langle b| \otimes |a\rangle\langle d|$. From Eqs. (6) and (7) one gets the identity

$$A = \mathrm{Tr}_1[E\, A \otimes 1] \;, \tag{9}$$

which, using Eq. (5) rewrites as follows

$$A = \mathrm{Tr}_1\left[\int \mathrm{d}g\, R^\dagger(g) A \otimes R(g)\right] \;. \tag{10}$$

A comparison with Eq. (2) shows that Eq. (10) is already an estimation rule for a *quorum* of observables $\mathcal{Q} \simeq \mathbf{t} \doteq \mathrm{Lie}(\mathbf{T})$. For this reason, I will call $\mathbf{T}$ a *tomographic group* for the system $\mathcal{S}$. However, the above *quorum* is redundant, because any two linearly dependent elements in $\mathbf{t}$ are proportional to the same observable, and, hence, they correspond to the same measurement. We can do much better, determining a smaller *quorum* that, at the same time, is achieved through the action of a dynamical group $\mathbf{G}$ on a set of seed observables. We consider $\mathbf{G}$ as a group of derivations of $\mathbf{t}$ such that $\mathbf{t} = \mathbf{G}\mathbf{a}$ is the orbit of the maximal abelian subalgebra $\mathbf{a}$ of $\mathbf{t}$ spanned by the finite set of seed observables $\{H_\nu\}$. For simplicity, here I consider the case of a single seed observable (moreover, a set of seed observables can be treated as a single vector-valued observable). Consider the following "polar" parametrization of $\mathbf{T}$ as $g \in \mathbf{T}$, $g = g(\psi; \vec{n}) = \exp(i\psi\vec{n} \cdot \vec{T})$ where $\vec{T} \equiv \{T_i\}$ is a basis for the Lie algebra $\mathbf{t}$, and $\vec{n} \in \mathbf{G}/\mathbf{H}$ is a point on the homogeneous space $\mathbf{G}/\mathbf{H}$, $\mathbf{H}$ being the (generalized) stabilizer of the seed observable. The element $\psi\vec{n} \cdot \vec{T}$ of the (real) Lie algebra $\mathbf{t}$ is obtained as the rotation of the seed observable under the action of $\mathbf{G}$. Then Eq. (5) can be rewritten in the following (Mackey-Bruhat) polar decomposition

$$E = \int \mathrm{d}\mu(\psi) \int_{\mathbf{G}/\mathbf{H}} \mathrm{d}\vec{n}\, e^{-i\psi\vec{n}\cdot\vec{T}} \otimes e^{+i\psi\vec{n}\cdot\vec{T}} \,, \tag{11}$$

for suitable measure $\mathrm{d}\mu(\psi)$ and invariant measure $\mathrm{d}\vec{n}$ on $\mathbf{G}/\mathbf{H}$. Notice that for semisimple Lie group $\mathbf{T}$ all derivations are inner, whence a group of derivations of the Lie algebra $\mathbf{t}$ is $\mathrm{Exp}(\mathbf{t})$, i. e. the connected identity component of $\mathbf{T}$ (as a sign change belongs to the stability group, discrete subgroups of reflections and double coverings are factorized out). By exchanging the integrals, a comparison with Eq. (2) shows that Eq. (11) is equivalent to the following unbiased tomographic estimation rule

$$\mathcal{E}[A](\vec{n} \cdot \vec{T}) = \mathrm{Tr}_1 \left[\int \mathrm{d}\mu(\psi) e^{-i\psi\vec{n}\cdot\vec{T}} A \otimes e^{+i\psi\vec{n}\cdot\vec{T}} \right] \,, \tag{12}$$

if the integral converges, or, more generally, in some distribution sense. For traceclass operators A one has

$$\mathcal{E}[A](\vec{n} \cdot \vec{T}) = \int \mathrm{d}\mu(\psi)\, \mathrm{Tr}[e^{-i\psi\vec{n}\cdot\vec{T}} A] e^{+i\psi\vec{n}\cdot\vec{T}} \,. \tag{13}$$

However, in the following section, we will see examples in which neither A is traceclass, nor the integral in Eq. (12) is convergent, nevertheless an estimator exists: this is due to the existence of null estimators, since a divergent estimator maybe equal to a convergent one plus a divergent null estimator. Null estimators arise as a consequence of additional discrete symmetries that enter the stability group and reduce the *quorum*. This is the case, for example, of the Weyl group of a semisimple Lie algebra.

5 Examples

Example 1: *Homodyne tomography*. The quantum system $\mathcal{S}$ is the harmonic oscillator, with annihilation and creation operators $[a, a^\dagger] = 1$ acting on a infinite dimensional Hilbert space

$\mathcal{H}$ (the Fock space). The tomographic group $\mathbf{T}$ is the Heisenberg-Weyl group of displacement operators $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. The *quorum* is the set of field quadratures $X_\phi \doteq \frac{1}{2}\left(a^\dagger e^{i\phi} + a e^{-i\phi}\right)$ with phase $\phi \in [0,\pi]$, and uniform probability measure $\mathrm{d}\mu(\phi) = \frac{1}{\pi}\mathrm{d}\phi$. The physical group $\mathbf{G}$ is the group $U(1)$ of rotations of the quadrature phase ϕ, the maximal abelian algebra is just the span of a single "seed" quadrature. The stability group is the π-rotation of ϕ, which is equivalent to the inversion symmetry $X_{\phi+\pi} = -X_\phi$. For traceclass operators, an estimation rule is given by

$$\mathcal{E}[A](X_\phi) = \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}k\, |k| \mathrm{Tr}[A \exp(ikX_\phi)] \exp(-ikX_\phi)\ . \tag{14}$$

The tomographic machine is a homodyne detector with tunable phase ϕ relative to the local oscillator (LO). The machine is robust to loss [10], nonunit quantum efficiency [11] and Gaussian noise [10], above a bound for noise that depends on the estimated operator A (see Refs. [6, 7]). For example, for Gaussian noise with variance Δ^2, one has $\Gamma[\exp(ikX_\phi)] = \exp(ikX_\phi - \frac{1}{4}k^2\Delta^2)$. The deconvolved estimation rule reads

$$\Gamma^{-1}\left(\mathcal{E}[A](X_\phi)\right) = \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}k\, |k| e^{-\frac{1}{4}k^2\left(\frac{1}{2}-\Delta^2\right)} \mathrm{Tr}[A\!: \exp(ikX_\phi)\!:] \exp(-ikX_\phi)\ . \tag{15}$$

where $: \ldots :$ denotes normal ordering. Nonunit quantum efficiency corresponds to Gaussian noise with $\Delta^2 = (1-\eta)/(2\eta)$. It is also possible to derive estimators for unbounded operators. In Refs. [6, 7] the estimators for monomials in the field operators are given. For example, one has $\mathcal{E}_\eta[a^\dagger a](X_\phi) = 2X_\phi^2 - 1/(2\eta)$, where $\mathcal{E}_\eta$ denotes the deconvolved estimator for quantum efficiency η. See also Ref. [4] for the estimators of density matrix elements. The bounds for η are $\eta > 1/2$ for estimating density matrix elements in the number representation $A = |n\rangle\langle m|$ [11], $\eta > 0$ for monomials in the field operators $A = a^{\dagger n} a^m$ [6], $\eta > 1$ for the parity operator $A = (-)^{a^\dagger a}$ [6], etc. The null estimators are the operators $N_\phi^{k,n} = X_\phi^k e^{\pm i(k+2+2n)\phi}$, $k, n \geq 0$: these are the basis for the *adaptive homodyne tomography* technique of Refs. [12, 13]. They are due to the discrete symmetry $X_{\phi+\pi} = -X_\phi$.

Example 2: *Angular momentum tomography*. The quantum system $\mathcal{S}$ is the angular momentum. For an elementary particle (i. e. a UIR of $SU(2)$) the tomographic group is $\mathbf{T} = SU(2)$ and the physical group is $\mathbf{G} = SO(3)$, the Hilbert space of the UIR $\mathcal{H}$ is finite dimensional, with dimension $2J + 1$, J being the particle spin. The *quorum* is the set of angular momentum operators on the Bloch sphere $\mathcal{Q} = \{\vec{J}\cdot\vec{n}, \vec{n} \in \mathbf{\Lambda} = \mathbf{S^2} \simeq \mathbf{SU(2)/U(1)}\}$ with probability measure $\mathrm{d}\mu(\vec{n}) = \frac{1}{4\pi}\mathrm{d}\vec{n}$ uniform in the solid angle. For the seed observable one can adopt J_z. The estimation rule is

$$\mathcal{E}[A](\vec{J}\cdot\vec{n}) = \frac{2J+1}{\pi}\int_0^{2\pi} \mathrm{d}\psi \sin^2\frac{\psi}{2} \mathrm{Tr}[A \exp(-i\psi\vec{J}\cdot\vec{n})] \exp(i\psi\vec{J}\cdot\vec{n})\ . \tag{16}$$

The tomographic machine is composed by a Stern-Gerlach machine [14] for the measurement of the seed observable J_z preceded by a uniform magnetic field in the xy plane which achieves the physical rotation group (the polar angles on the Bloch sphere are determined by the direction of the field in the xy plane (azimuth) and the field intensity or particle flying time in the field (zenith). The machine is robust to any kind of noise (as long as $J < \infty$, and the noise is at “finite temperature”). For composite systems, the case of distinguishable particles is simply achieved by a tensor-product tomography as in Eq. (4). For indistinguishable particles, see Ref. [8] (see also Ref. [8] for other details).

Example 3: *Pauli tomography*. For spin $J = 1/2$ a *minimal quorum* is given by the set of Pauli matrices $\mathcal{Q} = \{\sigma_x\ ,\sigma_y\ ,\sigma_z\}$. From the simple identity $\langle A\rangle = \frac{1}{2}\{\langle\vec{\sigma}\rangle\cdot \mathrm{Tr}[\vec{\sigma}A] + \mathrm{Tr}[A]\}$ one obtains the estimation rule

$$\mathcal{E}[A](\sigma_\alpha) = \frac{3}{2}\{\mathrm{Tr}[A\sigma_\alpha]\sigma_\alpha + \frac{1}{2}\mathrm{Tr}[A]\}\ . \tag{17}$$

The tomographic group is the discrete subgroup of SU(2) made by the Pauli matrices with the identity. The physical group is generated by the $\pi/2$ rotations along x, y and z axis. As a seed observable one can choose σ_z. The machine is robust to any kind of noise, for example, the “Pauli channel” noise $\Gamma_{\ p}(A) = (1-p)A + \frac{p}{2}\mathrm{Tr}[A]\ ,\quad 0 \le p \le 1$, where one has the deconvolved estimation rule

$$\mathcal{E}_p[A](\sigma_\alpha) = \frac{3}{2}\{(1-p)^{-1}\mathrm{Tr}[A\sigma_\alpha]\sigma_\alpha + \frac{1}{2}\mathrm{Tr}[A]\}\ . \tag{18}$$

Example 4: *One-LO multimode homodyne tomography*. The quantum system $\mathcal{S}$ is a multimode e.m. field, with annihilation operators $a_1, a_2, \ldots,\ a_{n+1}$ acting on the Hilbert space $\mathcal{H}^{\otimes(n+1)}$ tensor-product of harmonic-oscillator infinite dimensional Hilbert spaces $\mathcal{H}$. The tomographic group is $\mathbf{T} = SU(n+1)$. The *quorum* is given by the set of quadratures $X(\theta,\psi) = \frac{1}{2}\left[A^\dagger(\theta,\psi) + A(\theta,\psi)\right]$ where $A(\theta,\psi) = \sum_{l=0}^n e^{-i\psi_l}u_l(\theta)a_l$ are bosonic mode operators, with $\vec{u}$ representing a point on a Poincaré hyper-sphere (more precisely $\psi = \{\psi_l \in [0,2\pi]\}\ ,\theta = \{\theta_l \in [0,\pi/2]\}$) with probability measure $\mathrm{d}\mu(\theta,\psi) = \prod_{l=0}^n \frac{\mathrm{d}\psi_l}{2\pi}\,\mathrm{d}\vec{u}$ (for the explicit parametrization of $\vec{u}$ see Ref. [15]). The annihilation operators $A(\theta,\psi)$ of the *quorum* are achieved as the orbit of a fixed single mode, say a_0, under the action of the physical group $\mathbf{G} = SU(n+1)$, the orbit being the complex projective space $P_n\mathbf{C} \simeq SU(n+1)/U(n)$, and the *quorum* manifold being $\mathbf{\Lambda} = P_n\mathbf{C}\times[0,\pi]$. The tomographic machine is a homodyne detector with (non monochromatic) phase-tunable and mode-tunable LO. Similarly to the one mode case in Example 1 the tomographic machine is robust to loss, nonunit quantum efficiency and Gaussian noise, above noise bounds that depends on the estimated operator A. For nonunit quantum efficiency η, the estimation rule for traceclass operators is given by

$$\mathcal{E}_\eta[A](X(\theta,\psi)) = \frac{\kappa^{n+1}}{n!}\int_0^\infty \mathrm{d}t\, e^{-t}t^n\, e^{i2\sqrt{\kappa t}X(\theta,\psi)}\mathrm{Tr}[A{:}e^{-i2\sqrt{\kappa t}X(\theta,\psi)}{:}]\ , \tag{19}$$

where $\kappa = \frac{2\eta}{2\eta-1}$. As an example, from the general estimation rule (19) one can derive the estimator for the matrix element $\langle\{n_l\}|R|\{m_l\}\rangle$ of the joint density matrix of modes:

$$\mathcal{E}_\eta[|\{m_l\}\rangle\langle\{n_l\}|](x;\theta,\psi) = e^{-i\sum_{l=0}^n(n_l-m_l)\psi_l}\ \frac{\kappa^{n+1}}{n!}\prod_{l=0}^n\left\{[-i\sqrt{\kappa}u_l(\theta)]^{\mu_l-\nu_l}\sqrt{\frac{\nu_l!}{\mu_l!}}\right\}$$

$$\times\int_0^\infty \mathrm{d}t\, e^{-t+2i\sqrt{\kappa t}x}\, t^{n+\frac{1}{2}\sum_{l=0}^n(\mu_l-\nu_l)}\prod_{l=0}^n L_{\nu_l}^{\mu_l-\nu_l}[\kappa u_l^2(\theta)t]\ , \tag{20}$$

where $\mu_l = \max(m_l, n_l)$, $\nu_l = \min(m_l, n_l)$ and $L_n^l(x)$ denote customary generalized Laguerre polynomials. Other examples are: the estimator of the probability distribution of the total number of photons $N = \sum_{l=0}^n a_l^\dagger a_l$

$$\mathcal{E}_\eta[|p\rangle\langle p|](x;\theta,\psi) \quad = \quad \frac{\kappa^{n+1}}{n!} \int_0^\infty \mathrm{d}t \, e^{-t+2i\sqrt{\kappa t}x} \, t^n L_p^n[\kappa t] \ , \tag{21}$$

with $|p\rangle$ eigenvector of N with eigenvalue p; the generating function of the moments of N for two modes with annihilator operators a and b

$$\mathcal{E}_\eta[z^{a^\dagger a + b^\dagger b}](x;\theta,\psi) = \frac{1}{(z + \frac{1-z}{\kappa})^2} \Phi\left(2; \frac{1}{2}; -\frac{1-z}{z + \frac{1-z}{\kappa}} x^2\right) \ , \tag{22}$$

$\Phi(\alpha;\beta;z)$ denoting the customary confluent hypergeometric function. In particular, for the first two moments one has $\mathcal{E}_\eta[a^\dagger a + b^\dagger b](x;\theta,\psi) = 4x^2 + \frac{2}{\kappa} - 2$, $\mathcal{E}_\eta[(a^\dagger a + b^\dagger b)^2](x;\theta,\psi) = 8x^4 + \left(\frac{24}{\kappa} - 20\right) x^2 + \frac{6}{\kappa^2} - \frac{10}{\kappa} + 4$. Notice that both operators $a^\dagger a + b^\dagger b$ and $(a^\dagger a + b^\dagger b)^2$ are unbounded, and the general estimation rule (19) corresponds to an integral (12) that is a (not tempered) distribution; nevertheless, both operators have unbiased estimators. For more details on the one-LO multimode homodyne tomography see Ref. [15].

6 Other estimation methods

The group theoretical *quorum* can be used also for different estimation strategies. In the present method the adopted strategy is the averaging procedure, namely the estimated value is the mean value over many measurement results. Because of the existence of null estimators, there are many equivalent unbiased estimators, which lead to different r.m.s. statistical error in the estimation. An adaptive least-squares method has been presented in Refs. [12, 13], where the estimator is "adapted" to each set of measured data. In addition to the averaging strategy with minimum r.m.s. error, other strategies can be of interest in different situations. For example, in Ref. [16] a max-likelihood strategy has been presented, which allows the estimation of the diagonal density matrix of radiation using tomographic homodyne scans (for finite dimensional truncated Hilbert space). Recently it has been shown [17] that this method can be extended to the whole density matrix for arbitrary quantum system, also in the presence of noise. The basis of the method is the Cholesky decomposition $P = V^\dagger V$ of a positive operator P, where V is an upper-triangular matrix. We use this decomposition for both the density matrix $R = T^\dagger T$ and for a *quorum* of positive operator valued measures (POVM) $P_\lambda(q) = Z_\lambda^\dagger(q) Z_\lambda(q)$. Moreover, the CP map of the Γ-noise acting on a operator O can be written as $\Gamma[O] = \sum_k A_k^\dagger O A_k$, with $\sum_k A_k^\dagger A_k = 1$. In estimating the matrix elements of R, the Likelihood function is $\mathcal{L} = \sum_i \log \mathrm{Tr}\{R\Gamma(P_{\lambda_i}(q_i))\}$, where the sum runs over the label of the ith measurement. Using the above decompositions the trace argument of the logarithm can be written in the very convenient form

$$\mathrm{Tr}\{R\Gamma(P_{\lambda_i}(q_i))\} = \sum_k \sum_{nm} |\langle n|T A_k^\dagger Z_\lambda^\dagger(q)|m\rangle|^2 \ , \tag{23}$$

where $|n\rangle$ and $|m\rangle$ label basis on $\mathcal{H}$. Because the *quorum* unambiguously determines the state, the absolute maximum of the Likelihood function must be unique for sufficiently many measurements.

7 Conclusions, open problems and further developments

We have seen a group theoretical approach to quantum tomography for arbitrary quantum system, which provides a general method for estimating the ensemble average of all operators of the system from a set of measurements of a *quorum* of observables. This approach leads to a complete group theoretical classification of physically realizable quantum tomographic machines, which are made of a measuring apparatus for seed observable(s) and a transformation apparatus that achieves a dynamical group of physical transformations. A method for deconvolving noise of any kind in the measurement has been given within the CP map approach. Examples of application of this method are the customary homodyne tomography, the spin tomography, and the one-LO multimode tomography. New applications are possible to different fields of physics. Particularly interesting is the the Poincaré group tomography, because this case exhibits all aspects of the present method in its full generality [18]. Such case corresponds to the complete tomography of a relativistic elementary free particle. The tomographic machine is a kind of Mőssbauer variant of the Stern-Gerlach tomographic machine for the angular momentum, where, in addition, the energy of the particle is measured in a moving frame.

The method proposed in this paper opens a set of problems that are currently under study: I) Find the minimal *quorum* for a given $\mathcal{H}$. For example, for spin J=1/2 the Pauli matrices are a *quorum* for spin tomography. However, for larger spins, this is no longer a *quorum*, and one needs the spin operators along all directions on the Bloch sphere in order to make a *quorum* valid for arbitrary J. Thus, one can understand how the problem of finding a minimal *quorum* for given $\mathcal{H}$ generally will resort to the representation theory of discrete subgroups of Lie groups. II) Find the *optimal* estimator for A in the equivalence class, according to a given criterion/strategy. It would be also particularly interesting to generalize this approach to a generic cost-function strategy. Also, a general explicit analytical form for the estimation rule for non trace-class operators is still lacking. III) For each tomographic machine $\mathcal{T}$ find the class of noises Γ for which the machine is robust.

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