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Quantum statistics of photon cloning machines

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Abstract

We show an experiment to study the statistics of the output state of an optimal $N \to M$ cloning machine that produces indistinguishable clones. The experiment is based on a photonic implementation of universal cloning transformations. © 2001 Published by Elsevier Science B.V.

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Quantum mechanics protects its intrinsic statistical nature through the impossibility of determining the wave function of a single quantum system via any imaginable measurement scheme over a single or a finite number of copies of the same system. This well assessed result [1], logically consistent within the theoretical framework of quantum mechanics, is equivalent to the impossibility of perfect quantum cloning [2], which in turn is due to unitarity of quantum mechanical transformations [3]. On the other hand, for the same reason, even an approximate optimal cloning cannot be of any help in improving the statistics of an ideal quantum measurement, and this poses bounds on optimal quantum cloning. In this Letter we analyze the possibility of performing measurements on many output copies of an optimal cloning machine, and show how the cloning statistics prevents from improving any ideal quantum measurement exploiting the vanishing of statistical errors according to the central limit theorem. As we will see, this is due to the intrinsic super-Bernoullian statistics of clones, which originates from the unavoidable quantum correlations among them. A concrete experiment based on the proposal of Ref. [4] will be also analyzed, which allows measurements of permutation invariant observables on clones made of indistinguishable photons.

We consider optimal $N \to M$ cloning transformations for two-state quantum systems [5–7] and with the proposed experimental scheme of Refs. [4,8]. The scheme is based on spontaneous parametric down-conversion, with the qubit encoded on the polarization state of a photon. In the interaction picture and in the limit of a very intense classical undepleted pump, this process is described by the Hamiltonian

$$H = \gamma\left(a^\dagger_{V1}a^\dagger_{H2} - a^\dagger_{H1}a^\dagger_{V2}\right) + \text{h.c.}, \qquad (1)$$

where $a^\dagger_{V1,2}$ is the creation operator of a photon with vertical polarization V in beam 1 or 2, respectively, and analogously for H which denotes horizontal po-

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larization. As shown in Ref. [4], since the above Hamiltonian is invariant under a general joint $SU(2)$ transformation of all polarization modes, the achieved cloning transformation is universal, namely its quality does not depend on the input state.

In Ref. [4] it was shown that after detecting $M-N$ photons in beam 2 at the output of the down-converter, for an initial state with N photons with vertical polarization in beam 1, the state of the M photons in beam 1 is proportional to

$$\sum_{k=0}^{M-N} \binom{M-k}{N} |M-k\rangle_{V\,V}\langle M-k| \otimes |k\rangle_{H\,H}\langle k|. \quad (2)$$

The detection of $M-N$ photons in beam 2 occurs with probability

$$\Gamma^{2(M-N)} \binom{M+1}{N+1} \left(1-\Gamma^2\right)^{-N-2},$$

where $\Gamma = \tanh \gamma t$ (t is the interaction time). State (2) is the same as the output of the optimal $N \to M$ cloning transformation of Ref. [5] by establishing the correspondence between the state of M spins-1/2 of the symmetric multiplet with z-component equal to $M/2-k$ and the state $|M-k\rangle_V|k\rangle_H$ with $M-k$ photons vertically polarized and k horizontally polarized. Notice, however, that there are important physical differences between the photonic case and the case of M spins. First of all, the M photons in state (2) are by definition indistinguishable, whence only permutation invariant collective measurements are allowed. Moreover, the cloning transformation (2) is realized a posteriori, in the sense that the output number M of copies is a random variable that is selected as the result of the measurement of the photon number in beam 2.

Since this setup can be implemented in our lab, this gives us a unique opportunity to study the measurements of permutation invariant operators of the M spins by measuring the corresponding photonic observables. Let us consider for example the operator $S_z = \sum_{i=1}^{M} \sigma_{zi}$, where σ_{zi} is the Pauli operator corresponding to the ith spin. By exploiting the correspondence between photonic Fock states and the symmetric multiplet of M spins the operator S_z corresponds to the photonic operator $\hat{D} = a_{V1}^{\dagger} a_{V1} - a_{H1}^{\dagger} a_{H1}$, which is simply the difference of the photon numbers of the two polarizations modes in beam 1. Therefore, the statistics of $\hat{D}$ is the same as the statistics of S_z. Let us start from the simple case of an initial state of N photons all with vertical polarization. The output state corresponding to a detection of $M-N$ photons in beam 2 is given by Eq. (2). The probability $P(d)$ of detecting d as the result of the measurement of the photon difference $\hat{D}$ is given by

$$P(d) = \frac{(N+1)(M-N)!}{(M+1)!} \frac{\left(\frac{M+d}{2}\right)!}{\left(\frac{M+d}{2}-N\right)!}, \quad (3)$$

where d has the same parity of M and ranges from $2N-M$ to M. Notice that $P(d)$ is a polynomial of degree N as a function of d. For example, in the particular case of one input copy ($N=1$) it is a linear function.

In the general case where the vertical polarization component of the input state is v, namely each input photon is in state $(\sqrt{v}a_V^{\dagger} + \sqrt{1-v}e^{i\phi}a_H^{\dagger})|0\rangle$, the probability of detecting d is given by

$$P(d) = \frac{(N+1)(M-N)!\left(\frac{M+d}{2}\right)!\left(\frac{M-d}{2}\right)!}{(M+1)!} v^M \times \sum_{l=0}^{M-N} \frac{(M-l)!^2 l!}{(M-l-N)!} \left(\frac{v}{1-v}\right)^{(M-d)/2-l} c_l^2, \quad (4)$$

with

$$c_l = \sum_{k=\max\{0,(M-d)/2-l\}}^{\min\{(M-d)/2,M-l\}} \frac{(-1)^k}{k!((M-d)/2-k)!} \times \frac{1}{(M-l-k)!(l+k-(M-d)/2)!} \times \left(\frac{1-v}{v}\right)^k, \quad (5)$$

where d has the same parity of M and ranges from $-M$ to M. In terms of spin statistics, $P(d)$ is the probability of measuring the value d for S_z when the initial state of the N spins has mean value $2v-1$ for the operator σ_z. Notice that the probability distribution $P(d)$ is independent of the phase ϕ of the initial state.

The mean value observed in the measurement of $\hat{D}$ is given by

$$\langle \hat{D} \rangle = (2v-1)\frac{N(M+2)}{N+2}, \quad (6)$$

and the corresponding variance takes the form

$$\begin{aligned}\langle \Delta \hat{D}^2 \rangle &\equiv \langle \hat{D}^2 \rangle - \langle \hat{D} \rangle^2 \\ &= 4\frac{(2v-1)^2}{(N+2)^2(N+3)} \\ &\quad \times \left[M^2(N+1) + M\left(-N^2+10N+20\right) \right. \\ &\qquad \left. - 2N^2 - 11N - 18\right] \\ &\quad + \frac{4v(1-v)}{(N+2)(N+3)} \\ &\quad \times \left[2M^2(N+1) - M\left(N^2-5N+14\right) \right. \\ &\qquad \left. - 2N^2 + 2N + 18\right]. \end{aligned} \tag{7}$$

Notice that the above variance has a leading term which is proportional to M^2. As we will see in the following, this means that the statistical error associated to the estimation of $\langle \sigma_z \rangle$ from the sample of M clones does not vanish for large values of M, but remains a constant. In fact, when inferring $\langle \sigma_z \rangle$ from the average over clones $\sum \sigma_{zi}/M \equiv S_z/M$, the statistical error ϵ_M would be $\epsilon_M = \sqrt{\langle \Delta S_z^2 \rangle / M^2}$, which is constant in the limit $M \to \infty$, whereas for a Bernoulli statistics (which would be obtained by measuring uncorrelated clones) one has the error from the central limit theorem $\epsilon_M = \sqrt{\langle \Delta \sigma_z^2 \rangle / (M-1)}$. This is in agreement with the impossibility of determining the wave function of a single quantum system [1]. Actually, the estimation of $\langle \sigma_z \rangle$ from a sample of M clones can never lead to a better result for any M than performing this measurement directly on the N initial copies. This is due to the super-Bernoullian character of the probability distribution (4). In order to clarify this, let us consider the Bernoulli probability distribution $P_B(d)$ that would be obtained in the measurement of the operator $\hat{D}$ performed on M copies in the same pure state $(\sqrt{v} a_V^\dagger + \sqrt{1-v}\, e^{i\phi} a_H^\dagger)|0\rangle$, which is the state of each input copy of the cloner. Such a distribution is given by

$$P_B(d) = \binom{M}{\frac{M-d}{2}} v^{(M+d)/2} (1-v)^{(M-d)/2}. \tag{8}$$

It is clear that in this case the measurement of σ_z leads to a vanishing error for large values of M. In order to compare the probability distribution (4) with Bernoullian (8) we rescale the operator S_z as $\tilde{S}_z = S_z/\eta_{NM}$, where

$$\eta_{NM} = \frac{N}{N+2} \frac{M+2}{M}$$

is the shrinking factor [6] of the $N \to M$ cloning transformation. This rescaling is introduced so that the two probability distributions (4) and (8) have the same average value. The statistical fluctuations corresponding to the operator $\tilde{S}_z$ for the output state of the cloner can also be written as

$$\langle \Delta \tilde{S}_z^2 \rangle = \frac{M}{\eta_{MN}^2} \langle \Delta \sigma_{z1}^2 \rangle + \frac{M(M-1)}{2\eta_{MN}^2} \langle \Delta \sigma_{z1} \Delta \sigma_{z2} \rangle, \tag{9}$$

where $\langle \Delta \sigma_{z1}^2 \rangle = 1 - (2v-1)^2 \eta_{NM}^2$ is the statistical fluctuation corresponding to the σ_z operator for a single output copy and $\langle \Delta \sigma_{z1} \Delta \sigma_{z2} \rangle = \langle (\sigma_{z1} - \langle \sigma_{z1} \rangle)(\sigma_{z2} - \langle \sigma_{z2} \rangle) \rangle$ refers to the correlation between two output copies. As we can see by comparing the above equation to Eq. (7), the correlation terms are responsible for the fact that the statistical fluctuations of the operator $s_z = S_z/M$ are constant for large values of M, and therefore the σ_z component of the state cannot be measured perfectly by performing measurements on the output state of a cloner. On the other hand, a comparison between the variance in Eq. (7) with the Bernoulli variance for N spins proves the impossibility of improving the quantum measurement of σ_z through statistics on clones. Hence, the super-Bernoulli character of the clones statistics is an in-principle feature.

As an example, we show in Fig. 1 the probability distributions corresponding to Eqs. (4) and (8) for the case of the $N = 5 \to M = 30$ cloning transformation with initial states with $v = 1/2$. The super-Bernoullian character of the probability distribution (4) is striking.

Let us now discuss the experimental aspects of the measurement apparatus which is presently adopted within the present investigation. The universal quantum cloner is a *quantum-injected, entangled OPA* very similar to the apparatus aimed at the implementation of an all optical Schrödinger cat recently reported in Ref. [9]. Consider the diagram shown in Fig. 2. Two equal and equally oriented nonlinear crystals (e.g., beta-barium-borate) cut for type II phase matching are excited by two beams derived from a common UV laser beam at a wavelength of $\lambda_p = 400$ nm. In the present experiment the UV beam is supplied by sec-

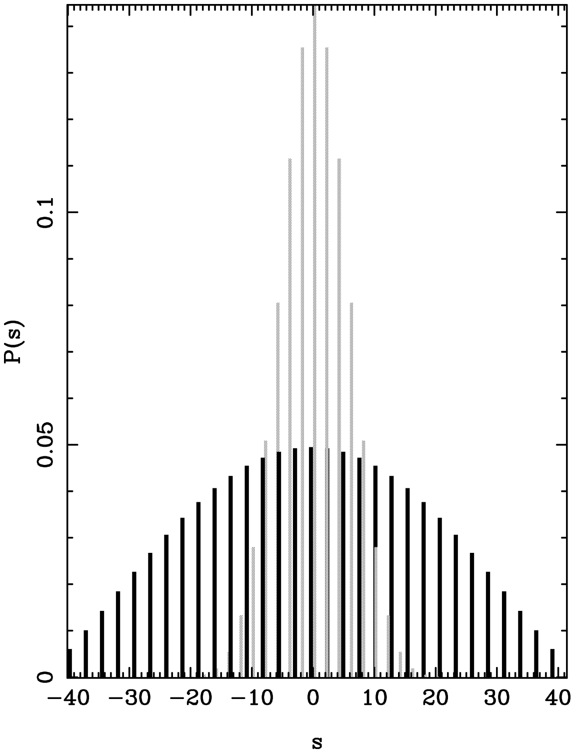

Fig. 1. Probability distributions corresponding to Eqs. (4) and (8) for the $5 \to 30$ cloning transformation with initial states with $v = 1/2$.

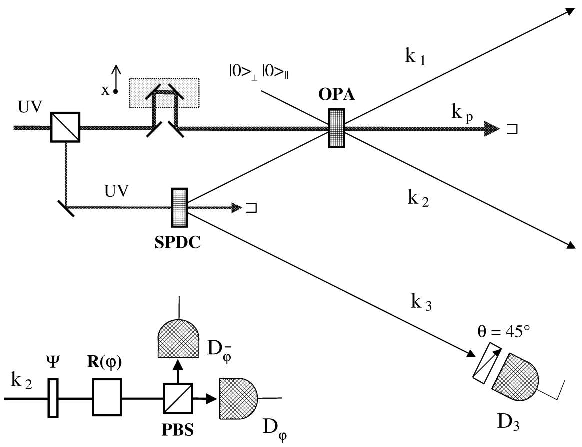

Fig. 2. The universal quantum cloner is a *quantum-injected, entangled OPA* very similar to the apparatus aimed at the implementation of an all optical Schrödinger cat of Ref. [9]. For the setup description see text.

ond harmonic generation of the output beam of a Coherent MIRA TI:SA mode-locked laser consisting of a train of 150 fs pulses emitted at a rate of 76 MHz. The average emitted power does not exceed 0.6 W and then the amplification gain is of the order $g \simeq 0.02$. Consequently a number of clones $M > 4$ cannot be easily produced for a number $N = 1$ of input qubits on mode $\mathbf{k}_1$. We expect a far larger efficiency by the next implementation within the apparatus of a regenerative OPA amplifier Coherent REGA9000. In this case the value of g will be multiplied by a factor about 20, and the cloning efficiency is expected to increase by the same factor. Crystal 1 is the spontaneous parametric down converter source of π-entangled photon couples emitted, with wavelength $2\lambda_p = 800$ nm, over the modes $\mathbf{k}_1$, $\mathbf{k}_3$ determined by two fixed 1 mm pinholes placed 2 m away from the nonlinear source crystal. In order to prevent any unwanted EPR type state reduction affecting the overall superposition process, the photon emitted over the output mode $\mathbf{k}_3$ is filtered by a polarization analyzer with axis oriented at 45° to the horizontal (t.h.) before being detected by D_3. A click at D_3 opens a gate selecting all registered outcomes, thus providing the *conditional* character of the overall cloning experiment. The photon emitted over $\mathbf{k}_1$ provides the quantum-injection into the OPA, physically consisting of the other nonlinear crystal, referred to as crystal 2. The measurement apparatus consists of equal, specially selected, low noise EGG SPCM-AQR14 cooled avalanche Silicon counters having a quantum efficiency Q.E. $\approx 60\%$ at the detection wavelength $\lambda = 850$ nm. The detectors are connected to a set of Stanford Research SR-400 gated photon counters and analyzed by a computer assisted multi channel analyzer (MCA) Canberra type Genie-2000. By this method the first experimental evidence of cloning single photon states ($N = 1$, $M \simeq 3$) has indeed been achieved recently in our Quantum Optics Laboratory in Rome [8].

In conclusion, we have shown how optimal quantum cloning protects the intrinsic uncertainty of quantum mechanics from being improved by means of statistics on many clones of the same quantum system. We have seen that this is due to the intrinsic super-Bernoullian statistics of clones, which originates from the unavoidable quantum correlations among them. This fundamental feature of quantum cloning is presently being investigated in our laboratory by the experimental method just outlined.

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