Evolution of twin-beam in active optical media

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Abstract. We study the evolution of twin-beam propagating inside active media that may be used to establish a continuous variable entangled channel between two distant users. In particular, we analyze how entanglement is degraded during propagation, and determine a threshold value for the interaction time, above which the state become separable, and thus useless for entanglement based manipulations. We explicitly calculate the fidelity for coherent state teleportation and show that it is larger than one half for the whole range of parameters preserving entanglemenent.

The crucial ingredient of quantum information is entanglement, which is the essential resource for quantum computing, teleportation, and cryptographic protocols. Entanglement is known to to be a valuable resource for improving measurements [1], spectroscopy [2, 3], lithography [4], interferometry [5], and tomography of quantum devices [6]. Most of these concepts were initially developed for discrete quantum variables, in particular quantum bits. Recently, however, much attention has been devoted to the use of continuous variables (CV), especially Gaussian state of light by means of linear optical circuits [7], since CV may be easier to manipulate than quantum bits in order to perform various information processes [8]. In the optical implementation of quantum information processing, the most relevant entangled channel is provided by the the twin-beam state (also called two-mode squeezed vacuum) of two modes of radiation, which, in the Fock bases, is expressed as

$$|X\rangle\rangle = \sqrt{1-x^2}\sum_p x^p\,|p\rangle\otimes|p\rangle\ . \tag{1}$$

The twin-beam is the maximally entangled state (for a given, finite, value of energy) of two modes of radiation. It can be produced either by mixing two single-mode squeezed vacuum (with orthogonal squeezing phases) in balanced beam splitter or, from the vacuum, by a nondegenerate parametric optical amplifier (NOPA). The evolution operator of the NOPA reads as follows $U_\lambda = \exp\left[\lambda\left(a^\dagger b^\dagger - ab\right)\right]$ where the "gain" λ is proportional to the interaction-time, the nonlinear susceptibility, and the pump intensity. We have $x = \tanh\lambda$, whereas the number of photons of the twin-beam is $N = 2\sinh^2\lambda = 2x^2/(1-x^2)$. In view of the duality squeezing/entanglement via balanced beam-splitter the parameter λ is sometimes referred to as the squeezing parameter of the twin-beam (1).

The Wigner function $W_0(x_1, y_1; x_2, y_2)$ of a twin-beam is Gaussian, and is given by (in the following we will omit the arguments, and denote the Wigner function by W)

$$W_0 = \left(2\pi\sigma_+^2\,2\pi\sigma_-^2\right)^{-1}\,\exp\left[-\frac{(x_1+x_2)^2}{4\sigma_+^2} - \frac{(y_1+y_2)^2}{4\sigma_-^2} - \frac{(x_1-x_2)^2}{4\sigma_-^2} - \frac{(y_1-y_2)^2}{4\sigma_+^2}\right]$$

where the variances are given by

$$\sigma_+^2 = 1/4\exp\{2\lambda\} \qquad \sigma_-^2 = 1/4\exp\{-2\lambda\}\ . \tag{2}$$

In applications such teleportation or cryptography one needs to transfer entanglement among distant partners, and therefore to transmit entangled states along some kind of channel. For optical implementation this is usually accomplished by means of (active) optical fibers. As a matter of fact, the propagation of twin-beam in optical media unavoidably lead to degradation of entanglement due to decoherence induced by losses and noise. In this paper, we study the evolution of twin-beam in active optical media, such the pair of optical fibers that may be used to transmit twin-beam, and analyze the separability of the evolved state as a function of the fiber parameters. A threshold value for the interaction time, above which the entanglement is destroyed, will be analytically derived.

In [9] and [10] the decoherence of twin-beam due to amplitude damping has been studied numerically in terms of the relative entropy of entanglement and the Bell nonlocality factor respectively. Here we analyze the propagation of twin-beam, in both active and passive fibers, using the Wigner function. In this way we are able to analytically study the evolution as well as the entanglement properties. In particular, we are able to check the positivity of the partial transpose (PPT), which is a necessary and sufficient condition for separability for two-mode Gaussian state of light. For some special choices of the parameters our results may be compared, and are in agreement, to that of Ref. [11] where the nonlocality Bell factor is computed from the Wigner function, and of Ref. [12] where the PPT condition is applied to the transformation induced by a generic four-port (amplifying or absorbing) optical device.

In the following we first study the propagation of twin-beam in active media by transforming the corresponding Master equation into a Fokker-Planck equation for the two-mode Wigner function. Then, we apply the PPT condition to check separability and determine a threshold value t_s for the interaction time, above which the state is separable and thus useless for the purposes of quantum information processing.

The propagation of a twin-beam inside active media can be modeled as the coupling of each part of the twin-beam with a non zero temperature reservoir. The dynamics can be described in terms of the two-mode Master equation

$$\frac{d\varrho_t}{dt} \equiv \mathcal{L}\varrho_t = \Gamma_a(1+M_a)L[a]\varrho_t + \Gamma_b(1+M_b)L[b]\varrho_t + \Gamma_a M_a L[a^\dagger]\varrho_t + \Gamma_b M_b L[b^\dagger]\varrho_t \tag{3}$$

where $\varrho_t \equiv \varrho(t)$, $\Gamma_a = \Gamma_b = \Gamma$ denotes the (equal) damping rate, $M_a = M_b = M$ the number of background thermal photons, and $L[O]$ is the Lindblad superoperator

$$L[O]\varrho_t = O\varrho_t O^\dagger - \frac{1}{2}O^\dagger O\varrho_t - \frac{1}{2}\varrho_t O O^\dagger\ .$$

The terms proportional to $L[a]$ and $L[b]$ describe the losses, whereas the terms proportional to $L[a^\dagger]$ and $L[b^\dagger]$ describe a linear phase-insensitive amplification process. This can be due either to fiber dynamics or to thermal hopping; in both cases no phase information is carried. Of course, the dynamics inside the two fibers are independent on each other.

The master equation (3) can be transformed into a Fokker-Planck equation for the two-mode Wigner function $W(x_1, y_1; x_2, y_2)$ Using the differential representation of the superoperators in Eq. (3) the corresponding Fokker-Planck equation reads as follows

$$\partial_\tau W_\tau = \left[\frac{1}{8}\left(\sum_{j=1}^{2}\partial^2_{x_j x_j} + \partial^2_{y_j y_j}\right) + \frac{\gamma}{2}\left(\sum_{j=1}^{2}\partial_{x_j} x_j + \partial_{y_j} y_j +\right)\right] W_\tau \,, \tag{4}$$

where τ denotes the rescaled time $\tau = \Gamma/\gamma\, t$, and the drift term γ is given by

$$\gamma = \frac{1}{2M+1}\,. \tag{5}$$

The solution of Eq. (4) can be written as

$$W_\tau = \int\!dx_1' \int\!dx_2' \int\!dy_1' \int\!dy_2' \;\; W_0(x_1', y_1'; x_2', y_2') \prod_{j=1}^{2} G_\tau(x_j|x_j') G_\tau(y_j|y_j') \tag{6}$$

where $W_0(x_1, y_1; x_2, y_2)$ is the Wigner function at $\tau = 0$, and the Green functions $G_\tau(x_j|x_j')$ are given by

$$G_\tau(x_j|x_j') = \frac{1}{\sqrt{2\pi D^2}} \exp\left[-\frac{(x_j - x_j' e^{-\frac{1}{2}\gamma\tau})^2}{2D^2}\right] \,, \quad D^2 = \frac{1}{4\gamma}(1 - e^{-\gamma\tau})\,. \tag{7}$$

Remarkably, the diffusion coefficients D^2 remains positive for all times. Eq. (4) admits a stationary solution, which can be easily derived from Eq. (4) and, independently on the initial state, it has the Gaussian form

$$W_{\rm stat} = \frac{2}{\pi(2M+1)} \exp\left(-2\,\frac{x_1^2 + y_1^2}{2M+1}\right) \times \frac{2}{\pi(2M+1)} \exp\left(-2\,\frac{x_2^2 + y_2^2}{2M+1}\right) \,, \tag{8}$$

corresponding to the (factorized) two-mode thermal density matrix given by

$$\varrho_{stat} = \mu_M \otimes \mu_M \,, \tag{9}$$

where μ_M is the density matrix of a thermal state with M thermal photons

$$\mu_M = \frac{1}{1+M}\left(\frac{M}{1+M}\right)^{a^\dagger a} \,. \tag{10}$$

Eq. (10) states that after a long interaction time entanglement is totally destroyed, independently on the amount of entanglement initially impinged into the channel. This is not unexpected since the losses and the amplification noise resulting from the propagation in the fibers induce decoherence on the (initially entangled) beams. The questions we want to answer are the following: given an initial twin-beam $|X\rangle\rangle$ how entanglement is degraded during the propagation ? And how long does the entanglement survive ? In order to answer to these questions, we need to solve Eq. (4) at a generic time τ. The Wigner function $W_\tau(x_1, y_1; x_2, y_2)$ can be obtained by the convolution (6), which can be easily evaluated since the initial Wigner function of the twin-beam is Gaussian. We have

$$W_\tau = \left(2\pi\Sigma_+^2\, 2\pi\Sigma_-^2\right)^{-1} \exp\left[-\frac{(x_1+x_2)^2}{4\Sigma_+^2} - \frac{(y_1+y_2)^2}{4\Sigma_-^2} - \frac{(x_1-x_2)^2}{4\Sigma_-^2} - \frac{(y_1-y_2)^2}{4\Sigma_+^2}\right] \,, \tag{11}$$

with the variances given by

$$\Sigma_+^2 = e^{-\gamma\tau}\sigma_+^2 + D^2 \qquad \Sigma_-^2 = e^{-\gamma\tau}\sigma_-^2 + D^2\,. \tag{12}$$

A quantum state of a bipartite system is *separable* if its density operator can be written as $\varrho = \sum_k p_k \sigma_k \otimes \tau_k$, where $\{p_k\}$ is a probability distribution and τ's and σ's are single-system density matrices. If a state is separable the correlations between the two systems are of purely classical origin. A quantum state which is not separable contains quantum correlations *i.e.* it is entangled. A necessary condition for separability is the positivity of the density matrix ϱ^T, obtained by partial transposition of the original density matrix (PPT condition) [13]. In general, PPT has been proved to be only a necessary condition for separability. However, for some specific sets of states PPT is also a sufficient condition. These include states of 2×2 and 2×3 Hilbert spaces [14], and Gaussian states (states with a Gaussian Wigner function) of a bipartite continuous variable system, *e.g.* the states of a two-mode radiation field [15, 16]. Our analysis is based on this results. In fact, the Wigner function of a twin-beam produced by a parametric source is Gaussian and the evolution inside active fibers preserves such Gaussian character. Therefore, we are able to characterize the entanglement at any time and to give conditions on the fiber parameters to preserve entanglement after a given fiber length. The PPT condition on the density matrix can be rephrased as a condition on the covariance matrix of the Wigner function $W(x_1,y_1;x_2,y_2)$ of the two modes. We have that a state is separable iff

$$V+\frac{i}{4}\Omega\geq 0 \qquad \Omega=\left(\begin{array}{cc} J & 0\\ 0 & -J\end{array}\right) \qquad J=\left(\begin{array}{cc} 0 & 1\\ -1 & 0\end{array}\right),$$

where

$$V_{pk}=\langle\Delta\xi_p\Delta\xi_k\rangle=\int d^4\xi\;\Delta\xi_p\Delta\xi_k\,W(\boldsymbol{\xi})\,,$$

with $\Delta\xi_j=\xi_j-\langle\xi_j\rangle$ and $\boldsymbol{\xi}=\{x_1,y_1,x_2,y_2\}$.

Given a Wigner function of the form (11) we have $\langle x_j\rangle=\langle y_j\rangle=\langle\Delta x_j\Delta y_k\rangle=0,\ \forall j,k$ and

$$\langle\Delta x_j^2\rangle=\langle\Delta y_j^2\rangle=1/2(\Sigma_+^2+\Sigma_-^2) \qquad \langle\Delta x_1\Delta x_2\rangle=\langle\Delta y_1\Delta y_2\rangle=1/2(\Sigma_+^2-\Sigma_-^2)\,,$$

such that

$$V=\frac{1}{2}\left(\begin{array}{cccc} \Sigma_+^2+\Sigma_-^2 & 0 & \Sigma_+^2-\Sigma_-^2 & 0\\ 0 & \Sigma_+^2+\Sigma_-^2 & 0 & \Sigma_+^2-\Sigma_-^2\\ \Sigma_+^2-\Sigma_-^2 & 0 & \Sigma_+^2+\Sigma_-^2 & 0\\ 0 & \Sigma_+^2-\Sigma_-^2 & 0 & \Sigma_+^2+\Sigma_-^2\end{array}\right).$$

In order to check separability, we diagonalize $V+i/4\Omega$ and impose that all its eigenvalues are greater or equal to zero. It turns out that the state described by W_τ is separable when *both* the variances satisfies the condition

$$\Sigma_+^2\geq\frac{1}{4} \qquad \Sigma_-^2\geq\frac{1}{4}\,.$$

The condition $\Sigma_+^2\geq\frac{1}{4}$ is weaker, and thus separability is determined by the condition $\Sigma_-^2\geq\frac{1}{4}$, which read as follows

$$\exp\left(-\gamma\tau-2\lambda\right)+\frac{1}{\gamma}\left[1-\exp\left(-\gamma\tau\right)\right]\geq 1\,.$$

Given the parameters M, Γ and λ the threshold value τ_s above which the state become separable is given by

$$\tau_s=-\frac{1}{\gamma}\log\left(\frac{1-\gamma}{1-\gamma e^{-2\lambda}}\right)=(2M+1)\log\left(1+\frac{1-e^{-2\lambda}}{2M}\right)\,. \qquad (13)$$

In terms of the unrescaled time t the threshold for separability reads as

$$t_s = \frac{1}{\Gamma}\,\log\left(1+\frac{1-e^{-2\lambda}}{2M}\right) = \frac{1}{\Gamma}\log\left(1+\frac{\sqrt{N(N+2)}-N}{2M}\right)\,. \tag{14}$$

In Fig. 1 we report t_s (in unit $1/\Gamma$) as a function of N for different values of the thermal number of photons M in the channel. The threshold t_s increases with N from zero to a saturation value given by $\log[1+(2M)^{-1}]/\Gamma$. This means that low thermal noise, and for an excited entangled state at the input, the entanglement is present on a long time scale comparable with the photon lifetime in the medium. The corresponding threshold $L_s = ct_s$ on the interaction length is of the order of many Kilometers assuming a small value for M (which is a reasonable approximation at room temperature and optical frequencies), and an attenuation factor about $0.3\,dB/Km$ for the optical fibers.

A question arises whether or not twin-beam after evolution can be used for entanglement based protocols such teleportation. If σ is the input state of a teleporting machine the Wigner function of the output state ϱ (the teleported state) is given by

$$W[\varrho](x,y) = \iiiint dx_1 dy_1 dx_2 dy_2\, W[\lambda](x+x_2, y+y_2, x_1+x_2, -y_1-y_2)\, W[\sigma](x_2,y_2)\,,$$

where $W[\sigma](x,y)$ is the Wigner function of the input and $W[\lambda]$ that of the entangled state providing the nonlocality for teleportation. For pure state at the input $\sigma = |\psi\rangle\langle\psi|$, the teleportation fidelity is given by $F = \langle\psi|\varrho|\psi\rangle$ and can be calculated as the overlap of the Wigner functions $F = \pi\int dxdy\; W[\sigma](x,y)\; W[\varrho](x,y)$. For the teleportation of a coherent state $|\psi\rangle = |z\rangle$ supported by evoluted twin-beam of the form (11) the fidelity is given by

$$F = \frac{1}{1+4\Sigma_-^2} = \frac{1}{1+e^{-2\lambda-\Gamma t}+(1-e^{-\Gamma t})(2M+1)+(1-\eta)/\eta}\,.$$

In the ideal case of entanglement provided by unperturbed twin-beam we have $\Sigma_- = \sigma_-$, and thus the known formula $F = (1+e^{-2\lambda})^{-1}$. Teleportation of a coherent state is a truly quantum (nonlocal) protocol if the fidelity is larger than one half [17]. Remarkably, the condition $F \geq 1/2$ is equivalent to the condition for entanglement survival $\Sigma_-^2 \geq \frac{1}{4}$. This means that evolved twin-beam can be used for teleportation in the whole range of parameters preserving entanglement, *i.e.* that no regimes of *unusable* entanglement are present.

The evolution and the degradation of continuous variable entanglement of twin-beam in optical fibers have been studied by means of two-mode Wigner function and PPT condition for separability. During the propagation, entanglement survives for a finite interval of time. In terms of the damping rate Γ and the thermal noise M, the threshold value for separability ranges from $t_s = 0$ to $\log[1+(2M)^{-1}]/\Gamma$, increasing with the initial squeezing parameter λ. For low thermal noise, and for an excited entangled state at the input, the entanglement is present on a long time scale compared with the photon lifetime in the medium. Evolved twin-beam supports coherent state quantum teleportation in the whole range of parameters preserving entanglement.

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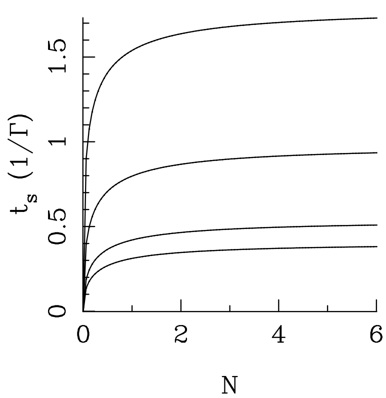

Figure 1. The threshold value t_s (in unit $1/\Gamma$) as a function of the mean photon number of the twin-beam N for different values of the thermal number of photons M in the channel. From top to bottom we have the curves for $M = 0.1, 0.3, 0.7, 1.0$.