

A condensate in a lossy cavity: collective atomic recoil and generation of entanglement

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Abstract. The interaction between a Bose–Einstein condensate and a single-mode quantized radiation field in the presence of a strong far-off-resonant pump laser generates, in proper regimes, atom–atom and atom–field entanglement. The effects of cavity losses are taken into account and an analytic solution of the corresponding master equation is given in terms of the Wigner function of the system.

The experimental realization of Bose–Einstein condensation opened the possibility to generate macroscopic atomic fields whose quantum statistical properties can in principle be manipulated and controlled [1]. The system usually considered for this purpose is a Bose–Einstein condensate driven by a far-off-resonant pump laser of wave vector \mathbf{k} and coupled to a single mode in an optical ring cavity. The mechanism at the basis of this kind of physics is the so-called collective atomic recoil lasing (CARL) in his full quantized version [2, 3]. This mechanism gives a reason for the gain in the cavity mode, as well as the generation of momentum side modes of the BEC. In this paper we review the resulting three-mode dynamics for an ideal cavity, showing the appearance of entanglement, and illustrate how to take into account the losses.

We consider a 1D geometry in which the off-resonant laser pulse is directed along the symmetry z -axis of an elongated BEC. The incident and the scattered wave vectors are $\mathbf{k}_s \approx -\mathbf{k}$. The dimensionless position and momentum of the j th atom along the axis \hat{z} directed along \mathbf{k} are $\theta_j = 2\mathbf{k} \cdot \mathbf{z}_j = 2kz_j$ and $p_j = mv_{zj}/2\hbar k$. The interaction time in units of the collective recoil bandwidth, $\rho\omega_r$, is $\tau = \rho\omega_r t$, where $\omega_r = 2\hbar k^2/M$ is the recoil frequency, M is the atomic mass and $\rho = (\Omega_0/2\Delta)^{2/3}(\omega\mu^2 n_s/\hbar\epsilon_0\omega_r^2)^{1/3}$ is the collective CARL parameter, $\Omega_0 = \mu\mathcal{E}_0/\hbar$ is the Rabi frequency of the pump, $n_s = N/V$ is the average atomic density of the sample (containing N atoms in a volume V), μ is the dipole matrix element, $\Delta = (\omega - \omega_s)/\rho\omega_r$ is the detuning and ϵ_0 is the permittivity of the free space.

In a second quantized model for CARL [2, 3] the atomic field operator $\hat{\Psi}(\theta)$ obeys the bosonic equal-time commutation relations $[\hat{\Psi}(\theta), \hat{\Psi}^\dagger(\theta')] = \delta(\theta - \theta')$,

$[\hat{\Psi}(\theta), \hat{\Psi}(\theta')] = [\hat{\Psi}^\dagger(\theta), \hat{\Psi}^\dagger(\theta')] = 0$ and the normalization condition $\int_0^{2\pi} d\theta \hat{\Psi}(\theta)^\dagger \hat{\Psi}(\theta) = N$. We assume that the atoms are delocalized inside the condensate and that, at zero temperature, the momentum uncertainty $\sigma_{p_z} \approx \hbar/\sigma_z$ can be neglected with respect to $2\hbar k$. This approximation is valid for $L \gg \lambda$, where L is the size of the condensate and λ is the wavelength of the incident radiation. In this limit, we can introduce creation and annihilation operators for the atoms of a definite momentum p , i.e. $\hat{\Psi}(\theta) = \sum_m c_m(\theta|m)$, where $p|m\rangle = m|m\rangle$ (with $m = -\infty, \dots, \infty$), $\langle \theta|m\rangle = (2\pi)^{-1/2} \exp(im\theta)$ and c_m are bosonic operators obeying the commutation relations $[c_m, c_{m'}^\dagger] = \delta_{mm'}$ and $[c_m, c_{m'}] = 0$. The Hamiltonian in this case is [4, 5]

$$\hat{H} = \sum_{n=-\infty}^{\infty} \left\{ \frac{n^2}{\rho} c_n^\dagger c_n + ig(a^\dagger c_n^\dagger c_{n+1} - \text{h.c.}) \right\} - \Delta a^\dagger a \tag{1}$$

where $g = \sqrt{\rho/2N}$ is the coupling. The Heisenberg equations for c_n and a are:

$$\frac{dc_n}{d\tau} = -i[c_n, \hat{H}] = -i \frac{n^2}{\rho} c_n + g(a^\dagger c_{n+1} - ac_{n-1}) \tag{2}$$

$$\frac{da}{d\tau} = -i[a, \hat{H}] = i\Delta a + g \sum_{n=-\infty}^{\infty} c_n^\dagger c_{n+1}. \tag{3}$$

The source of the field equation (3) is the bunching operator, defined by $\hat{B} = (1/N) \int_0^{2\pi} d\theta \hat{\Psi}(\theta)^\dagger e^{-i\theta} \hat{\Psi}(\theta) = (1/N) \sum_n c_n^\dagger c_{n+1}$. We note that equations (2) and (3) conserve the number of atoms, i.e. $\sum_n c_n^\dagger c_n = N$, and the total momentum, $Q = a^\dagger a + \sum_n n c_n^\dagger c_n$.

Let us now consider the equilibrium state with no probe field and all the atoms in the same initial momentum state n_0 , i.e. $c_n \approx \sqrt{N} e^{-in^2\tau/\rho} \delta_{n,n_0}$. This is equivalent to assuming the temperature of the system is zero and all the atoms are moving with the same momentum $n_0 2\hbar k$, without spread. The system is unstable for certain values of detuning Δ . In fact, by linearizing equations (2) and (3) around the equilibrium state, the only equations depending linearly on the radiation field are those for c_{n_0-1} and c_{n_0+1} . Hence, in the linear regime, the only transitions involved are those from the state n_0 toward the levels $n_0 - 1$ and $n_0 + 1$. By introducing the operators

$$a_1 = c_{n_0-1} e^{i(n_0^2\tau/\rho + \Delta\tau)} \quad a_2 = c_{n_0+1} e^{i(n_0^2\tau/\rho - \Delta\tau)} \quad a_3 = a e^{-i\Delta\tau}, \tag{4}$$

the atomic field operator reduces to the sum of only three contributions, while equations (2) and (3) reduce to linear equations for the three coupled harmonic oscillator operators [6]:

$$\frac{da_1^\dagger}{d\tau} = -i\delta_- a_1^\dagger + \sqrt{\rho/2} a_3 \quad \frac{da_2}{d\tau} = -i\delta_+ a_2 - \sqrt{\rho/2} a_3 \quad \frac{da_3}{d\tau} = \sqrt{\rho/2} (a_1^\dagger + a_2), \tag{5}$$

with Hamiltonian

$$H = \delta_+ a_2^\dagger a_2 - \delta_- a_1^\dagger a_1 + i\sqrt{\rho/2} [(a_1^\dagger + a_2) a_3^\dagger - (a_1 + a_2^\dagger) a_3], \tag{6}$$

where $\delta_\pm = \delta \pm 1/\rho$ and $\delta = \Delta + 2n_0/\rho = (\omega - \omega_s + 2n_0\omega_r/\rho\omega_r)$. The dynamics of the system is that of three parametrically coupled harmonic oscillators [7]. Note that the Hamiltonian (6) commutes with the constant of motion

$\mathcal{N} = a_2^\dagger a_2 - a_1^\dagger a_1 + a_3^\dagger a_3$. The exact solution of equation (5) can be obtained using the Laplace transform [3, 7]. After some algebra we have

$$a_1^\dagger = e^{-i\delta\tau} [g_1 a_1^\dagger(0) + g_2 a_2(0) + g_3 a_3(0)] \tag{7}$$

$$a_2 = e^{-i\delta\tau} [h_1 a_1^\dagger(0) + h_2 a_2(0) + h_3 a_3(0)] \tag{8}$$

$$a_3 = e^{-i\delta\tau} [f_1 a_1^\dagger(0) + f_2 a_2(0) + f_3 a_3(0)], \tag{9}$$

where the explicit expressions for f_i , g_i and h_i are given in [3], while the initial values verify the initial conditions for a_i . The functions f_i , g_i and h_i are the sum of three terms proportional to $e^{i\lambda_j\tau}$, where λ_j are the complex roots of the cubic equations: $(\lambda - \delta)(\lambda^2 - 1/\rho^2) + 1 = 0$. This characteristic equation has either three real solutions, or one real and a pair of complex conjugate solutions. In the first case, the system is stable and exhibits only small oscillations around its initial state. In the second case, the system is unstable and grows exponentially, even from noise.

The evolution operator $U(\tau) = \exp(-iH\tau)$, where H is given by equation (6), can be disentangled into those of individual operators [3]. This allows us to calculate how the state $|\psi_\tau\rangle$ evolves from the vacuum state $|0, 0, 0\rangle$. The calculation yields

$$|\psi_\tau\rangle = \frac{1}{\sqrt{1 + \langle n_1 \rangle}} \sum_{n,m=0}^{\infty} \alpha_1^m \alpha_2^n \sqrt{\frac{(m+n)!}{m!n!}} |m+n, n, m\rangle, \tag{10}$$

where

$$\alpha_1 = \frac{f_1 g_1^*}{1 + \langle n_1 \rangle} \quad \alpha_2 = \frac{h_1 g_1^*}{1 + \langle n_1 \rangle} \quad |\alpha_{1,2}|^2 = \langle n_{3,2} \rangle / (1 + \langle n_1 \rangle). \tag{11}$$

The state in equation (10) is a fully inseparable three-mode Gaussian state.

Two-mode entangled states between the modes 1 and 2 or the modes 1 and 3 can be obtained for interaction times leading to $\langle n_3 \rangle \ll \langle n_1 \rangle \approx \langle n_2 \rangle$ or $\langle n_2 \rangle \ll \langle n_1 \rangle \approx \langle n_3 \rangle$ respectively. In these cases one has

$$|\psi_{1,2}\rangle = \frac{1}{\sqrt{1 + \langle n_1 \rangle}} \sum_{n=0}^{\infty} \alpha_2^n |n, n, 0\rangle, \quad |\psi_{1,3}\rangle = \frac{1}{\sqrt{1 + \langle n_1 \rangle}} \sum_{n=0}^{\infty} \alpha_1^n |n, 0, n\rangle. \tag{12}$$

The pure states in (12) are maximally entangled bipartite states, as can be shown by evaluating the reduced density operators $\rho_i = \text{Tr}_1[\rho_{1i}]$, where $\rho_{1i} = |\psi_{1i}\rangle\langle\psi_{1i}|$ and $i = 2, 3$. In fact, in both cases we obtain a thermal state for which the von Neumann entropy $S_i = -\text{Tr}[\rho_i \ln \rho_i]$ is maximum [9]. In general, the presence of the third mode reduces the entanglement between the other two modes [10]. We also observe that no two-mode entanglement is possible between states 2 and 3.

In practice, there exist two different regimes of CARL dynamics in which the initial vacuum state evolves into a two-mode entangled state [3]. In particular, atom–atom entanglement can be obtained in the limit $\rho \gg 1$ and in a detuned, not fully exponential regime. On the contrary, in the limit $\rho < 1$, atom–photon entanglement can be obtained when the average occupation number $\langle n_2 \rangle$ remains smaller than one. Recently, the atom–field entanglement of state $|\psi_{1,3}\rangle$ has been exploited to suggest an interspecies teleportation protocol between a radiation beam and a condensate side beam [11].

We have considered so far an ideal optical cavity. In order to have a more realistic description of the entanglement generation we now want to take into

account losses from the cavity. The dynamics of the system is described by the master equation $\dot{\rho} = -i[H, \rho] + 2\kappa L[a_3]\rho$, where 2κ is the damping rate and $L[a_3]$ is the Lindblad superoperator $L[a_3]\rho = a_3\rho a_3^\dagger - \frac{1}{2}a_3^\dagger a_3\rho - \frac{1}{2}\rho a_3^\dagger a_3$. The master equation can be transformed into a Fokker–Planck equation for the Wigner function $W(\alpha_1, \alpha_2, \alpha_3, \tau)$. Using the differential representation of the Lindblad superoperator the Fokker–Planck equation is given by

$$\frac{\partial W}{\partial \tau} = -\{\mathbf{u}'^T \mathbf{A} \mathbf{u} + \mathbf{u}'^* T \mathbf{A}^\dagger \mathbf{u}^*\} W + \mathbf{u}'^T \mathbf{D} \mathbf{u}'^* W \quad (13)$$

where

$$\begin{aligned} \mathbf{u}'^T &= (\alpha_1^*, \alpha_2, \alpha_3) & \mathbf{u}'^* T &= \left(\frac{\partial}{\partial \alpha_1^*}, \frac{\partial}{\partial \alpha_2}, \frac{\partial}{\partial \alpha_3} \right) \\ \mathbf{A} &= \begin{pmatrix} i\delta_- & 0 & -g \\ 0 & i\delta_+ & g \\ -g & g & \kappa \end{pmatrix} & \mathbf{D} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix}. \end{aligned} \quad (14)$$

The solution of the Fokker–Planck is given by the convolution $W(\mathbf{u}, \tau) = \int d^2 \mathbf{u}_0 W(\mathbf{u}_0, 0) G(\mathbf{u}, \tau; \mathbf{u}_0, 0)$ where $W(\mathbf{u}_0, 0)$ is the Wigner function for the initial state (the vacuum) and the Green function $G(\mathbf{u}, t; \mathbf{u}_0, 0)$ is given by

$$G(\mathbf{u}, \tau; \mathbf{u}_0, 0) = \frac{1}{\pi^3 \det \mathbf{Q}} \exp\{-(\mathbf{u} - e^{\mathbf{A}\tau} \mathbf{u}_0)^\dagger \mathbf{Q}^{-1} (\mathbf{u} - e^{\mathbf{A}\tau} \mathbf{u}_0)\}. \quad (15)$$

where

$$\mathbf{Q} = \int_0^\tau d\tau' e^{\mathbf{A}\tau'} \mathbf{D} (e^{\mathbf{A}\tau'})^\dagger \quad e^{\mathbf{A}\tau} = e^{-i\delta\tau} \begin{pmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \\ f_1 & f_2 & f_3 \end{pmatrix}. \quad (16)$$

The initial Wigner function is Gaussian and the evolution preserves this character. The covariance matrix at time τ is given by $\mathbf{C} = \mathbf{Q} + (1/2) \exp(\mathbf{A}\tau) \exp(\mathbf{A}^\dagger \tau)$. In order to quantify the detrimental effect of losses we employ the fidelity $F = \langle \psi_\tau | \varrho | \psi_\tau \rangle$ between the ideal pure state (10), obtained with an ideal cavity, and the state ϱ , corresponding to the evolved Wigner function $W(\mathbf{u}, t)$. We have $F = [\det(\mathbf{C} + \mathbf{C}_\psi)]^{-1}$ where \mathbf{C}_ψ is the covariance matrix for $|\psi_\tau\rangle$ with respect to the variables \mathbf{u} . A systematic numerical study of the fidelity is in progress and results will be published elsewhere [12]. Here we give a first approximate result calculating the fidelity F to the first order in time τ . For the involved matrices we have $\exp(\mathbf{A}\tau) \approx \mathbf{I} + \mathbf{A}\tau$ and $\mathbf{Q} \approx \mathbf{D}\tau$. Therefore, for small τ we have $\mathbf{C} \approx \mathbf{I} + [\mathbf{D} + \frac{1}{2}(\mathbf{A} + \mathbf{A}^\dagger + \mathbf{A}_{id} + \mathbf{A}_{id}^\dagger)]\tau$ and $F \approx 1 - 3\kappa\tau$. Notice that these results do not depend on atomic parameters. Notice also that $\mathbf{C} \rightarrow \mathbf{C}_\psi$ for $\kappa \rightarrow 0$ and that $\det(\mathbf{C}_\psi) = \frac{1}{2}$ at any time.

In this paper we have analysed the interaction between a Bose–Einstein condensate and a single-mode quantized radiation field in the presence of a strong far-off-resonant pump laser. In the so-called linear regime, i.e. for situations where atomic ground state depletion and saturation of the radiation mode can be neglected, the generation of atom–atom and atom–field entanglement have been considered taking into account the effects of cavity imperfections. As a preliminary result we have suggested fidelity to quantify the detrimental effects of losses.

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