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Quantum and classical properties of the fields generated by two interlinked second-order non-linear interactions

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Abstract. We consider two interlinked non-linear interactions occurring simultaneously in a single $\chi^{(2)}$ crystal. Classical and quantum working regimes are considered and their peculiar properties analysed. In particular, we describe an experiment, realized in the classical regime, that verifies the holographic nature of the process, and predict, for the quantum regime, the generation of a fully inseparable tripartite Gaussian state of light that can be used to support a general $1 \rightarrow 2$ continuous variable telecloning protocol.

Parametric processes show a rich phenomenology that is at the heart of nonlinear and quantum optics. In the last decade, multiple non-linear processes involving several modes of radiation have become essential for the realization of all-optical quantum information processing and for the generation of non-classical states of light [1]. These kinds of systems, which need the simultaneous phase-matching of the different interactions, have been recently realized by periodically and aperiodically poled crystals with quasi-phase matching conditions [2] or by self-phase-locked parametric oscillators [3]. In this paper we analyse a different implementation that consists of the experimental realization of two second-order interactions simultaneously phase-matched in a single crystal, in a non-collinear interaction geometry [4] (see figure 1). By assuming that two of the five fields involved are non-depleted during the process, the classical evolution of the field modes is described by the following system of non-linear Maxwell equations

$$\begin{aligned} da_1/dz &= -ig_1 a_3^*, \qquad da_2/dz = -ig_2 a_3, \\ da_3/dz &= -ig_1 a_1^* - ig_2^* a_2 \,, \end{aligned} \tag{1}$$

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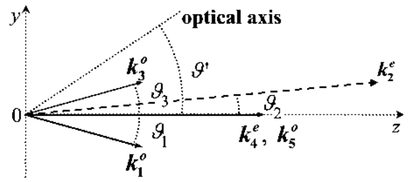

Figure 1. Schematic diagram of the interaction scheme.

where the coupling constants g_1 and g_2 are proportional to the pump amplitudes a_{40} and a_{50} respectively. The solution of (1) reads as follows

$$\begin{aligned} a_{1z} &= \frac{1}{\Gamma^2}\left[|g_2|^2-|g_1|^2\cos(\Gamma z)\right]a_{10} - \frac{g_1g_2}{\Gamma^2}[\cos(\Gamma z)-1]a_{20}^* - i\frac{g_1}{\Gamma}\sin(\Gamma z)a_{30}^*, \\ a_{2z} &= \frac{g_1g_2}{\Gamma^2}[\cos(\Gamma z)-1]a_{10}^* - \frac{1}{\Gamma^2}\left[|g_1|^2-|g_2|^2\cos(\Gamma z)\right]a_{20} - i\frac{g_2}{\Gamma}\sin(\Gamma z)a_{30}, \qquad (2) \\ a_{3z} &= -i\frac{g_1}{\Gamma}\sin(\Gamma z)a_{10}^* - i\frac{g_2^*}{\Gamma}\sin(\Gamma z)a_{20} + \cos(\Gamma z)a_{30}, \end{aligned}$$

where $\Gamma = (|g_2|^2 - |g_1|^2)^{1/2}$ and a_{j0} denotes the initial amplitude of mode a_j. Notice that equations (2) describe the fields' evolution both for real and imaginary values of Γ *i.e.* both in the oscillatory and in the amplification regimes.

In order to verify some theoretical predictions of the classical model, we have realized an experimental set-up in which the non-linear crystal was a type I β-BaB_2O_4 (cut angle 32 deg, cross-section $10\times10\,\mathrm{mm}^2$ and 4 mm thickness), while the interacting fields were provided by the harmonics of a Q-switched amplified Nd:YAG laser (7 ns pulse duration) at the wavelengths $\lambda_1=\lambda_3=1064\,\mathrm{nm}$, $\lambda_4=\lambda_5=532\,\mathrm{nm}$ and $\lambda_2=355\,\mathrm{nm}$. The fields at the wavelengths λ_4 and λ_5 were considered as pump fields and superimposed in a single beam with mixed polarization. At first we checked the holographic nature of the generated fields: in fact, by setting $a_{10}\neq0$, $a_{20}=0$ and $a_{30}=0$, equations (2) show that both the generated fields are holographic phase-conjugated replicas of the input signal, i.e. $a_{2z}\propto a_{10}^*$ and $a_{3z}\propto a_{10}^*$ [5]. The results are summarized in figure 2: we inserted the object $\mathbf{O}$ on the seed field a_1 and obtained two real holographic images $\mathbf{O}'$ and $\mathbf{O}''$ reconstructed by the two generated fields a_3 and a_2 respectively at the distances and with the transversal dimensions predicted by the holographic theory of three-wave mixing [6].

As a second check, we have verified the dependence of the generated fields on one of the two pump fields: using the field a_1 as a seed, we measured the energy E_2, corresponding to the field a_2, as a function of the energy E_5 of the ordinarily polarized pump field a_5 for fixed values of the energy E_4 of the other pump field a_4 and of the seed field a_1 (see figure 3a). We compared the experimental results with the field evolution calculated according to the classical solution (2) and obtained an excellent agreement. We also repeated this kind of measurement by interchanging the roles of the modes a_1 and a_2 reaching a good agreement also in this case (see figure 3b).

The present classical scheme can also be used in the realization of all-optical addressing and switching, and in the implementation of logic gates. Work along these lines is in progress, and the results will be presented elsewhere.

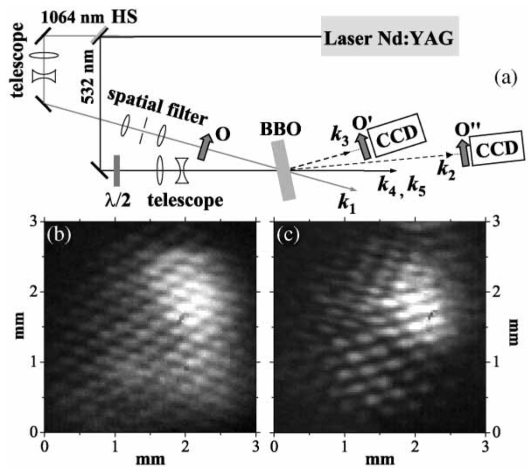

Figure 2. (a) Experimental set-up for the verification of the holographic properties of the generated fields. **O**, object; **O**$'$ and **O**$''$, the reconstructed real holographic images at ω_3 and ω_2, respectively; (b) Holographic image at ω_3; (c) Holographic image at ω_2.

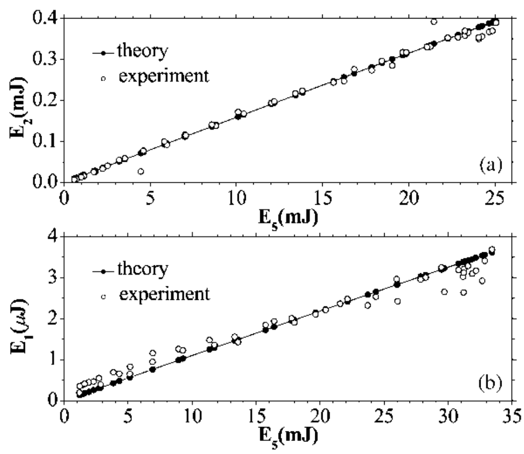

Figure 3. (a) Measured (open circles) and calculated [from equation (2), full circles] energy at ω_2 (case *a*) as a function of the measured energy of the pump at ω_5; (b) Same as in case *b*.

Besides applications in the classical regime, the three mode dynamics given by equation (1) is of interest for the quantum properties of the output fields. Indeed, when the power of the seed a_1 is decreased the quantum description of the process becomes unavoidable. The system, using the parametric approximation for the pump modes a_4 and a_5 is governed by the following interaction Hamiltonian

$$H_{int} = g_1 a_1^\dagger a_3^\dagger + g_2 a_2^\dagger a_3 + h.c. \,. \tag{3}$$

The evolution of the field-operators in the Heisenberg picture is still given by equations (1) and (2), where the complex field-amplitudes a_j are substituted by their boson operator analogues. Considering, as in the classical regime, that only one of the modes is initially *seeded* by a coherent signal, the input state is given by $|\alpha, 0, 0\rangle$, and the evolved state $|\mathbf{T}_\alpha\rangle = \exp(-iH_{int}t)|\alpha, 0, 0\rangle$ by

$$|\mathbf{T}_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{npq} \frac{\alpha^n}{n!} \sqrt{\frac{b_{2\alpha}^p b_{3\alpha}^q}{b_{1\alpha}^{1+n+p+q}} \frac{(n+p+q)!}{p!q!}} |n+p+q, p, q\rangle, \qquad (4)$$

where the coefficients $b_{j\alpha}$ are given by

$$b_{2\alpha} = \frac{|g_1|^2 |g_2|^2}{\Gamma^4} [\cos \Gamma t - 1]^2, \qquad b_{3\alpha} = \frac{|g_1|^2}{\Gamma^2} \sin^2(\Gamma t), \qquad (5)$$

with $b_{1\alpha} = 1 + b_{2\alpha} + b_{3\alpha}$, and the populations $N_{j\alpha} = \langle \mathbf{T}_\alpha | a_j^\dagger a_j | \mathbf{T}_\alpha \rangle$, as calculated from the Heisenberg evolution of modes, leads to $N_{j\alpha} = (1 + |\alpha|^2) b_{j\alpha}$, $j = 2, 3$, $N_{1\alpha} = N_{2\alpha} + N_{3\alpha} + |\alpha|^2$. Notice that when $\alpha = 0$ we recover the result of reference [7]. It can be shown [8] that $|\mathbf{T}_\alpha\rangle$ is a fully inseparable Gaussian state, *i.e.* a state that is inseparable with respect to any grouping of the modes, thus permitting realizations of truly tripartite quantum protocols such as conditional twin-beam generation and cloning at distance, *i.e.* telecloning [9]. We emphasize the fact that in the scheme presented here the entanglement arises as a consequence of two non-linear interactions that took place in a single crystal. As a consequence, our source of tripartite entanglement is a more compact one compared to other schemes [10] suggested and demonstrated so far, in which both parametric sources and linear optical devices have to be used.

We now illustrate how state $|\mathbf{T}_\alpha\rangle$ provides a support for telecloning of coherent states, also in the general case where the two clones share different amounts of the information contained in the input state (asymmetric cloning). In figure 4 a schematic diagram of our scheme is depicted. The coherent input state $\sigma = |z\rangle\langle z|$, *i.e.* the state to be teleported and cloned, is mixed with mode a_1 in such a way as to perform a double-homodine measurement described by the σ-dependent POVM $\Pi(\beta) = (1/\pi) D(\beta) \sigma^T D^\dagger(\beta)$, where $D(\beta) = \exp\{\beta a^\dagger - \bar{\beta} a\}$ is the displacement operator and β is a complex number, labelling the possible outcomes of the measurement. As a consequence of the measurement on the mode a_1 we have a projection on the state of modes a_2 and a_3. Since the POVM is pure, such a conditional state is also pure, and it can be demonstrated that it is the product of

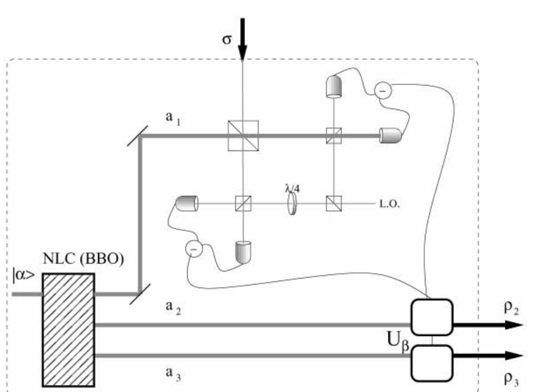

Figure 4. Schematic diagram of the telecloning scheme.

two independent coherent states whose amplitude can be calculated analytically. The result of the measurement may now be sent through a classical channel to the parties which want to prepare approximate clones, where the conditional state may be transformed by a further unitary operation U_β depending on the outcome β. In our case, it consists of a suitable two-mode product displacement, independent of the initial amplitude z, which generalizes the procedure already used for the original CV $1 \to 1$ teleportation protocol. The overall state of the two modes is now obtained by averaging over the possible outcomes β, which occur with probability $P_z(\beta)$. After some operator algebra one has

$$\varrho_j = \int_C d^2\beta P_z(\beta) |z\kappa_j + \bar{\beta}(\kappa_j - 1)\rangle\langle z\kappa_j + \bar{\beta}(\kappa_j - 1)|, \tag{6}$$

where $\kappa_j = \sqrt{b_{j\alpha}/b_{1\alpha}}$, with $j = 2, 3$. We see from the teleported states (6) that it is possible to engineer a symmetric cloning protocol if $b_{2\alpha} = b_{3\alpha}$, otherwise we have an asymmetric cloning machine. In this case the fidelities $F_j = \langle z|\varrho_j|z\rangle$ of the two clones are given by

$$F_j = \frac{1}{2 + b_{i\alpha} + 2b_{j\alpha} - 2\sqrt{b_{j\alpha}(b_{j\alpha} + b_{i\alpha} + 1)}}, \tag{7}$$

where $j, i = 2, 3$ ($j \neq i$). Notice that the fidelities (7) do not explicitly depend on α. A remarkable feature of this protocol is that it is possible to obtain a fidelity larger than the bound $F = 2/3$ for one of the clones, say ϱ_2, while accepting a decreased fidelity for the other one. In particular if we impose $F_3 = 1/2$, *i.e.* the minimum value to assure the genuine quantum nature of the telecloning protocol, we can maximize F_2 by varying the value of the coupling constants g_1 and g_2. The maximum value turns out to be $F_{2\max} = 4/5$ and it corresponds to the choice $b_{3\alpha} = 1/4$ and $b_{2\alpha} = 1$. More generally one can fix F_3, then the maximum value of F_2 is obtained by choosing $b_{2\alpha} = (1/F_3 - 1)$ and $b_{3\alpha} = 1/(4/F_3 - 4)$. The relation between the fidelities is then

$$F_2 = 4\frac{(1 - F_3)}{(4 - 3F_3)}, \tag{8}$$

which shows that F_2 is a decreasing function of F_3 and that $2/3 < F_2 < 4/5$ for $1/2 < F_3 < 2/3$ (see figure 5). The sum of the two fidelities $F_2 + F_3 = 1 + 3F_2F_3/4$ is maximized in the symmetric case in which optimal fidelity $F_2 = F_3 = 2/3$ can be reached. The role of ϱ_2 and ϱ_3 can be exchanged, and the above considerations still hold.

In conclusion, we have analysed the phenomena arising from two interlinked non-linear interactions occurring simultaneously in a single $\chi^{(2)}$ crystal. Phase and amplitude properties of the output fields have been experimentally demonstrated in the classical regime, whereas the quantum regime has been suggested for the generation of a fully inseparable tripartite Gaussian state of light that can be used to support a general $1 \to 2$ continuous variable telecloning protocol. We mention that the generation of the state $|\mathbf{T}_\alpha\rangle$ can be achieved by implementing the same experimental set-up as in figure 1 with a different laser source able to deliver a higher intensity. In fact, we plan to use a mode-locked Nd:YLF laser with a

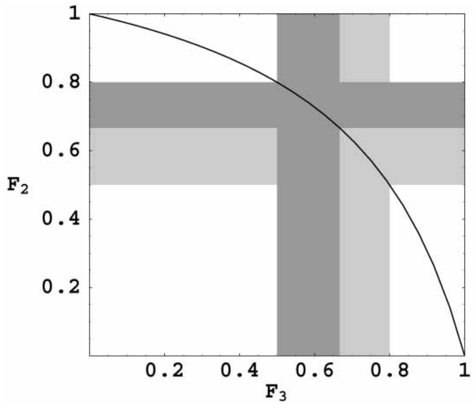

Figure 5. Relation between the fidelities of the two clones in the asymmetric telecloning protocol (see text for details).

regenerative amplifier operating at a repetition rate of 500 Hz (IC-500, HIGH Q Laser Production, Hohenems, Austria) with which it is easy to achieve an intensity value of 50 GW cm^2 in a collimated beam.

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