JOURNAL OF MODERN OPTICS, 15 APRIL–10 MAY 2004
VOL. 51, NO. 6–7, 1057–1061

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Degradation of continuous variable entanglement in a phase-sensitive environment

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(*Received 10 January 2004*)

Abstract. The propagation of a twin-beam state of radiation through Gaussian phase-sensitive channels, *i.e.* noisy channels with *squeezed* fluctuations is addressed. It is found that squeezing the environment always reduces the survival time of entanglement in comparison to the case of simple dissipation and thermal noise. It is also shown that the survival time is further reduced if the squeezing phase of the fluctuations is different from the twin-beam phase.

1. Introduction

The use of phase-sensitive environments has been addressed by many authors [1–3] for preservation of macroscopic quantum coherence. In fact, by adding *squeezed* fluctuations to dissipation, superpositions of states preserve their coherence longer than in the presence of dissipation alone. A question arises whether or not phase-sensitive environments may also be used to preserve entanglement. In this paper, in particular, we study the behaviour of a twin-beam state of radiation (TWB) propagating through a Gaussian noisy phase-sensitive channel, in order to investigate whether or not squeezing the environment is useful to preserve continuous variable entanglement. The answer turns out to be negative. The survival time of entanglement in a squeezed bath is always smaller than in purely dissipative or thermal ones, and the degradation is more pronounced the further the bath fluctuations are out of the TWB phase.

2. TWB in a Gaussian bath

The propagation of a TWB interacting with a general Gaussian environment, can be modelled as the coupling of each part of the state with a non-zero temperature squeezed reservoir. The dynamics can be described by the two-mode master equation

$$\frac{d\rho_t}{dt} = \{\Gamma(1+N)L[a] + \Gamma(1+N)L[b] + \Gamma N L[a^\dagger] + \Gamma N L[b^\dagger] + \Gamma M \mathcal{M}[a^\dagger] + \Gamma M^* \mathcal{M}[a] + \Gamma M \mathcal{M}[b^\dagger] + \Gamma M^* \mathcal{M}[b]\}\rho_t, \qquad (1)$$

where $\rho_t \equiv \rho(t)$ is the system's density matrix at the time t, Γ is the damping rate and N and M are the effective photons number and the squeezing parameter of the bath respectively (which are assumed to be equal for the two channels). $L[O]$ is the Lindblad superoperator, $L[O]\rho_t = O\rho_t O^\dagger - \frac{1}{2}O^\dagger O\rho_t - \frac{1}{2}\rho_t O^\dagger O$, and

Journal of Modern Optics ISSN 0950–0340 print/ISSN 1362–3044 online © 2004 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/09500340410001664476

$\mathcal{M}[O]\rho_t = O\rho_t O - \frac{1}{2}OO\rho_t - \frac{1}{2}\rho_t OO$. Of course, the dynamics of the two modes are independent of each other.

Using the differential representation of the superoperators in equation (1), the corresponding Fokker–Planck equation for the two-mode Wigner function $W \equiv W(x_1, y_1; x_2, y_2)$ is given by [4]

$$\partial_\tau W = \left\{ -\sum_{j=1}^{4} \partial_{x_j} a_j(\underline{x}) + \frac{1}{2}\sum_{i,j=1}^{4} \partial^2_{x_i x_j} d_{ij} \right\} W, \tag{2}$$

where, for the sake of simplicity, we put $\underline{x} = (x_1, y_1; x_2, y_2) \equiv (x_1, x_2; x_3, x_4)$, $\tau = \Gamma t/\gamma$ and $\gamma = (2N+1)^{-1}$. In equation (2) $a_j(\underline{x})$ and d_{ij} are the matrix elements of the drift and diffusion matrices $\mathbf{A}(\underline{x})$ and $\mathbf{D}$ respectively, which are given by

$$\mathbf{A}(\underline{x}) = -\frac{1}{2}\gamma \underline{x}, \tag{3}$$

$$\mathbf{D} = \frac{1}{2}\begin{pmatrix} \frac{1}{2} + \gamma\,\Re\mathrm{e}[M] & \gamma\,\Im\mathrm{m}[M] & 0 & 0 \\ \gamma\,\Im\mathrm{m}[M] & \frac{1}{2} - \gamma\,\Re\mathrm{e}[M] & 0 & 0 \\ 0 & 0 & \frac{1}{2} + \gamma\,\Re\mathrm{e}[M] & \gamma\,\Im\mathrm{m}[M] \\ 0 & 0 & \gamma\,\Im\mathrm{m}[M] & \frac{1}{2} - \gamma\,\Re\mathrm{e}[M] \end{pmatrix}. \tag{4}$$

Notice that the drift term is linear in $\underline{x}$ and the diffusion matrix does not depend on $\underline{x}$. The positivity of $\mathbf{D}$ requires that $|M| < (2N+1)/2$. Moreover, to ensure positivity of the density matrix $\rho_t \geq 0$ we need $|M|^2 \leq N(N+1)$, which includes the former condition.

The solution of the Fokker–Planck equation (2) can be calculated analytically. For the case $\Im\mathrm{m}[M] = 0$, and considering (without loss of generality) TWB with real parameter as the initial state, *i.e.* $\rho_0 \equiv \rho_{\mathrm{TWB}} = |\mathrm{TWB}\rangle\rangle\langle\langle\mathrm{TWB}|$, where $|\mathrm{TWB}\rangle\rangle = \sqrt{1-\xi^2}\sum_p \xi^p |p\rangle|p\rangle$, $\xi \in R$, the solution assumes the simple form [4]

$$W_\tau(x_1, y_1, x_2, y_2) = \frac{1}{(2\pi)^2 \Sigma_1\Sigma_2\Sigma_3\Sigma_4} \times \exp\left\{ -\frac{(x_1+x_2)^2}{4\Sigma_1^2} - \frac{(y_1+y_2)^2}{4\Sigma_2^2} - \frac{(x_1-x_2)^2}{4\Sigma_3^2} - \frac{(y_1-y_2)^2}{4\Sigma_4^2} \right\} \tag{5}$$

where $\Sigma_j^2 = \Sigma_j^2(r, \Gamma, n_{\mathrm{th}}, n_{\mathrm{s}})$, $j = 1, 2, 3, 4$, are

$$\begin{aligned} \Sigma_1^2 &= \sigma_+^2 e^{-\Gamma t} + D_+^2(t), & \Sigma_2^2 &= \sigma_-^2 e^{-\Gamma t} + D_-^2(t), \\ \Sigma_3^2 &= \sigma_-^2 e^{-\Gamma t} + D_+^2(t), & \Sigma_4^2 &= \sigma_+^2 e^{-\Gamma t} + D_-^2(t), \end{aligned} \tag{6}$$

with $\sigma_\pm^2 = \frac{1}{4}e^{\pm 2\lambda}$, $\xi = \tanh\lambda$, and

$$D_\pm^2(t) = \frac{1 + 2N \pm 2M}{4}\left(1 - e^{-\Gamma t}\right). \tag{7}$$

For the general case (M complex) the analytical solution of equation (2) is quite cumbersome, and we do not explicitly write it here.

Notice that if we suppose the environment composed of a set of oscillators excited in a squeezed-thermal state of the form $\nu = S(\zeta)\rho_{\mathrm{th}}S^\dagger(\zeta)$, with $\zeta = |\zeta|e^{i\theta}$, $S(\zeta) = \exp\{\frac{1}{2}[\zeta^* a^{\dagger 2} - \zeta a^2]\}$ and $\rho_{\mathrm{th}} = (1+n_{\mathrm{th}})^{-1}[n_{\mathrm{th}}/(1+n_{\mathrm{th}})]^{a^\dagger a}$, then we can rewrite the parameters N and M in terms of the squeezing and thermal number of photons $n_s = \sinh^2|\zeta|$ and n_{th} respectively. We have $M = |M|e^{i\theta}$ [5] and

$$|M| = (1 + 2\,n_{th})\sqrt{n_s(1 + n_s)} \quad \text{and} \quad N = n_{th} + n_s(1 + 2\,n_{th}). \tag{8}$$

This parametrization automatically guarantees the semipositivity of ρ_t.

3. Separability

A quantum state of a bipartite system is *separable* if its density operator can be written as $\varrho = \sum_k p_k \sigma_k \otimes \tau_k$, where $\{p_k\}$ is a probability distribution and τ and σ are single-system density matrices. If a state is separable the correlations between the two systems are of purely classical origin, otherwise it is entangled. A necessary and sufficient condition for separability of Gaussian states is the positivity of the density matrix ϱ^T, obtained by partial transposition of the original density matrix (PPT condition) [6–8]. Notice that the Wigner function of a twin-beam is Gaussian and the evolution in a Gaussian environment preserves such character. Therefore, we are able to characterize the entanglement at any time and discuss its degradation as a function of bath's parameters. The PPT condition for a density matrix can be rephrased as a condition on the covariance matrix $\mathbf{V}$ of the two-mode Wigner function $W(x_1, y_1; x_2, y_2)$. After defining

$$\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & -\mathbf{J} \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9}$$

and

$$V_{pk} = \langle \Delta\xi_p \Delta\xi_k \rangle = \int_{\mathbb{R}^4} d^4\xi\, \Delta\xi_p \Delta\xi_k W(\xi), \tag{10}$$

with $\Delta\xi_j = \xi_j - \langle \xi_j \rangle$, and $\underline{\xi} = \{x_1, y_1, x_2, y_2\}$, we have that a state is separable iff

$$\mathbf{S} \equiv \mathbf{V} + \frac{i}{4}\,\boldsymbol{\Omega} \geq 0 \tag{11}$$

For the state (5), the condition (11) can be rewritten as [4]

$$\Sigma_1^2 \Sigma_4^2 \geq \frac{1}{16}, \qquad \Sigma_2^2 \Sigma_3^2 \geq \frac{1}{16}. \tag{12}$$

Recall that in this case the squeezing parameter M is real, *i.e.* it has the same phase as the TWB parameter ($\theta = 0$). By solving inequalities (12) with respect to time t, we find that the TWB becomes separable for $t > t_s$, where the survival time $t_s = t_s(\lambda, \Gamma, n_{th}, n_s)$ is given by

$$t_s = \frac{1}{\Gamma} \log\left(f + \frac{1}{1 + 2n_{th}} \sqrt{f^2 + \frac{n_s(1 + n_s)}{n_{th}(1 + n_{th})}} \right), \tag{13}$$

where we have defined

$$f \equiv f(\lambda, n_{th}, n_s) = \frac{(1 + 2\,n_{th})\big[1 + 2\,n_{th} - e^{-2\lambda}(1 + 2\,n_s)\big]}{4\,n_{th}(1 + n_{th})}. \tag{14}$$

As one may expect, t_s decreases as n_{th} and n_s increase. Moreover, in the limit $n_s \to 0$, the threshold time reduces to the value t_0 pertaining to a non-squeezed bath [4]. In order to see the effect of squeezing the bath on the survival time we introduce the function

$$G(\lambda, n_{\mathrm{th}}, n_{\mathrm{s}}) \equiv \frac{t_{\mathrm{s}} - t_0}{t_0}. \tag{15}$$

$G > 0$ means that squeezing leads to a longer survival time, shorter otherwise. Results are illustrated in figure 1, where we plot G as a function of n_{s} for different values of n_{th} and λ. Since G is always negative, we conclude that coupling a TWB with an *in-phase* squeezed bath destroys entanglement faster than the coupling with a non-squeezed environment.

We have also evaluated the threshold time for separability in the case of an *out-of-phase* squeezed bath, *i.e.* for complex $M = |M|\, e^{i\theta}$. The analytical expression is quite cumbersome and will not be reported here. However, in order to investigate the positivity of $\mathbf{S}$ as a function of θ, it suffices to consider the characteristic polynomial $q_{\mathbf{S}}(x)$ associated to $\mathbf{S}$, and study the sign of its roots. A numerical analysis shows that this polynomial has four *real* roots and three of them are always *positive*. We focus our attention on the other one. In figure 2, we plot the characteristic polynomial for different values of the parameters n_{th}, n_{s} and θ (we put $e^{-\Gamma t} = 0.55$): it is apparent that by adding thermal noise and squeezed fluctuations the sign of the smallest root changes from negative (entangled state)

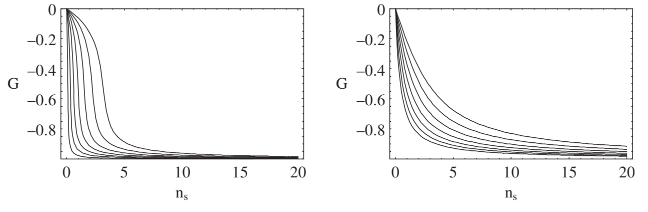

Figure 1. Plots of the ratio $G = (t_{\mathrm{s}} - t_0)/t_0$ as a function of the number of squeezed photons n_{s} for different values of the TWB parameter λ and of the number of thermal photons n_{th} when M is real. The values of n_{th} are $n_{\mathrm{th}} = 10^{-3}$ (left) and $n_{\mathrm{th}} = 1$ (right), while the solid lines, from bottom to top, refer to λ varying between 0.1 to 1.0 in steps of 0.15.

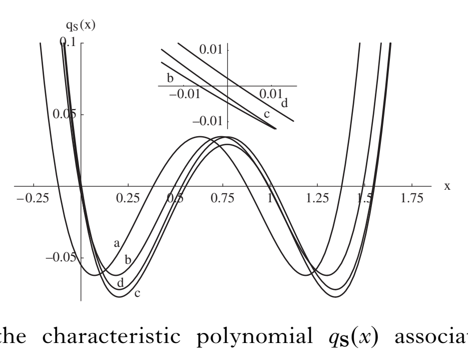

Figure 2. Plots of the characteristic polynomial $q_{\mathbf{S}}(x)$ associated to $\mathbf{S}$ for a TWB propagating in a non-classical environment with squeezing parameter $M = |M|\, e^{i\theta}$. We set $e^{-\Gamma t} = 0.55$, $\lambda = 1$, and (a) $n_{\mathrm{th}} = n_{\mathrm{s}} = 0$, (b) $n_{\mathrm{th}} = 0.5$, $n_{\mathrm{s}} = 0$, (c) $n_{\mathrm{th}} = 0.5$, $n_{\mathrm{s}} = 0.07$ and $\theta = 0$, (d) $n_{\mathrm{th}} = 0.5$, $n_{\mathrm{s}} = 0.07$ and $\theta = \pi/5$. Notice that there are always four real roots and three of them are positive. The inset is a magnification of the region near to 0: the presence of thermal noise and squeezed fluctuations with non zero phase reduces the survival time.

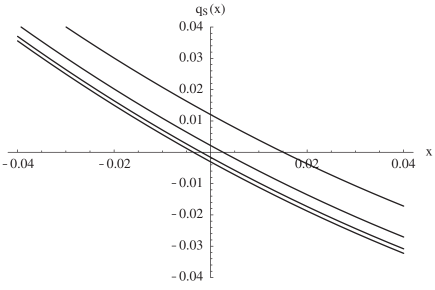

Figure 3. Plots of the characteristic polynomial $q_{\mathbf{S}}(x)$ associated to **S** for a TWB propagating in a non-classical environment with squeezing parameter $M = |M|\, e^{i\theta}$. This plot shows only the region near to $x = 0$ (the other three roots are always positive). We set $e^{-\Gamma t} = 0.55$, $\lambda = 1$, $n_{\text{th}} = 0.5$, $n_{\text{s}} = 0.07$ and, from left to right, $\theta = 0, \pi/10, \pi/5$ and $\pi/2$: a non-real squeezing parameter of the bath always reduces the survival time.

to positive (separable state). In other words, the survival time becomes shorter. Moreover, in figure 3 we show that increasing θ from 0 to $\pi/2$ the threshold time is further reduced. The behaviour of the polynomial roots for different values of λ and Γt is analogous.

4. Conclusions

In this paper we have analysed the propagation of a TWB through Gaussian phase-sensitive noisy channels, and have evaluated the threshold (survival) time for the state to become separable.

We found that the survival time in a squeezed environment is always shorter than in a purely dissipative or thermal ones. In addition, the survival time is further reduced if the squeezing phase of the fluctuations is different from the TWB phase.

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