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Applications of the Group $\mathbb{SU}(1, 1)$ for Quantum Computation and Tomography

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Abstract—This paper collects miscellaneous results about the group $\mathbb{SU}(1, 1)$ that are helpful in applications in quantum optics. Moreover, we derive two new results, the first of which is about the approximability of $\mathbb{SU}(1, 1)$ elements by a finite set of elementary gates and the second of which is about the regularization of group identities for tomographic purposes.

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1. INTRODUCTION

In the last two decades, many achievements in quantum optics have come from nonlinear effects in crystals (for a review on the topic, see [1]). Nonlinear crystals made it possible to produce both single-mode squeezed states, which carry attenuated quadrature noise and constitute good carriers for classical information [2–4], and two-mode squeezed states, such as the twin beam, which is a prototype for harmonic-oscillator entangled states and is useful in many applications, such as continuous variables teleportation [5]. On mathematical grounds, the action of nonlinear crystals can be described by parametric unitary transformations in which the pump mode is considered as a classical field and its creation and annihilation operators are replaced by the complex amplitude. The effective Hamiltonian makes possible parametric down-conversion, which is the process by which a photon with higher frequency is annihilated and two photons with lower frequencies are created. This process gives rise to time evolution that can be described through unitaries in the Schwinger representation of the group $\mathbb{SU}(1,1)$, namely, exponentials of linear combinations of the three generators

$$K_+ = a^\dagger b^\dagger, \quad K_- = ab, \quad K_z = \frac{1}{2}(a^\dagger a + b^\dagger b + 1), \quad (1)$$

where a and b are the annihilation operators for the two modes. The degenerate parametric down conversion happens when the two created photons are in the same mode with a frequency which is half of the annihilated photon frequency, and this particular case giving rise to single-mode squeezing corresponds to $a = b$ with the three generators

$$K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2, \quad K_z = \frac{1}{2}(a^\dagger a + 1/2). \quad (2)$$

Squeezed states and twin beams are nowadays widely used in experimental quantum optics, and it is clear that the ability to manipulate radiation modes by unitaries of the group $\mathbb{SU}(1, 1)$ is crucial. In this paper, we consider some general aspects of the group $\mathbb{SU}(1, 1)$ that can be exploited on the physical ground in order to approximately simulate any $\mathbb{SU}(1, 1)$ transformation by a finite set of elementary *gates*, namely, unitary transformations which can be applied in a given succession in order to approach a target unitary in the representation of $\mathbb{SU}(1, 1)$. This is very useful in a situation in which an experimenter needs a flexible setup which allows him to simulate to within some accuracy any possible gate. A similar situation holds for qubits, where a very powerful theorem from Solovay and Kitaev states that any gate can be efficiently approximated by a finite set of elementary gates. In the case of harmonic oscillators, however, the theorem still lacks an important part, which states that the amount of elementary gates needed in order to approximate any gate grows logarithmically with the accuracy. This fact is due to the dimension of the Hilbert space, and some intermediate result towards the analogue of the qubit Solovay–Kitaev theorem in the case of harmonic oscillators can be derived with the reasonable assumption that the states of interest, to which the gates have to be applied, have finite average energy and finite variance of the energy distribution. In the paper, we will also discuss severe limitations that make it impossible to find a power law which is independent of the group element that one wants to approximate.

Besides the problem of approximation of squeezed states, we can consider the problem of classifying and analyzing the performances of covariant measurements and tomographic measurements. The first ones are an idealization of physical measurements, which turns out to be interesting, because they saturate the bounds on the precision for the estimation of squeezing parameters, thus providing an absolute standard for rating actual detectors. As regards the tomographic measurements, their statistics allows one to completely determine the state of radiation modes—up to statistical errors. In [6], a particular tomographic measurement has been proposed for states with even or odd parity based on the properties of the Schwinger representation of $\mathbb{SU}(1, 1)$. In this paper, we will discuss the possibility of deriving similar tomographic identities from group integrals. Moreover, an interesting mechanism, due to which the "natural" group integral does not converge for physical representations and a sort of regularization is needed, is shown. This analysis provides a whole range of tomographic POVMs corresponding to different regularizations, which can be studied in order to optimize the performances of $\mathbb{SU}(1, 1)$ tomography. The technique is general and can be applied to many tomographic measurements originating from other groups. The core of the regularization technique consists in modifying the invariant (Haar) measure on the group manifold; this modification gives rise to a generalization of the Duflo–Moore [7, 8] operator, which is typical in groups which are not unimodular, namely, for which the Haar measure does not exist. This fact implies some complication in the data processing with respect to the usual homodyne tomography but on the other hand allows the group measure to be optimized in order to minimize the statistical errors.

In Section 2, we discuss some general aspects of the group $\mathbb{SU}(1, 1)$, considering its defining representation, The results derived there will be exploited in subsequent sections. In Section 3, we prove the existence of a set of three elementary gates and discuss the possibility to use them for the approximation of target group elements under reasonable assumptions on the physical states. We also discuss the impossibility of having the exact analogue of the Solovay–Kitaev theorem for the quantum optical representations of $\mathbb{SU}(1, 1)$. In Section 5, we show that the physical representations of $\mathbb{SU}(1, 1)$ are not square-summable, and we show how one can modify the group theoretical identities for group integrals in order to obtain converging integrals which are useful for group tomography. In Section 6, we close the paper with a summary of the contents and concluding remarks.

2. GENERAL ASPECTS OF THE GROUP $\mathbb{SU}(1, 1)$

$\mathbb{SU}(1, 1)$ is the group of complex 2×2 matrices M with unit determinant that satisfy the relation

$$M^{\dagger}PM = P, \tag{3}$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

This relation implies that the elements of $\mathbb{SU}(1, 1)$ preserve the Hermitian form $\omega(v_1, v_2) \doteq v_1^{\dagger}Pv_2$ for arbitrary column vectors $v_i \in \mathbb{C}^2$.

From the above definition, it is simple to show that any matrix $M \in \mathbb{SU}(1, 1)$ has the form

$$M = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \tag{5}$$

for α, β complex numbers such that $|\alpha|^2 - |\beta|^2 = 1$. Notice that the columns M_1, M_2 of M are orthogonal and normalized with respect to the form ω, namely, $\omega(M_1, M_2) = 0$, $\omega(M_1, M_1) = 1$, and $\omega(M_2, M_2) = -1$. By writing $\alpha = t + iz$ and $\beta = x + iy$, we obtain

$$\begin{aligned} M &= t\mathbb{1} + iz\sigma_z + x\sigma_x + y\sigma_y, \\ t^2 + z^2 - x^2 - y^2 &= 1, \end{aligned} \tag{6}$$

$\mathbb{1}$ and σ_x, σ_y, σ_z being the identity and the three Pauli matrices, respectively. In other words, the elements of $\mathbb{SU}(1, 1)$ are parametrized by points of a hyperboloid in $\mathbb{R}^4$. This makes $\mathbb{SU}(1, 1)$ a Lie group, namely, a group which is also a differentiable manifold. The above parametrization clearly exhibits three relevant facts: (i) a group element is in one-to-one correspondence with three real parameters (x, y, and z, for example); i.e., the group manifold is three-dimensional, (ii) the group $\mathbb{SU}(1, 1)$ is not compact, and (iii) it is not simply connected [12].

Given a parametrization $M(\mathbf{r})$, where the element $M(\mathbf{r}) \in \mathbb{SU}(1, 1)$ is specified by the triple $\mathbf{r} \in \mathbb{R}^3$, the matrix multiplication induces a composition law in the parameter space: $(\mathbf{r}, \mathbf{s}) \mapsto \mathbf{r} \circ \mathbf{s}$, where $\mathbf{r} \circ \mathbf{s}$ is defined by the relation $M(\mathbf{r} \circ \mathbf{s}) = M(\mathbf{r})M(\mathbf{s})$. In particular, if $\mathbf{r} = (x, y, z)$, with x, y, z as in Eq. (6), we can define the invariant measure

$$d\mu(\mathbf{r}) = \frac{1}{\sqrt{1 + x^2 + y^2 - z^2}}dxdydz. \tag{7}$$

Invariance of the measure means that the action of the group does not change the volume of regions in the parameter space, namely, for any $\mathbf{r}$, $\mathbf{s}$, $d\mu(\mathbf{r} \circ \mathbf{s}) = d\mu(\mathbf{s} \circ \mathbf{r}) = d\mu(\mathbf{r})$. Expression (7) of the invariant measure $d\mu(x, y, z)$ is particularly useful, since it allows the invariant measure in any parametrization of the group to be obtained just by performing a change of variables. For example, a useful alternative parametrization of a group element $M \in \mathbb{SU}(1, 1)$ is given by

$$M(\theta, \phi, \psi) = \begin{pmatrix} \cosh\theta e^{i\phi} & \sinh\theta e^{-i\psi} \\ \sinh\theta e^{i\psi} & \cosh\theta e^{-i\phi} \end{pmatrix}, \tag{8}$$

for $\theta \in [0, +\infty)$, $\phi \in [0, 2\pi)$, $\psi \in [0, 2\pi)$. The change of parametrization from Eq. (6) to (8) corresponds to the change of variables $x = \sinh\theta\cos\psi$, $y = \sinh\theta\sin\psi$, $z = \cosh\theta\sin\phi$. Performing the change of variables in Eq. (7), we obtain the expression of the invariant measure in the parametrization $M = M(\theta, \phi, \psi)$, namely,

$$d\nu(\theta, \phi, \psi) = \sinh\theta\cosh\theta d\theta d\phi d\psi. \tag{9}$$

2.1. The Lie Algebra $\mathbb{SU}(1, 1)$

Since $\mathbb{SU}(1, 1)$ is a real three-dimensional manifold, its Lie algebra $su(1, 1)$—the tangent space in the identity—is a three-dimensional vector space. As usual, a basis of the Lie algebra is obtained by differentiating curves passing through the identity. Differentiation with respect to the parameters x, y, z in the identity provides the generators

$$i\sigma_x = i\left[\frac{d}{dx}M(x, y, z)\right]_{x=y=z=0}, \tag{10}$$

$$i\sigma_y = i\left[\frac{d}{dy}M(x, y, z)\right]_{x=y=z=0}, \tag{11}$$

$$-\sigma_z = i\left[\frac{d}{dz}M(x, y, z)\right]_{x=y=z=0}, \tag{12}$$

where $M(x, y, z)$ is defined by Eq. (6). Hence, the Lie algebra $su(1, 1)$ is the real vector space spanned by the matrices $i\sigma_x$, $i\sigma_y$, and σ_z. By defining $k_x = i\frac{\sigma_x}{2}$, $k_y = i\frac{\sigma_y}{2}$, $k_z = \frac{\sigma_z}{2}$, and $k_\pm = k_x \pm ik_y$, we obtain the standard commutation relations

$$\begin{cases} [k_+, k_-] = -2k_z \\ [k_z, k_\pm] = \pm k_\pm. \end{cases} \tag{13}$$

By definition, an operator representation of the algebra $su(1, 1)$ is given by the assignment of three operators K_x, K_y, and K_z that satisfy the above commutation relations with $K_\pm = K_x \pm iK_y$. From such relations, it follows that, in any representation of $su(1, 1)$, the Casimir operator

$$\mathbf{K}\cdot\mathbf{K} \doteq K_z^2 - K_x^2 - K_y^2 \tag{14}$$

commutes with the whole algebra spanned by K_x, K_y, K_z.

2.2. The Exponential Map

A way of writing the group elements in any representation in terms of the Lie algebra generators is through the exponential map. The exponential map $M = e^{im}$ is the map that associates an element $m \in su(1, 1)$ of the Lie algebra with an element $M \in \mathbb{SU}(1, 1)$ of the group. In order to discuss the exponential map, it is suitable to write the elements of the algebra as $m = \chi\mathbf{n}\cdot\mathbf{k}$, where $\chi \in \mathbb{R}$, $\mathbf{n}\cdot\mathbf{k} \doteq n_z k_z - n_x k_x - n_y k_y$ and $\mathbf{n} \in \mathbb{R}^3$ is a normalized vector. In this context, *normalized* means that the product $\mathbf{n}\cdot\mathbf{n} \doteq n_z^2 - n_x^2 - n_y^2$ can assume only the values +1, –1, and 0. Then, the exponentiation of the element $m \in su(1, 1)$ is easily performed by using the relation

$$(\mathbf{n}\cdot\mathbf{k})^2 = \frac{\mathbf{n}\cdot\mathbf{n}}{4}\mathbb{1}, \tag{15}$$

which follows directly from the properties of Pauli matrices. In the following, we analyze the three cases $\mathbf{n}\cdot\mathbf{n} = \pm 1, 0$ separately.

Case 1: $\mathbf{n}\cdot\mathbf{n} = +1$. The exponentiation gives

$$M_+ = e^{i\chi\mathbf{n}\cdot\mathbf{k}} = \cos\left(\frac{\chi}{2}\right)\mathbb{1} + i\sin\left(\frac{\chi}{2}\right)2\mathbf{n}\cdot\mathbf{k}. \tag{16}$$

Notice that, for any fixed direction $\mathbf{n}$, we have a one-parameter subgroup, which is compact and isomorphic to $U(1)$.

The group elements of the form (16) form a region $\Omega_+ \subset \mathbb{SU}(1, 1)$, which contains $\pm\mathbb{1}$ and all the matrices $M \in \mathbb{SU}(1, 1)$ such that $|\mathrm{Tr}[M]| < 2$.

Case 2: $\mathbf{n}\cdot\mathbf{n} = -1$. Exponentiating the generator $\mathbf{n}\cdot\mathbf{k}$, we obtain

$$M_- = e^{i\chi\mathbf{n}\cdot\mathbf{k}} = \cosh\left(\frac{\chi}{2}\right)\mathbb{1} + i\sinh\left(\frac{\chi}{2}\right)2\mathbf{n}\cdot\mathbf{k}. \tag{17}$$

In this case, for a fixed direction $\mathbf{n}$, we have a one-parameter subgroup, which is not compact and is isomorphic to $\mathbb{R}$. The elements M_- form a region $\Omega_- \subset \mathbb{SU}(1, 1)$, which contains the identity and all the matrices $M \in \mathbb{SU}(1, 1)$ such that $\mathrm{Tr}[M] > 2$.

Case 3: $\mathbf{n}\cdot\mathbf{n} = 0$. In this case, the exponentiation gives

$$M_0 = e^{i\chi\mathbf{n}\cdot\mathbf{k}} = \mathbb{1} + i\chi\mathbf{n}\cdot\mathbf{k}. \tag{18}$$

The elements M_0 form a region $\Omega_0 \subset \mathbb{SU}(1, 1)$, which contains all matrices $M \in \mathbb{SU}(1, 1)$ such that $\mathrm{Tr}[M] = 2$. The region Ω_0 is a two-dimensional surface and therefore has zero volume.

We want to stress that the exponential map does not cover the whole group $\mathbb{SU}(1, 1)$. The region of $\mathbb{SU}(1, 1)$ covered by the exponential map is $\Omega = \Omega_+ \cup \Omega_- \cup \Omega_0$ and contains matrices $M \in \mathbb{SU}(1, 1)$ such that $\mathrm{Tr}[M] \geq -2$. However, according to parametrization (6), the trace of the matrix $M \in \mathbb{SU}(1, 1)$ is $\mathrm{Tr}[M] = 2t$, $t \in \mathbb{R}$. Therefore, the group $\mathbb{SU}(1, 1)$ also contains elements with the trace $\mathrm{Tr}[M] < -2$, which cannot be obtained with the exponential map. Nevertheless, any matrix $M \in \mathbb{SU}(1, 1)$ with $\mathrm{Tr}[M] < -2$ can be written as $M = -M_-$ for some $M_- \in \Omega_-$, and any matrix $M \in \mathbb{SU}(1, 1)$ with $\mathrm{Tr}[M] = -2$ can be written as $M = -M_0$

for some $M_0 \in \Omega_0$. Defining $-\Omega_- \doteq \{-M_-|M_- \in \Omega_-\}$ and $-\Omega_0 \doteq \{-M_0|M_0 \in \Omega_0\}$, we have

$$\mathbb{SU}(1, 1) = \Omega \cup -\Omega_- \cup -\Omega_0. \quad (19)$$

Notice that, since Ω_0 and $-\Omega_0$ have zero measure, any group integral can be written as the sum of only three contributions coming from Ω_+, Ω_-, and $-\Omega_-$, respectively.

Even though the exponential map does not cover the whole group $\mathbb{SU}(1, 1)$, any group element $M(\theta, \phi, \psi)$—parametrized as in Eq. (8)—can be written as a product of exponentials, for example, as

$$M(\theta, \phi, \psi) = e^{\xi k_+ - \bar{\xi} k_-} e^{2i\phi k_z}, \quad \xi = -i\theta e^{-i(\psi - \phi)}. \quad (20)$$

Relation (20) is particularly useful, since it allows one to construct from any representation of the Lie algebra $su(1, 1)$ a representation of the group $\mathbb{SU}(1, 1)$. In particular, for the physical realizations of the group $\mathbb{SU}(1, 1)$, where the generators k_x, k_y, k_z are represented by Hermitian operators K_x, K_y, K_z in an infinite-dimensional Hilbert space, relation (20) provides the *unitary* representation

$$U_{\theta, \phi, \psi} = e^{\xi K_+ - \bar{\xi} K_-} e^{2i\phi K_z}, \quad \xi = -i\theta e^{-i(\psi - \phi)}. \quad (21)$$

2.3. Baker–Campbell–Hausdorff Formula

The exponential with k_+, k_- in Eq. (21) can be further decomposed according to the *Baker–Campbell–Hausdorff (BCH) formula*. The BCH formula is the fundamental relation holding *for any* representation of the algebra $su(1, 1)$, given by [9]

$$e^{\xi K_+ - \bar{\xi} K_-} = \exp^{\frac{\xi}{|\xi|}\tanh|\xi| K_+} \left(\frac{1}{\cosh|\xi|}\right)^{2K_z} \exp^{-\frac{\bar{\xi}}{|\xi|}\tanh|\xi| K_-} \quad \forall \xi \in \mathbb{C}. \quad (22)$$

This formula can be simply proven by verifying it in the case of the two-by-two matrices $k_+, k_-, k_z \in su(1, 1)$.

A version of the BCH formula in "antinormal order" is given by the relation

$$e^{\xi K_+ - \bar{\xi} K_-} = \exp^{-\frac{\bar{\xi}}{|\xi|}\tanh|\xi| K_-} (\cosh|\xi|)^{2K_z} \exp^{\frac{\xi}{|\xi|}\tanh|\xi| K_+} \quad \forall \xi \in \mathbb{C}, \quad (23)$$

which follows from expression (22) with the change of representation $K'_+ = -K_-$, $K'_- = -K_+$, $K'_z = -K_z$.

ELEMENTARY GATES

Parametrization (8) makes it evident that any element of $\mathbb{SU}(1, 1)$ can be obtained as a product of exponentials of the generators k_z and k_x. In fact, Eq. (8) is equivalent to the decomposition

$$M(\theta, \phi, \psi) = e^{i(\phi - \psi)k_z} e^{-2ik_x} e^{i(\phi + \psi)k_z}. \quad (24)$$

As a consequence, we have the following approximation theorem:

Theorem 1. (Approximation of group elements) Any element of $M \in \mathbb{SU}(1, 1)$ can be approximated with arbitrary precision by a finite product involving only three elements $G_1, G_2, G_3 \in \mathbb{SU}(1, 1)$. A possible choice is

$$G_1 = e^{\theta_1 \sigma_x}, \quad G_2 = e^{-\theta_2 \sigma_x}, \quad G_3 = e^{i\phi_3 \sigma_z}, \quad (25)$$

with $\theta_1, \theta_2 > 0$, $\theta_1/\theta_2 \notin \mathbb{Q}$, and $\phi_3/2\pi \notin \mathbb{Q}$.

Proof. Due to decomposition (24), it is enough to show that all elements of the form $e^{i\phi\sigma_z}$ and of the form $e^{\theta\sigma_x}$ can be approximated with a product of G_1, G_2, and G_3. First, any point of the circle $\mathscr{C} = \mathbb{R} \bmod 2\pi$ can be approximated by a multiple of an angle ϕ_3, provided that ϕ_3 is not rational with 2π. Approximating ϕ as $\phi \approx N_3\phi_3$, we see that $N_3 \in \mathbb{N}$ corresponds to approximating the exponential $e^{i\phi\sigma_z}$ as $G_3^{N_3}$. In the same way, any point of the circle $\mathscr{C}' = \mathbb{R} \bmod \theta_1$ can be approximated by a multiple of $-\theta_2$, provided that θ_2 is not rational with θ_1. Since any real number $\theta \in \mathbb{R}$ can be written as $\theta = M\theta_1 + \theta \bmod \theta_1$, by approximating $\theta \bmod \theta_1 \approx N_2\theta_2 \bmod \theta_1$, we obtain $\theta \approx N_1\theta_1 - N_2\theta_2$ for some $N_1 \in \mathbb{N}$. This corresponds to approximating the exponential $e^{\theta\sigma_x}$ as $G_1^{N_1} G_2^{N_2}$.

The previous theorem is particularly important in consideration of physical realizations, where the group $\mathbb{SU}(1, 1)$ acts unitarily on an infinite-dimensional Hilbert space. In this case, the previous result shows that any unitary transformation representing an element of $\mathbb{SU}(1, 1)$ can be arbitrarily approximated by a finite circuit made only of three elementary gates. However, if we thoroughly define a parameter for the rating of the approximation, we find that the accuracy is arbitrarily small; this is due to the unboundedness of the generators for the unitary representations of $\mathbb{SU}(1, 1)$ of physical interest. In particular, we are interested in the two representations in which

$$K_z = \frac{1}{2}(a^\dagger a + b^\dagger b + 1), \quad K_+ = a^\dagger b^\dagger, \quad K_- = ab,$$
$$K_z = \frac{1}{2}(a^\dagger a + 1/2), \quad K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2. \quad (26)$$

The parameter for the approximation rating is the accuracy ϵ^{-1}, with

$$\epsilon \doteq \sup_{\|\psi\| = 1} \|(U_1 - U_2)|\psi\rangle\|, \quad (27)$$

where U_1 is the target element, U_2 is the product of elementary gates that approximates U_1. However, since we are considering infinite-dimensional representations, the difference $U_1 - U_2$ has eigenvalues arbitrarily near 2. The supremum is then always 2, and in order to find some approximation criterion we have to impose some constraints on the states that we are considering. For example, we will impose that the average and second moment of the photon number distribution are finite, which are reasonable physical assumptions. Suppose now that we have a sufficiently long sequence of elementary gates, in such a way that, using the decomposition of Eq. (24), $U_1 = e^{-i\alpha K_z}e^{-i\beta K_x}e^{-i\gamma K_z}$ and $U_2 = e^{-i(\alpha+\delta_\alpha)K_z}e^{-i(\beta+\delta_\beta)K_x}e^{-i(\gamma+\delta_\gamma)K_z}$, and only first-order terms in δ_x are relevant thanks to the constraint on states. After some algebra, and exploiting Eq. (63), one can verify that the supremum of $\langle\psi|2I - U_1^\dagger U_2 - U_2^\dagger U_1|\psi\rangle$ is almost equal to the supremum of $\langle\psi|\Delta|\psi\rangle$, where

$$\Delta = \begin{cases}(\delta_\alpha^2 + \delta_\gamma^2 + 2\cosh\beta\delta_\alpha\delta_\gamma - \delta_\beta^2)K_z^2 \\ (\delta_\beta^2 - \delta_\alpha^2 - \delta_\gamma^2 - 2\cosh\beta\delta_\alpha\delta_\gamma)K_x^2,\end{cases} \tag{28}$$

depending on the sign of $\delta_\alpha^2 + \delta_\gamma^2 + 2\cosh\beta\delta_\alpha\delta_\gamma - \delta_\beta^2$. This equation implies two facts. First of all, we can easily verify that the physical constraint on states is necessary in order to guarantee the boundedness of ϵ. However, it is not sufficient because of the presence of $\cosh\beta$ in the expression. This is due to the noncompactness of the group, which implies that, even in the defining representation, the approximation is worse as one goes further along the direction of a noncompact parameter. This is a fatal flaw of any analogy to the Solovay–Kitaev theorem for the qubit case, and in order to have a similar result one must also restrict the set of unitaries that he wants to approximate. Otherwise, a power law for the number of gates as a function of ϵ^{-1} can be sought which also contains an explicit dependence on the parameter β. Suppose that we have $|\delta_\alpha^2 + \delta_\gamma^2 + 2\cosh\beta\delta_\alpha\delta_\gamma - \delta_\beta^2| \doteq f(N)$, where N is the number of elementary gates needed to approximate the target group element within the defining representation. Then, for $\Delta \propto K_z^2$ in Eq. (28), we have $\epsilon = f(N)(\langle E^2\rangle + 2\lambda\langle E\rangle + \lambda^2)$, where E is the total number of photons and $\lambda = \frac{1}{4}$ for the single-mode representation and $\lambda = \frac{1}{2}$ for two modes. The function f is clearly nonincreasing; supposing that it is strictly monotonic, we can invert it, obtaining

$$N = f^{-1}\left(\frac{\epsilon}{\langle E^2\rangle + 2\lambda\langle E\rangle + \lambda^2}\right). \tag{29}$$

4. UNITARY REPRESENTATIONS OF $\mathbb{SU}(1,1)$

Given a representation of the $su(1, 1)$ algebra where the generators K_x, K_y, K_z are Hermitian operators acting in an infinite-dimensional Hilbert space $\mathcal{H}$, we consider the unitary representation $U_{\theta,\phi,\psi}$ of the group $\mathbb{SU}(1, 1)$ defined by Eq. (21). In general, such a representation is reducible, and it can be decomposed into unitary irreducible representations (UIRs).

A UIR $U_{\theta,\phi,\psi}$ is called *square-summable* if there is a nonzero vector $|v\rangle \in \mathcal{H}$ such that

$$\int_{\mathbb{SU}(1,1)} d\nu(\theta,\phi,\psi)|\langle v|U_{\theta,\phi,\psi}|v\rangle|^2 < \infty, \tag{30}$$

where $d\nu$ is the invariant measure defined in Eq. (9). Moreover, since the group $\mathbb{SU}(1, 1)$ is unimodular, if the above integral converges for one vector $|v\rangle \neq 0$, then it converges for any vector in $\mathcal{H}$ [10].

Square-summable representations enjoy the important property expressed by the following.

Theorem 2 (Formula for the group average). If the irreducible representation $U_{\theta,\phi,\psi}$ is square-summable, then for any operator $A \in \mathcal{B}(\mathcal{H})$ the following relation holds:

$$\int_{\mathbb{SU}(1,1)} d\nu(\theta,\phi,\psi)U_{\theta,\phi,\psi}AU_{\theta,\phi,\psi}^\dagger = \mathrm{Tr}[A]\frac{\mathbb{1}}{d}. \tag{31}$$

Here, $\mathbb{1}$ is the identity in $\mathcal{H}$ and d is the formal dimension defined by

$$d \doteq \left(\int_{\mathbb{SU}(1,1)} d\nu(\theta,\phi,\psi)|\langle v|U_{\theta,\phi,\psi}|v\rangle|^2\right)^{-1}, \tag{32}$$

where $|v\rangle$ is any normalized vector $|v\rangle \in \mathcal{H}$, $\langle v|v\rangle = 1$.

The formula for the group average is fundamental in the contexts of quantum estimation and tomography, since it allows resolutions of the identity to be constructed via a group integral. In the context of quantum estimation, Eq. (31) ensures that the operators

$$P(\theta,\phi,\psi) = U_{\theta,\phi,\psi}\xi U_{\theta,\phi,\psi}^\dagger, \tag{33}$$

where ξ is any operator satisfying $\xi > \mathrm{Tr}[\xi] = d$, provide a *positive operator valued measure* (POVM) for the joint estimation of the three parameters θ, ϕ, ψ. In fact, such operators satisfy the normalization condition

$$\int_{\mathbb{SU}(1,1)} d\nu(\theta,\phi,\psi)P(\theta,\phi,\psi) = \mathbb{1}, \tag{34}$$

which guarantees that the total probability of all possible outcomes is one. In particular, if $\xi = d|v\rangle\langle v|$ for some state $|v\rangle$, the above formula gives the completeness of the set of $\mathbb{SU}(1, 1)$ coherent states [11]:

$$|v_{\theta,\phi,\psi}\rangle \doteq U_{\theta,\phi,\psi}|v\rangle. \quad (35)$$

4.1. Examples

4.1.1. Single-mode squeezing. The representation of the $su(1, 1)$ algebra, given by

$$K_+ = \frac{a^{\dagger 2}}{2}, \quad K_- = \frac{a^2}{2}, \quad K_z = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right) \quad (36)$$

is reducible in the Hilbert space $\mathcal{H}$ of a single harmonic oscillator. In fact, the subspaces $\mathcal{H}_{\text{even}} = \text{Span}\{|2n\rangle | n \in \mathbb{N}\}$ and $\mathcal{H}_{\text{odd}} = \text{Span}\{|2n+1\rangle | n \in \mathbb{N}\}$, defined in terms of the Fock basis $|n\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n}|0\rangle$, are invariant under the application of K_x, K_y, K_z. The unitary representation of $\mathbb{SU}(1, 1)$ defined by Eq. (21) acts irreducibly in the subspaces $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$. There is a substantial difference between the two UIRs acting in $\mathcal{H}_{\text{odd}}$ and $\mathcal{H}_{\text{even}}$; in fact, the first is square-summable, with formal dimension $d_{\text{odd}} = 1/(4\pi^2)$, while the latter is not. A surprising consequence of the non-square-summability in $\mathcal{H}_{\text{even}}$ is that the squeezed states $|\xi\rangle = e^{\xi K_+ - \bar{\xi} K_-}|0\rangle$—which are the coherent states of $\mathbb{SU}(1, 1)$ commonly considered in quantum optics—do not provide a resolution of the identity.

4.1.2. Two-mode squeezing. The representation of the Lie algebra $su(1, 1)$ given by the operators

$$K_+ = a^\dagger b^\dagger, \quad K_- = ab, \quad K_z = \frac{1}{2}(a^\dagger a + b^\dagger b + 1) \quad (37)$$

is reducible in the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ of two harmonic oscillators. It is indeed immediately evident that, for any $\delta \in \mathbb{Z}$, the subspaces $\mathcal{H}_\delta = \text{Span}\{|m\rangle|n\rangle | m, n \in \mathbb{N}, m - n = \delta\}$ are invariant under application of the operators K_+, K_-, K_z. The unitary representation of $\mathbb{SU}(1, 1)$ given by Eq. (21) is irreducible in each subspace $\mathcal{H}_\delta$. The two UIRs acting in $\mathcal{H}_\delta$ and $\mathcal{H}_{-\delta}$ are unitarily equivalent, while for different values of $|\delta|$ one has nonequivalent UIRs. All UIRs in the two-mode realization are square-summable, with the only exception being the case $\delta = 0$.

5. $\mathbb{SU}(1, 1)$ TOMOGRAPHY

5.1. Reconstruction Formula for Square-Summable Representations

Let us now consider the group as a tool for quantum tomography. In order to do that, it is useful to consider the set of operators on a Hilbert space $\mathcal{H}$ as a Hilbert space itself, isomorphic to $\mathcal{H} \otimes \mathcal{H}$, and to look for spanning sets in this Hilbert space. An immediate and handy way of defining the isomorphism between operators and bipartite vectors is through the definition

$$|A\rangle\rangle \doteq \sum_{m,n} \langle m|A|n\rangle |m\rangle \otimes |n\rangle, \quad (38)$$

where A is an operator on $\mathcal{H}$ and $|n\rangle$ are elements of a fixed basis for $\mathcal{H}$. This definition implies the following useful identities:

$$A \otimes B|C\rangle\rangle = |ACB^\tau\rangle\rangle, \quad (39)$$

$$\langle\langle A|B\rangle\rangle = \text{Tr}[A^\dagger B], \quad (40)$$

where X^τ denotes the transpose of X in the basis $|n\rangle$.

For a square-summable UIR $U_{\theta,\phi,\psi}$, we can obtain a resolution of the identity by simply exploiting Eqs. (31) and (39), namely,

$$\mathbb{1} \otimes \mathbb{1} = d \int_{\mathbb{SU}(1,1)} d\nu(\theta,\phi,\psi) |U_{\theta,\phi,\psi}\rangle\rangle\langle\langle U_{\theta,\phi,\psi}|, \quad (41)$$

which shows that the unitaries $U_{\theta,\phi,\psi}$ form a spanning set for the space of operators.

Tomographing the state ρ is equivalent to reconstructing the ensemble average $\text{Tr}[\rho A] = \langle\langle \rho|A\rangle\rangle$ of any operator A on the state ρ. This can be done using the reconstruction formula

$$\begin{aligned} &\text{Tr}[\rho A] \\ &= d \int_{\mathbb{SU}(1,1)} d\nu(\theta,\phi,\psi) \text{Tr}[\rho U_{\theta,\phi,\psi}] \text{Tr}[U^\dagger_{\theta,\phi,\psi} A], \end{aligned} \quad (42)$$

which directly follows by inserting the resolution of identity (41) into the product $\text{Tr}[\rho A] = \langle\langle \rho|A\rangle\rangle$. In a real tomographic scheme, the traces $\text{Tr}[\rho U_{\theta,\phi,\psi}]$ have to be evaluated by experimental data and subsequently averaged with the *processing function* $f_A(\theta, \phi, \psi) = \text{Tr}[U^\dagger_{\theta,\phi,\psi} A]$ in order to obtain the expectation value $\text{Tr}[\rho A]$. A feasible scheme for evaluating the traces $\text{Tr}[\rho U_{\theta,\phi,\psi}]$ by experimental data is discussed in Subsection 5.4.

5.2. Non-Square-Summable Representations: Regularization

The representations of $\mathbb{SU}(1, 1)$ that are common in quantum optics are single-mode and two-mode Schwinger representations, analyzed in paragraphs 4.1.1 and 4.1.2, respectively. In particular, the irreducible subspaces $\mathcal{H}_{\text{even}}$ in the single-mode case, and $\mathcal{H}_0$ in the two-mode case, are particularly interesting, since coherent states with an even photon number in the single-mode case (or, alternatively, zero difference of photon numbers in the two-mode case) are experimentally achievable by simple vacuum squeezing.

The problem now is that the single-mode representation in $\mathcal{H}_{\text{even}}$ and the two-mode representation in $\mathcal{H}_0$ are not square-summable; therefore, the group integral in Eq. (30) diverges. In order to circumvent this problem, we address here a technique that consists in modifying the invariant measure $d\nu(\theta, \phi, \psi)$ by a regularization factor $g(\theta, \phi, \psi)$, which is positive almost everywhere. The modification of the measure makes it noninvariant, and consequently the group average identities become similar to those of nonunimodular groups, where there is no invariant measure.

Using the regularization factor instead of the resolution of identity (41), we have a positive invertible operator

$$F = \int_{\mathbb{SU}(1,1)} d\nu(\theta, \phi, \psi) g(\theta, \phi, \psi) |U_{\theta, \phi, \psi}\rangle\rangle\langle\langle U_{\theta, \phi, \psi}|, \quad (43)$$

and the ensemble average of any operator A can be obtained by writing $\mathrm{Tr}[\rho A] = \langle\langle \rho | F F^{-1} | A \rangle\rangle$. In this way, we can provide a *regularized reconstruction formula*

$$\mathrm{Tr}[\rho A] = \int_{\mathbb{SU}(1,1)} d\nu(\theta, \phi, \psi) g(\theta, \phi, \psi) f_A(\theta, \phi, \psi) \mathrm{Tr}[U_{\theta, \phi, \psi}\rho], \quad (44)$$

involving the processing function

$$f_A(\theta, \phi, \psi) = \langle\langle U_{\theta, \phi, \psi}| F^{-1} | A \rangle\rangle, \quad (45)$$

instead of $\langle\langle U_{\theta, \phi, \psi}| A \rangle\rangle$. Notice that identity Eq. (44) can also be used in the square-summable case with $g(\theta, \phi, \psi) \equiv 1$.

5.3. A Relevant Example

Here we consider in detail the case of the UIRs with even photon number (single-mode case) and with $\delta = 0$ (two-mode case) representations, providing an example of the general method discussed above. We will start from the following integral:

$$S(m, n; m', n') = \int_{\mathbb{SU}(1,1)} d\nu(\theta, \phi, \psi) \langle 2m | U_{\theta, \phi, \psi} | 2n \rangle \langle 2n' | U^{\dagger}_{\theta, \phi, \psi} | 2m' \rangle, \quad (46)$$

where $|2m\rangle$ denotes either the even eigenstate of $a^{\dagger}a$ or the zero-difference eigenstate $|m\rangle|m\rangle$ of $a^{\dagger}a + b^{\dagger}b$. This can be evaluated by exploiting Eqs. (21) and (22), thus obtaining the following expressions for the matrix element $\langle 2m | U_{\theta, \phi, \psi} | 2n \rangle$:

$$\langle 2m | U_{\theta, \phi, \psi} | 2n \rangle = e^{i\phi(2n+\kappa)} e^{i(\psi - \phi)(n-m)} \times \sum_{p=0}^{n} c_{\kappa}(p) (-i \tanh\theta)^{2p+m-n} \left(\frac{1}{\cosh\theta}\right)^{2n-2p+\kappa}, \quad (47)$$

where $\kappa = 1/2$ for even single-mode states and $\kappa = 1$ for $d = 0$ two-mode states, and

$$c_{\kappa}(p) = \begin{cases} \dfrac{\sqrt{2n!2m!}}{p!(p+m-n)!(2n-2p)!2^{2p+m-n}}, & \kappa = \dfrac{1}{2} \\ \dfrac{n!m!}{p!(p+m-n)!(n-p)!^2}, & \kappa = 1. \end{cases} \quad (48)$$

By exploiting the integral in ψ and then in ϕ, we obtain

$$S(m, n; m', n') = 4\pi^2 \delta_{m, m'} \delta_{n, n'} \sum_{p, p'=0}^{n} c_{\kappa}(p) c_{\kappa}(p') (-1)^{p+p'} I_{\kappa}(m, n, p, p'), \quad (49)$$

where

$$I_{\kappa}(m, n, p, p') = \int_0^{\infty} d\theta \sinh\theta \cosh\theta \frac{(\tanh\theta)^{2(p+p')+2m-2n}}{(\cosh\theta)^{4n-2p-2p'+2\kappa}}, \quad (50)$$

and for $2n = p + p'$ this is clearly divergent. Moreover, for $m < n$ and $p = p' = 0$, the integral diverges because of the singularity in $\theta = 0$. If we introduce the regularization factor $g(\theta, \phi, \psi) = g(\theta) \doteq \dfrac{e^{-1/(\tanh\theta)^2}}{(\cosh\theta)^3}$, by exploiting the same calculations we get the same result with $I_{\kappa}(m, n, p, p')$ replaced by

$$I_{\kappa, g}(m, n, p, p') = \int_0^1 dx\, x^{2n-p-p'+\kappa} (1-x)^{p+p'+m-n} e^{-\frac{1}{1-x}}, \quad (51)$$

which is derived from Eq. (50) by the change of variable $(1/\cosh\theta)^2 \longrightarrow x$ and which is finite. As a consequence,

$$\int_{\mathbb{SU}(1,1)} d\nu(\theta, \phi, \psi) g(\theta) |U_{\theta, \phi, \psi}\rangle\rangle\langle\langle U_{\theta, \phi, \psi}| = \sum_{m, n = 0}^{\infty} F_{m, n}^{(\kappa)} |2m\rangle\langle 2m| \otimes |2n\rangle\langle 2n|, \tag{52}$$

$$F_{m, n}^{(\kappa)} = 4\pi^2 \sum_{p, p' = 0}^{n} (-1)^{p+p'} c_\kappa(p) c_\kappa(p') I_{\kappa, g}(m, n, p, p')$$
$$= \int_0^1 dx \left| \sum_p (-1)^p c_\kappa(p) \left(\frac{1-x}{x}\right)^p \right|^2 x^{2n} (1-x)^{m-n} e^{-\frac{1}{1-x}},$$

and since the coefficients $c_\kappa(p)$ are not null, clearly $0 < F_{m, n}^{(\kappa)} < \infty$. This implies that the operator F in Eq. (43) is actually invertible, and we can safely use the processing function in Eq. (45) for tomographic reconstruction of the operator A. Notice that this formula is very close to the group-average formula for nonunimodular groups, where the Duflo–Moore operator C is involved. The group average identity in that case is similar to Eq. (52) with $F_{m, n}^{(\kappa)} = (C^\dagger C)_{m, m}$.

5.4. How to Make Tomography Experimentally

In quantum tomography, the expectation value $\langle A \rangle = \mathrm{Tr}[\rho A]$ of any observable is reconstructed by exploiting integral (44). Moreover, in order to have a feasible tomography, it is essential to devise a method to evaluate the traces $\mathrm{Tr}[\rho U_{\theta, \phi, \psi}]$ from experimental data. To do this, it is useful to break the integral over $\mathbb{SU}(1, 1)$ into the sum of the contributions coming from the regions Ω_+, Ω_-, and $-\Omega_-$ introduced in Subsection 2.2. It is not difficult to see that the regions Ω_- and $-\Omega_-$ give the same contribution to the tomographic integrals, whence we have

$$\mathrm{Tr}[\rho A] = \int_{\Omega_+} d\nu(\theta, \phi, \psi) g(\theta, \phi, \psi) f_A(\theta, \phi, \psi) \mathrm{Tr}[U_{\theta, \phi, \psi}\rho] + 2\int_{\Omega_-} d\nu(\theta, \phi, \psi) g(\theta, \phi, \psi) f_A(\theta, \phi, \psi) \mathrm{Tr}[U_{\theta, \phi, \psi}\rho]. \tag{53}$$

By definition, any element in $\Omega_+(\Omega_-)$ can be obtained by the exponential map as $e^{i\chi \mathbf{n} \cdot \mathbf{K}}$ for some χ and $\mathbf{n}$ with $\mathbf{n} \cdot \mathbf{n} = +1$ (-1). In addition, it is possible to show that any exponential $e^{i\chi \mathbf{n} \cdot \mathbf{K}}$ can be written as

$$e^{i\chi \mathbf{n} \cdot \mathbf{K}} = \begin{cases} V(\mathbf{n})^\dagger e^{i\chi K_z} V(\mathbf{n}), & \mathbf{n} \cdot \mathbf{n} = +1 \\ W(\mathbf{n})^\dagger e^{i\chi K_x} W(\mathbf{n}), & \mathbf{n} \cdot \mathbf{n} = -1, \end{cases} \tag{54}$$

where $V(\mathbf{n})$ and $W(\mathbf{n})$ are suitable unitaries in the group representation. A detailed proof of this result is given in the Appendix. Thank to this observation, the trace $\mathrm{Tr}[\rho e^{i\chi \mathbf{n} \cdot \mathbf{K}}]$ can be evaluated by performing a unitary transformation on the state ρ (either $V(\mathbf{n})$ or $W(\mathbf{n})$) and subsequently by measuring one of the observables K_z and K_x.

Finally we observe that, since a real experiment produces only a finite array of data, integral (53) has to be approximated by a statistical average over the experimental results obtained by measuring a large number N of identically prepared systems. This introduces the need for a randomization in the experimental setup that produces the unitaries $V(\mathbf{n})$, $W(\mathbf{n})$ according to some probability distribution. Notice that the most natural choice, which would be to take $d\rho(\mathbf{n})$ as the measure over the space of directions $\mathbf{n}$ induced by the invariant measure $d\nu(\theta, \phi, \psi)$, is not possible, since such a measure cannot be normalized (the space of directions is noncompact). The form of Eq. (53) suggests then to take as a measure $d\nu(\theta, \phi, \psi)g(\theta, \phi, \psi)$, and in the example we considered this actually works. However, it may happen that regularizing the integral in Eq. (44) is not sufficient to regularize the group measure also. In this case, it is convenient to modify $g(\theta, \phi, \psi)$ in such a way that both the measure itself and the group integrals converge. This implies in particular that the choice $g(\theta, \phi, \psi) \equiv 1$ for square-summable representations has to be changed. Finally, the ensemble average $\langle A \rangle$ can be approximated by the expression

$$\langle A \rangle \simeq \frac{1}{N} \sum_{j=1}^{N} f_A(\theta_j, \phi_j, \psi_j) \mathrm{Tr}[\rho U(\theta_j, \phi_j, \psi_j)], \tag{55}$$

where θ_j, ϕ_j, ψ_j are the randomly extracted parameters. Notice that the expression on the right-hand side of Eq. (55) reasonably converges to the left-hand side if the variance of the processing function is finite, namely, if $f_A(\theta, \phi, \psi)$ is square-summable. By Eqs. (43) and (45), this condition is equivalent to

$$\int_{\mathbb{SU}(1,1)} d\nu(\theta, \phi, \psi) g(\theta, \phi, \psi) |f_A(\theta, \phi, \psi)|^2 = \langle\langle A| F^{-1} |A\rangle\rangle < \infty. \tag{56}$$

6. CONCLUSIONS

This paper collects a large number of useful results about the group $\mathbb{SU}(1, 1)$ that are dispersed in the literature and also contains some novel applications regarding the use of $\mathbb{SU}(1, 1)$ for quantum computation and tomography with nonlinear optics. The main issues we addressed here are (i) the approximation of $\mathbb{SU}(1, 1)$ gates in the quantum optical representations and (ii) the tomographic state reconstruction exploiting group theoretical methods. As regards the first topic, we gave an approximability theorem and discussed the limits under which it holds. The theorem provides a useful result in the search for an elementary set of gates that can be used to universally approximate any $\mathbb{SU}(1, 1)$ gate with arbitrary accuracy. To complete the analogy with the Solovay–Kitaev theorem for qubit gates, the power law of the number of elementary gates as a function of the accuracy should be evaluated, and due to noncompactness we expect that the law would depend on the parameters of the target group element.

In the context of quantum estimation and tomography, we showed a technique for regularization of the group integral for physically relevant representations that are not square-summable. The core of the regularization technique is a modification of the Haar measure over the group, such that the regularized measure is no longer invariant. This makes the integrals for tomographic reconstruction convergent but radically modifies the processing functions in a way that can be seen as a generalization of the group integral identity for nonunimodular groups. Such a regularization technique is very powerful, since it contains a freedom in the choice of the regularization factor, which allows for a further optimization of the processing. Moreover, the mentioned scheme can be applied not only to the case of $\mathbb{SU}(1, 1)$, but also to any other tomographic setup.

APPENDIX A

For a given representation of the $su(1, 1)$ algebra, consider the real vector space $\mathcal{V}$ spanned by the generators K_x, K_y, K_z. Of course, $\mathcal{V}$ is isomorphic to $\mathbb{R}^3$ via the correspondence

$$K_x \longleftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad K_y \longleftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad K_z \longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (57)$$

The action of the group $\mathbb{SU}(1, 1)$ on the space $\mathcal{V}$, given by $\mathcal{V} \ni m \mapsto e^{i\chi\mathbf{n}\cdot\mathbf{K}} m e^{-i\chi\mathbf{n}\cdot\mathbf{K}}$, can be obtained by exponentiating the adjoint action on the algebra, namely,

$$e^{i\chi\mathbf{n}\cdot\mathbf{K}} m e^{-i\chi\mathbf{n}\cdot\mathbf{K}} = e^{i\chi\mathbf{n}\cdot\mathrm{Ad}(\mathbf{K})} m, \quad (58)$$

where $\mathrm{Ad}(K_i)$ is defined by $\mathrm{Ad}(K_i)K_j \doteq [K_i, K_j]$. Moreover, using the commutation relations of $su(1, 1)$, we immediately find

$$\mathrm{Ad}(K_x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad \mathrm{Ad}(K_y) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\mathrm{Ad}(K_z) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (59)$$

Therefore, we find that a generic element of $\mathbb{SU}(1, 1)$—parametrized as $M(\theta, \phi, \psi) = e^{i(\phi-\psi)k_z} e^{-2ik_x} e^{i(\phi+\psi)k_z}$ as in Eq. (24)—is represented in the space $\mathcal{V}$ by the matrix

$$R(\theta, \phi, \psi) = e^{i(\phi-\psi)\mathrm{Ad}(K_z)} e^{-2i\,\mathrm{Ad}(K_x)} e^{i(\phi+\psi)\mathrm{Ad}(K_z)}, \quad (60)$$

whose explicit expression is rather lengthy but is easily computable by exponentiating the matrices in Eq. (59).

It is not difficult to see that the matrices $R(\theta, \phi, \psi)$ given by Eq. (60) form a subgroup of the group $\mathbb{SO}(2, 1)$; namely, they all have unit determinant and preserve the form $\mathbf{v}\cdot\mathbf{w} = v_z w_z - v_x w_x - v_y w_y$. More precisely, the matrices $R(\theta, \phi, \psi)$ coincide with the group $\mathbb{SO}^+(2, 1)$, which contains all matrices $R \in \mathbb{SO}(2, 1)$ such that $R_{33} \geq 1$. Incidentally, we notice that the correspondence $\mathbb{SU}(1, 1) \longrightarrow \mathbb{SO}^+(2, 1)$ is not one-to-one, since both $\pm\mathbb{1} \in \mathbb{SU}(1, 1)$ are mapped into the identity in $\mathbb{SO}^+(2, 1)$. One has indeed the group homeomorphism $\mathbb{SO}^+(2, 1) \simeq \mathbb{SU}(1, 1)/\mathbb{Z}_2$ [14], which is exactly the same relation occurring between the groups $\mathbb{SU}(2)$ and $\mathbb{SO}(3)$, namely, $\mathbb{SO}(3) \simeq \mathbb{SU}(2)/\mathbb{Z}_2$.

Similarly to the case of $\mathbb{SO}(3)$, where any spatial direction $\mathbf{n}$ can be conjugated with the direction of the z axis by a suitable rotation, in the case of $\mathbb{SO}^+(2, 1)$, any direction $\mathbf{n}$ with $\mathbf{n}\cdot\mathbf{n} = +1$ can be conjugated with the z axis, and any direction with $\mathbf{n}\cdot\mathbf{n} = -1$ can be conjugated with the x axis. For example, the matrix $R(\theta, \phi, \psi)$ in Eq. (60) transforms the direction of the z axis as

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \mathbf{n} = \begin{pmatrix} -\sinh(2\theta)\sin(\phi-\psi) \\ -\sinh(2\theta)\cos(\phi-\psi) \\ \cosh(2\theta) \end{pmatrix}, \quad (61)$$

and it is clear that here $\mathbf{n}$ can be any direction with $\mathbf{n}\cdot\mathbf{n} = 1$ (modulo an overall phase factor). Therefore, we have, for any $\mathbf{n}$ with $\mathbf{n}\cdot\mathbf{n} = +1$,

$$\mathbf{n}\cdot\mathbf{K} = U^\dagger_{\theta,\phi,\psi} K_z U_{\theta,\phi,\psi}, \quad (62)$$

for suitable θ, ϕ, ψ. In conclusion,

$$e^{i\chi\mathbf{n}\cdot\mathbf{K}} = U^\dagger_{\theta,\phi,\psi} e^{i\chi K_z} U_{\theta,\phi,\psi}. \quad (63)$$

The same reasoning holds for any direction with $\mathbf{n}\cdot\mathbf{n} = -1$.

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12. For example, the unit circle given by $x = y = 0$, $t^2 + z^2 = 1$ cannot be contracted to a point. Similarly to $U(1)$, the group $\mathbb{SU}(1, 1)$ is infinitely many times connected, and its covering group is made by sewing together infinitely many sheets, each of them is omeomorphic to $\mathbb{SU}(1, 1)$.
13. A unimodular group **G** is a group that admits an invariant Haar measure $d\mu(g)$, namely a measure which is both left- and right-invariant: $d\mu(hg) = d\mu(gh) = d\mu(g)$, $\forall g, h \in \mathbf{G}$. For an irreducible unitary representation of an unimodular there are only two alternatives: the integral (30) converges either for any vector $|v\rangle \in \mathcal{H}$, or for none (see, for example [10]).
14. This fact is not mentioned in [11], where actually the correspondence is mistakenly stated as $\mathbb{SO}(2, 1) \simeq \mathbb{SU}(1, 1)/\mathbb{Z}_2$.