Entanglement can enhance the distinguishability of entanglement-breaking channels

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We show the rather counterintuitive result that entangled input states can strictly enhance the distinguishability of two entanglement-breaking channels.

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The class of entanglement-breaking channels—trace-preserving completely positive maps for which the output state is always separable—has been extensively studied [1–8]. More precisely, a quantum channel $\mathcal{E}$ is called entanglement breaking if $(\mathcal{E}\otimes I)(\Gamma)$ is always separable, i.e., any entangled density matrix Γ is mapped to a separable one. The convex structure of entanglement-breaking channels has been thoroughly analyzed in Refs. [1,2]. Moreover, the properties of such a kind of channel have allowed to obtain a number of results for the hard problem of additivity of capacity in quantum information theory [3–11].

Channels that break entanglement are particularly noisy in some sense. In order to check if a channel is entanglement breaking, it is sufficient to look at the separability of the output state corresponding just to an input maximally entangled state [1]; namely $\mathcal{E}$ is entanglement breaking if and only if $(\mathcal{E}\otimes I)(|\beta\rangle\langle\beta|)$ is separable for $|\beta\rangle = d^{-1/2}\sum_{j=0}^{d-1}|j\rangle\otimes|j\rangle$, d being the dimension of the Hilbert space. Another equivalent condition [1] is that the channel $\mathcal{E}$ can be written as

$$\mathcal{E}(\rho) = \sum_k \langle\phi_k|\rho|\phi_k\rangle|\psi_k\rangle\langle\psi_k|, \qquad (1)$$

where $\{|\phi_k\rangle\langle\phi_k|\}$ gives a positive operator-valued measure (POVM), namely $\sum_k|\phi_k\rangle\langle\phi_k| = I$ [12]. The last formulation has an immediate physical interpretation: an entanglement-breaking channel can be simulated by a classical channel, in the sense that the sender can make a measurement on the input state ρ by means of a POVM $\{|\phi_k\rangle\langle\phi_k|\}$, and send the outcome k via a classical channel to the receiver who then prepares an agreed-upon pure state $|\psi_k\rangle$. For the above reason one could think that entanglement—the peculiar trait of quantum mechanics—may not be useful when one deals with entanglement-breaking channels. In fact, entanglement breaking channels have zero quantum capacity [10].

In this Brief Report, however, we will show a situation in which the use of entanglement can be relevant for entanglement-breaking channels, such as when one is asked to optimally discriminate two entanglement-breaking channels, as in the quantum hypothesis testing scenario [13]. What we mean is that an entangled input state can *strictly* enhance the distinguishability of two given entanglement-breaking channels. We will make use of some recent results [14] on the optimal discrimination of two given quantum operations. In particular, a complete characterization of the optimal input states to achieve the minimum-error probability has been given for Pauli channels [14], along with a necessary and sufficient condition for which entanglement strictly improves the discrimination; such a condition follows.

Given with *a priori* probability p_1 and $p_2 = 1-p_1$, two Pauli channels

$$\mathcal{E}_i(\rho) = \sum_{\alpha=0}^{3} q_i^{(\alpha)}\sigma_\alpha\rho\sigma_\alpha, \quad i = 1,2, \qquad (2)$$

where $\{\sigma_1,\sigma_2,\sigma_3\} = \{\sigma_x,\sigma_y,\sigma_z\}$ denote the customary spin Pauli matrices, $\sigma_0 = I$, and $\sum_{\alpha=0}^{3} q_i^{(\alpha)} = 1$, the use of entanglement strictly improves the discrimination if and only if [14]

$$\prod_{\alpha=0}^{3} r_\alpha < 0, \qquad (3)$$

with

$$r_\alpha = p_1 q_1^{(\alpha)} - p_2 q_2^{(\alpha)}. \qquad (4)$$

Moreover, the optimal input state can always be chosen as a maximally entangled state.

In the following we explicitly show the case of two entanglement-breaking channels that are strictly better discriminated by means of a maximally entangled input state. Let us consider for simplicity two different depolarizing channels

$$\mathcal{E}_i^D(\rho) = q_i\rho + \frac{1-q_i}{3}\sum_{\alpha=1}^{3}\sigma_\alpha\rho\sigma_\alpha, \quad q_1 \neq q_2. \qquad (5)$$

The two channels are supposed to be given with *a priori* probability $p_1 = p$ and $p_2 = 1-p$, respectively. The coefficients r_α of Eq. (4) are given in this case by

$$r_0 = pq_1 - (1-p)q_2,$$

$$r_1 = r_2 = r_3 = p\frac{1-q_1}{3} - (1-p)\frac{1-q_2}{3}. \qquad (6)$$

Hence, entanglement strictly enhances the distinguishability of the two channels $\mathcal{E}_1^D$ and $\mathcal{E}_2^D$ if and only if

$$[pq_1 - (1-p)q_2]\left[p\frac{1-q_1}{3} - (1-p)\frac{1-q_2}{3}\right] < 0, \qquad (7)$$

or equivalently

$$(q_1+q_2)(2-q_1-q_2)p^2-(q_1-2q_1q_2+3q_2-2q_2^2)p + q_2(1-q_2)<0. \tag{8}$$

The solution of Eq. (8) for the prior probability p vs q_1 and q_2 is given by

$$\frac{1-q_2}{2-q_1-q_2}<p<\frac{q_2}{q_1+q_2} \quad \text{for } q_1<q_2,$$

$$\frac{q_2}{q_1+q_2}<p<\frac{1-q_2}{2-q_1-q_2} \quad \text{for } q_1>q_2. \tag{9}$$

A depolarizing channel is entanglement breaking if and only if $q \leqslant 1/2$, where q is the probability pertaining to the identity transformation. This fact can be easily checked by applying the positive-partial-transpose (PPT) condition [15,16] to the Werner state [17] $(\mathcal{E}\otimes I)(|\beta\rangle\langle\beta|)$, where $|\beta\rangle$ denotes the maximally entangled state $|\beta\rangle=(1/\sqrt{2})(|00\rangle+|11\rangle)$. It follows that the solution in Eq. (9) for $q_1,q_2\leqslant 1/2$ gives examples of situations where a maximally entangled input state strictly improves the distinguishability of two entanglement-breaking channels.

In Fig. 1 we plot such a set of solutions for the *a priori* probability p in the case of discrimination between an entanglement-breaking depolarizing channel with $q_1=q \leqslant 1/2$ and a completely depolarizing channel $q_2=1/4$.

In conclusion, in the problem of discriminating two quantum operations, the relevant object is the map corresponding to their differences, which is not a completely positive map.

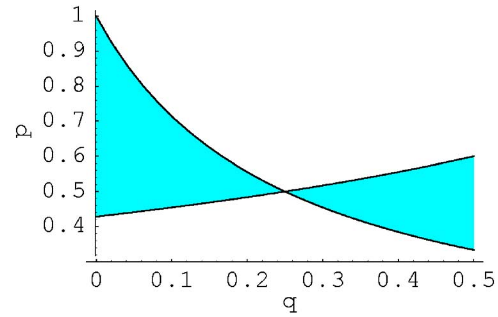

FIG. 1. (Color online) The gray region represents the value of the *a priori* probability p for which the discrimination between a depolarizing channel with $q\leqslant 1/2$ (an entanglement-breaking channel) and a completely depolarizing channel is strictly enhanced by using a maximally entangled input state.

Using entangled states at the input of entanglement-breaking channels gives output separable states that, however, can be better discriminated since they live in a higher dimensional Hilbert space. Curiously, we note that, on the other hand, when we are asked to optimally discriminate two arbitrary unitary transformations—which are of course entanglement-preserving operations—entanglement never enhances the distinguishability [18–20].

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