

## Information-disturbance tradeoff for spin coherent state estimation

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We show how to quantify the optimal tradeoff between the amount of information retrieved by a quantum measurement in estimating an unknown spin coherent state and the disturbance on the state itself, and how to derive the corresponding minimum-disturbing measurement.

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### I. INTRODUCTION

The tradeoff between information retrieved from a quantum measurement and the disturbance caused on the state of a quantum system is a fundamental concept of quantum mechanics and has received a lot of attention in the literature [1–20]. Such an issue is studied for both foundations and its enormous relevance in practice, in the realm of quantum key distribution and quantum cryptography [21,22].

Quantitative derivations of such a tradeoff have been obtained in the scenario of quantum state estimation [23,24]. The optimal tradeoff has been derived in the following cases: In estimating a single copy of an unknown pure state [7], many copies of identically prepared pure qubits [9] and qubits [14], a single copy of a pure state generated by independent phase shifts [13], an unknown maximally entangled state [18], an unknown coherent state [17], and a Gaussian state [19]. Experimental realization of minimal-disturbing measurements has also been reported [15,17]. Recently, the optimal tradeoff has also been derived for quantum state discrimination [20].

The problem is typically the following. One performs a measurement on a quantum state picked (randomly, or according to an assigned *a priori* distribution) from a known set and evaluates the retrieved information along with the disturbance caused on the state. To quantify the tradeoff between information and disturbance, one can adopt two mean fidelities [7]: The estimation fidelity  $G$ , which evaluates on average our best guess of the original state on the basis of the measurement outcome, and the operation fidelity  $F$ , which measures the average resemblance of the state of the system after the measurement to the original one.

In this paper, we study the optimal tradeoff between estimation and operation fidelities when the state is a completely unknown spin coherent state.

Our results will be obtained by exploiting the group symmetry of the problem, which allows us to restrict our analysis on *covariant measurement instruments*. In fact, the property of covariance generally leads to a striking simplification of problems that may look intractable, and has been thoroughly used in the context of state and parameter estimation [23,24].

The derivation of the optimal tradeoff for spin coherent state estimation might find application in the problem of how to achieve a secure distribution of a private shared directional reference frame [25,26]. This task can be achieved by setting up a secure classical key using regular quantum key

distribution, and then converting this into a private shared reference frame using the technique of Ref. [26]. However, it is conceivable that there may be some benefit to using a different, more direct protocol wherein one sends the systems by encoding the directional information over the public channel and monitors for eavesdropping upon them. If the signal states were SU(2) coherent states, then the question of how much security can be achieved depends on the nature of the information gain–disturbance tradeoff.

The paper is organized as follows. In Sec. II we briefly review the concept of spin coherent states. In Sec. III we show that the tradeoff between estimation and disturbance can be studied without loss of generality by considering measurement instruments with a covariant symmetry with respect to the rotation group. In Sec. IV we show how to quantify the optimal information-disturbance tradeoff and to obtain the corresponding minimum-disturbing measurement. We close the paper in Sec. V with concluding remarks.

### II. SPIN COHERENT STATES

In the infinite dimensional space of the harmonic oscillator states we can construct the creation and annihilation operators,  $a^\dagger$ ,  $a$ , obeying the boson commutation relation  $[a, a^\dagger]=1$ . The coherent states of such a system (harmonic oscillator coherent states) are eigenvectors of the annihilation operator  $a|\alpha\rangle=\alpha|\alpha\rangle$  and can be obtained as displacements of the ground state  $|0\rangle$ :

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \quad (1)$$

An important property of such coherent states is that they satisfy the lower bound on the product of the dispersions of the position and momentum operators (or the quadrature operators) required by the Heisenberg uncertainty relation.

The concept of coherent states is not restricted to the infinite dimensional space. In a finite dimensional space one can introduce different kinds of coherent states [27]. In this paper we shall concentrate on so-called spin coherent states [SU(2) coherent states], which we define below.

Let us consider the Hilbert space of spin states with total spin  $j$ , hence with dimension  $d=2j+1$ . By  $|m\rangle$ ,  $m=-j, -j+1, \dots, j-1, j$ , we denote the basis consisting of the eigenvectors of  $J_z$  operator. Spin coherent states are defined as rotations of the “ground” state  $|{-j}\rangle$  by unitary operators from the irreducible SU(2) representation in the  $(2j+1)$ -dimensional space:

$$|\theta, \phi\rangle = R_{\theta, \phi}|-j\rangle, \quad R_{\theta, \phi} = e^{i\theta(J_x \sin \phi - J_y \cos \phi)}. \quad (2)$$

The operator  $R_{\theta, \phi}$  corresponds to a rotation by the angle  $\theta$  around the axis  $\vec{n} = [\sin \phi, -\cos \phi, 0]$ . For  $j=1/2$  the dimension of the space is  $d=2$  (qubit). In this case spin coherent states are actually all the pure states in the space (every pure state can be described by a direction on the Bloch sphere). In higher dimensions, however, spin coherent states constitute only a subset in the set of all states of a given Hilbert space and, moreover, they approach harmonic oscillator coherent states when the dimension of the space tends to infinity [28].

One can decompose a spin coherent state  $|\theta, \phi\rangle$  in the  $J_z$ -eigenvectors basis as follows [27]:

$$\langle m|\theta, \phi\rangle = \binom{2j}{j+m}^{1/2} \left(-\sin \frac{\theta}{2}\right)^{j+m} \left(\cos \frac{\theta}{2}\right)^{j-m} e^{-i(j+m)\phi}. \quad (3)$$

A spin coherent state in a Hilbert space  $\mathcal{H}$ , with  $\dim(\mathcal{H}) = 2j+1$ , will be written as  $|\psi_g\rangle = U_g| -j\rangle$ , where  $U_g$  is a unitary irreducible representation of the  $SU(2)$  group in dimension  $d$  [29]. When performing averages on group parameters, for convenience we will take the normalized invariant Haar measure  $dg$  over the group, i.e.,  $\int_{SU(2)} dg = 1$ , and we will also omit  $SU(2)$  from the symbol of integral.

### III. COVARIANT INSTRUMENTS FOR THE ROTATION GROUP

A measurement process on a quantum state  $\rho$  with outcomes  $\{r\}$  is described by an *instrument* [30], namely a set of trace-decreasing completely positive (CP) maps  $\{\mathcal{E}_r\}$ . Each map is a superoperator that provides the state after the measurement

$$\rho_r = \frac{\mathcal{E}_r[\rho]}{\text{Tr}\{\mathcal{E}_r[\rho]\}}, \quad (4)$$

along with the probability of outcome

$$p_r = \text{Tr}\{\mathcal{E}_r[\rho]\}. \quad (5)$$

The set of positive operators  $\{\Pi_r = \mathcal{E}_r^\vee[I]\}$ , where  $\vee$  denotes the dual map satisfying  $\text{Tr}\{\mathcal{E}^\vee[A]B\} = \text{Tr}\{A\mathcal{E}[B]\}$  for all  $A$  and  $B$ , is known as positive operator-valued measure (POVM), and normalization requires the completeness relation  $\sum_r \mathcal{E}_r^\vee = I$ . This is equivalent to requiring that the map  $\sum_r \mathcal{E}_r$  is trace-preserving. In the following we will denote a  $SU(2)$  unitary map as  $U_g$ , namely  $U_g[\rho] = U_g \rho U_g^\dagger$ . Notice that  $U_g^\vee = U_{g^{-1}}$ .

The operation fidelity  $F$  evaluates on average how much the state after the measurement resembles the original one, in terms of the squared modulus of the scalar product. Hence, for a measurement of an unknown spin coherent state, one has

$$F = \int dg \sum_r \langle -j|U_g^\vee \circ \mathcal{E}_r \circ U_g[|-j\rangle\langle -j|] |-j\rangle. \quad (6)$$

By adopting a guess function  $f$ , for each measurement outcome  $r$  one guesses a spin coherent state  $|\psi_{f(r)}\rangle$ , and the

corresponding average estimation fidelity is given by

$$G = \int dg \sum_r \text{Tr}\{\mathcal{E}_r \circ U_g[|-j\rangle\langle -j|]\} \times \langle \psi_{f(r)}|U_g[|-j\rangle\langle -j|]|\psi_{f(r)}\rangle. \quad (7)$$

We are interested in the optimal tradeoff between  $F$  and  $G$  and, without loss of generality, we can restrict our attention to *covariant* instruments that satisfy

$$U_g^\vee \circ \mathcal{E}_h \circ U_g = \mathcal{E}_{g^{-1}h}. \quad (8)$$

In fact, for any instrument  $\{\mathcal{E}_r\}$  and guess function  $f$ , it turns out that the covariant instrument

$$\tilde{\mathcal{E}}_h = \sum_r U_h \circ U_{f(r)}^\vee \circ \mathcal{E}_r \circ U_{f(r)} \circ U_h^\vee \quad (9)$$

with continuous outcome  $h \in SU(d)$ , along with the guess  $|\psi_h\rangle$ , provides the same values of  $F$  and  $G$  as the original instrument  $\{\mathcal{E}_r\}$ . Moreover, for covariant instruments the optimal guess function automatically turns out to be the identity function.

It is useful now to consider the Jamiolkowski representation [31,32] that gives a one-to-one correspondence between a CP map  $\mathcal{E}$  from  $\mathcal{H}_{\text{in}}$  to  $\mathcal{H}_{\text{out}}$  and a positive operator  $R$  on  $\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}$  through the equations

$$\mathcal{E}(\rho) = \text{Tr}_{\text{in}}[(I_{\text{out}} \otimes \rho^\tau)R],$$

$$R = (\mathcal{E} \otimes I_{\text{in}})|\Phi\rangle\langle\Phi|, \quad (10)$$

where  $|\Phi\rangle$  represents the (unnormalized) maximally entangled state of  $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{in}}$  and  $\tau$  denotes the transposition on the fixed basis. When  $\mathcal{E}$  is trace preserving, correspondingly one has  $\text{Tr}_{\text{out}}[R] = I_{\text{in}}$ . For covariant instruments  $\mathcal{E}_g$  as in Eq. (8), the operator  $R_g$  acts on  $\mathcal{H}^{\otimes 2}$  and has the form

$$R_g = U_g \otimes U_g^* R_0 U_g^\dagger \otimes U_g^\tau \quad (11)$$

(where  $*$  denotes complex conjugation) with  $R_0 \geq 0$ , and the trace-preserving condition

$$\int dg \text{Tr}_2[R_g] = I. \quad (12)$$

From Eq. (11) and the identity (Schur's lemma for irreducible group representations [33])

$$\int dg U_g X U_g^\dagger = \frac{\text{Tr}[X]}{\text{Tr}[I]} I, \quad (13)$$

it follows that condition (12) is equivalent to

$$\text{Tr}[R_0] = 2j + 1. \quad (14)$$

### IV. OPTIMAL INFORMATION-DISTURBANCE TRADEOFF

For covariant instruments, the expressions of the fidelities  $F$  and  $G$  in Eqs. (6) and (7) can be rewritten as follows:

$$\begin{aligned}
 F &= \int dg \int dh \langle -j | \mathcal{U}_g^\dagger \circ \mathcal{E}_h \circ \mathcal{U}_g [|-j\rangle\langle -j|] - j \rangle \\
 &= \int dg \langle -j | \mathcal{E}_g [|-j\rangle\langle -j|] - j \rangle, \quad (15)
 \end{aligned}$$

$$\begin{aligned}
 G &= \int dg \int dh \text{Tr} \{ \mathcal{E}_h \circ \mathcal{U}_g [|-j\rangle\langle -j|] \} \\
 &\quad \times \langle -j | \mathcal{U}_h^\dagger \circ \mathcal{U}_g [|-j\rangle\langle -j|] - j \rangle \\
 &= \int dg \text{Tr} \{ \mathcal{E}_g [|-j\rangle\langle -j|] \} \langle -j | \mathcal{U}_g [|-j\rangle\langle -j|] - j \rangle, \quad (16)
 \end{aligned}$$

where the covariance property (8) and the invariance of the Haar measure have been used. Moreover, using the isomorphism (10), we can write  $F$  and  $G$  as  $F = \text{Tr}[R_F R_0]$  and  $G = \text{Tr}[R_G R_0]$ , where  $R_F$  and  $R_G$  are the following positive operators:

$$R_F = \int dg U_g \otimes U_g^* |-j\rangle\langle -j|^{\otimes 2} U_g^\dagger \otimes U_g^\tau, \quad (17)$$

$$\begin{aligned}
 R_G &= \int dg | \langle -j | U_g | -j \rangle |^2 I \otimes (U_g^* |-j\rangle\langle -j| U_g^\tau) \\
 &= I \otimes \text{Tr}_1 [ (|-j\rangle\langle -j| \otimes I) R_F ]. \quad (18)
 \end{aligned}$$

Using Schur's lemma for reducible group representations [33], one can evaluate the group integral in Eq. (17) from the identity

$$\begin{aligned}
 \int dg U_g \otimes U_g^* Y U_g^\dagger \otimes U_g^\tau &= \left( \int dg U_g \otimes U_g Y^\theta U_g^\dagger \otimes U_g^\tau \right)^\theta \\
 &= \sum_{l=0}^{2j} \text{Tr}[Y^\theta P_l] \frac{P_l^\theta}{\text{Tr} P_l}, \quad (19)
 \end{aligned}$$

where  $\theta$  denotes the partial transpose on the second Hilbert space and  $P_l$  represents the projector on the subspace of  $\mathcal{H} \otimes \mathcal{H}$  with total spin  $l$ . Then, one has

$$R_F = \frac{1}{4j+1} P_{2j}^\theta, \quad (20)$$

$$R_G = \frac{1}{4j+1} I \otimes \text{Tr}_1 [ (|-j\rangle\langle -j| \otimes I) P_{2j} ]. \quad (21)$$

The optimal tradeoff between  $F$  and  $G$  can be found by looking for a positive operator  $R_0$  that satisfies the trace-preserving condition (14) and maximizes a convex combination

$$pG + (1-p)F = \text{Tr} \{ [pR_G + (1-p)R_F] R_0 \}, \quad (22)$$

where  $p \in [0, 1]$  controls the tradeoff between the quality of the state estimation and the quality of the output replica of the state. Then,  $R_0$  will provide a covariant instrument that achieves the optimal tradeoff. In fact, we are interested in maximizing the operation fidelity  $F = \text{Tr}[R_F R_0]$ , for a fixed value of the estimation fidelity  $G = \text{Tr}[R_G R_0]$ . This is equivalent

to maximizing the convex combination (22). Indeed, suppose that, for a given value of  $p$ , we find  $R_0$  that maximizes Eq. (22). It is clear that for  $G = \text{Tr}[R_G R_0]$  this map yields maximum possible  $F$ , because any higher  $F$  would increase Eq. (22).

It turns out that for any  $p$  the eigenvector corresponding to the maximum eigenvalue of  $C(p) \equiv pR_G + (1-p)R_F$  is non-degenerate and of the form [34,35]

$$|\chi\rangle = \sum_{n=-j}^j a_n |n\rangle |n\rangle, \quad (23)$$

with suitable positive  $\{a_n\}$ . Upon taking  $R_0$  proportional to  $|\chi\rangle\langle\chi|$  and satisfying Eq. (14), the corresponding covariant instrument will then be optimal.

Notice that one has

$$F_{\min} \equiv \frac{2j+1}{4j+1} \leq F \leq 1, \quad (24)$$

$$G_{\min} \equiv \frac{1}{2j+1} \leq G \leq \frac{2j+1}{4j+1} \equiv G_{\max}, \quad (25)$$

where  $G_{\max}$  is the optimal estimation fidelity with corresponding operation fidelity  $F_{\min}$ , and  $G_{\min}$  corresponds to the value of  $G$  for a random guess of the unknown state. The values  $G_{\max}$ ,  $F_{\min}$  for the optimal estimation are obtained for  $R_0 = (2j+1) |-j\rangle\langle -j|^{\otimes 2}$ , corresponding to a quantum measurement described by spin coherent state POVM, i.e.,

$$\mathcal{E}_g(\rho) = (2j+1) |\psi_g\rangle\langle\psi_g| \rho |\psi_g\rangle\langle\psi_g|. \quad (26)$$

On the other hand, the values  $F=1$  and  $G=G_{\min}$  are obtained for  $R_0 = (2j+1) \sum_{n,m=-j}^j |n, n\rangle\langle m, m|$ , which corresponds to the identity operation.

Once one recognizes that the eigenvector  $|\chi\rangle$  is of the form as in Eq. (23), the optimization problem can be rewritten as

$$F = \frac{1}{4j+1} \max_{\{a_n\}} \sum_{n,m=-j}^j a_n a_m c_{n,m}^2, \quad (27)$$

with the constraints

$$\frac{1}{4j+1} \sum_{n=-j}^j a_n^2 c_{n,-j}^2 = G, \quad \sum_{n=-j}^j a_n^2 = 2j+1, \quad (28)$$

where  $c_{n,m}$  denotes Clebsch-Gordan coefficients  $c_{n,m} = \langle j, n; j, m | (2j, n+m) \rangle$ . Notice that from the property

$$\langle j_1, n; j_2, m | j, n+m \rangle = (-1)^{j_1+j_2-j} \langle j_2, m; j_1, n | j, n+m \rangle, \quad (29)$$

it follows that the matrix  $(c_{n,m})$  is symmetric. One can solve numerically such a constrained maximization, thus obtaining the optimal tradeoff between the operation and the estimation fidelities, along with the corresponding optimal measurement instrument

$$\mathcal{E}_g(\rho) = U_g \xi U_g^\dagger \rho U_g \xi U_g^\dagger, \quad (30)$$

with  $\xi = \sum_{n=-j}^j a_n |n\rangle\langle n|$ .

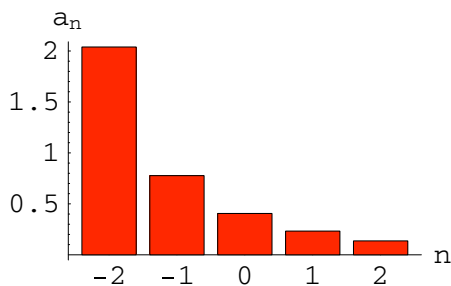


FIG. 1. (Color online) Optimal coefficients  $\{a_n\}$  that provide through Eq. (30) the minimum-disturbing measurement for the estimation of a spin coherent state with  $j=2$  and fixed estimation fidelity  $G=1/2$ , for which the maximum value of the operation fidelity  $F \approx 0.795$  is achieved.

For example, in the histogram of Fig. 1, we plot the optimal coefficients  $\{a_n\}$  for the minimum-disturbing measurement of a spin coherent state with  $j=2$  and fixed estimation fidelity  $G=1/2$ , for which the maximum value of the operation fidelity  $F \approx 0.795$  is achieved.

We can introduce two normalized quantities that can be interpreted as the average information  $I$  retrieved from the quantum measurement and the average disturbance  $D$  affecting the original quantum state as follows:

$$I = \frac{G - G_{\min}}{G_{\max} - G_{\min}} \quad (31)$$

and

$$D = \frac{1 - F}{1 - F_{\min}}. \quad (32)$$

Clearly, one has  $0 \leq I \leq 1$  and  $0 \leq D \leq 1$ .

In Fig. 2 (solid line) we plot the optimal information-disturbance tradeoff, for  $j=1$ . The curve  $D(I)$  represents a lower bound for the disturbance of any measurement instrument that gathers information  $I$ . The bound is achieved by a covariant instrument as in Eq. (30). The optimal tradeoff  $D(I)$  depends very slightly on the value of the spin  $j$ . In fact,

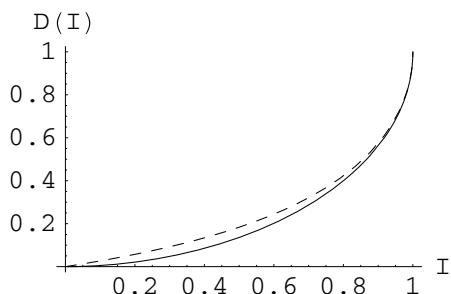


FIG. 2. Optimal information-disturbance tradeoff in estimating an unknown spin coherent state for  $j=1$  (solid line), and a harmonic oscillator coherent state (dotted line), where  $I$  and  $D$  are defined through Eqs. (31) and (32) in terms of the estimation and operation fidelities  $G$  and  $F$ , respectively. For a given value of the retrieved information  $I$ , the curve  $D(I)$  is a lower bound for the disturbance of any measurement instrument.

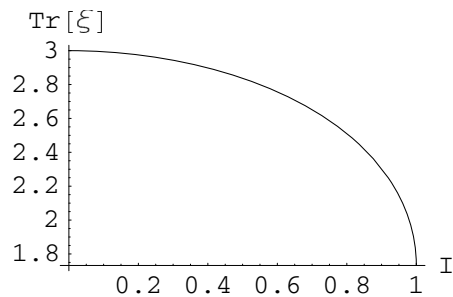


FIG. 3. The quantity  $\text{Tr}[\xi]$  for the measurement instrument of Eq. (30) that achieves the optimal information-disturbance tradeoff versus the information  $I$ , for spin  $j=1$ .

it is known that, for  $j \rightarrow \infty$ , spin coherent states approach the standard coherent states of the harmonic oscillator [28]. From Eq. 5 of Ref. [17], for harmonic oscillator coherent states one can obtain the following expression for the optimal information-disturbance tradeoff:

$$D(I) = 1 - \sqrt{\frac{2(1-I)}{2-I}}, \quad (33)$$

which has been plotted in the dotted line in Fig. 2.

We can consider the quantity  $\text{Tr}[\xi] = \sum_{n=-j}^j a_n$  as a global quantity that characterizes the measurement instrument of Eq. (30) achieving the optimal tradeoff. In fact, using Jensen's inequality, one has

$$\sqrt{2j+1} \leq \text{Tr}[\xi] \leq 2j+1, \quad (34)$$

where the lower and upper bound correspond to the optimal estimation map (26) and the identity map (with neither information nor disturbance), respectively. Notice that  $\text{Tr}[\xi]$  is related to the projection of the optimal bipartite vector  $R_0$  on the maximally entangled vector  $|\Phi\rangle = \sum_{n=-j}^j |n\rangle|n\rangle$  by the relation  $\text{Tr}[\xi] = \sqrt{\langle \Phi | R_0 | \Phi \rangle}$ . In Fig. 3, we plot the value of  $\text{Tr}[\xi]$  for the minimum-disturbing measurement versus the information, for spin  $j=1$ .

## V. CONCLUSIONS

In conclusion, we have shown how to derive the optimal tradeoff between the quality of estimation of an unknown spin coherent state and the degree the initial state has to be changed by this operation. The optimal tradeoff can be achieved by a covariant measurement instrument as in Eq. (30). By suitable normalization of the estimation and operation fidelities, the optimal tradeoff is shown to be almost independent of the value of the spin  $j$ .

In the case of estimation of an unknown pure state [7] or maximally entangled state [18], the minimum-disturbing measurement is simply the ‘‘coherent superposition’’ of the measurement instrument for optimal estimation and the identity map; namely, the Kraus operators achieving the optimal information-disturbance tradeoff are just the sum of those corresponding to maximum information extraction and minimum disturbance. For a spin coherent state, the solution is more complex. This is due to the fact that the derivation of

the tradeoff in a covariant estimation problem involves the evaluation of group integrals as in  $R_F$  and  $R_G$ . Generally, such integrals give a sum of  $N$  operators, where  $N$  is the number of invariant subspaces of the representation  $U_g \otimes U_g^*$ , and the optimal Kraus operators are the sum of  $N$  corresponding terms. For pure states or maximally entangled states,  $U_g \in \text{SU}(d)$  in dimension  $d$  and  $N=2$  (the symmetric and antisymmetric subspaces). For spin coherent states,  $U_g$  is a unitary irreducible representation of  $\text{SU}(2)$  in dimension  $2j+1$ , and the invariant subspaces of  $U_g \otimes U_g^*$  are  $N=2j+1$ .

Correspondingly, the Kraus operators of the minimum-disturbing measurement are given by a sum of  $2j+1$  operators.

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- [1] W. Heisenberg, *Z. Phys.* **43**, 172 (1927).  
 [2] K. Wódkiewicz, *Phys. Lett. A* **124**, 207 (1987).  
 [3] M. O. Scully, B.-G. Englert, and H. Walther, *Nature (London)* **351**, 111 (1991).  
 [4] S. Stenholm, *Ann. Phys. (N.Y.)* **218**, 233 (1992).  
 [5] B.-G. Englert, *Phys. Rev. Lett.* **77**, 2154 (1996).  
 [6] C. A. Fuchs and A. Peres, *Phys. Rev. A* **53**, 2038 (1996).  
 [7] K. Banaszek, *Phys. Rev. Lett.* **86**, 1366 (2001).  
 [8] C. A. Fuchs and K. Jacobs, *Phys. Rev. A* **63**, 062305 (2001).  
 [9] K. Banaszek and I. Devetak, *Phys. Rev. A* **64**, 052307 (2001).  
 [10] H. Barnum, e-print quant-ph/0205155.  
 [11] G. M. D'Ariano, *Fortschr. Phys.* **51**, 318 (2003).  
 [12] M. Ozawa, *Ann. Phys. (N.Y.)* **311**, 350 (2004).  
 [13] L. Mišta, Jr., J. Fiurášek, and R. Filip, *Phys. Rev. A* **72**, 012311 (2005).  
 [14] L. Mišta, Jr. and J. Fiurášek, *Phys. Rev. A* **74**, 022316 (2005).  
 [15] F. Sciarrino, M. Ricci, F. De Martini, R. Filip, and L. Mišta, Jr., *Phys. Rev. Lett.* **96**, 020408 (2006).  
 [16] L. Maccone, *Phys. Rev. A* **73**, 042307 (2006).  
 [17] U. L. Andersen, M. Sabuncu, R. Filip, and G. Leuchs, *Phys. Rev. Lett.* **96**, 020409 (2006).  
 [18] M. F. Sacchi, *Phys. Rev. Lett.* **96**, 220502 (2006).  
 [19] M. G. Genoni and M. G. A. Paris, *Phys. Rev. A* **74**, 012301 (2006).  
 [20] F. Buscemi and M. F. Sacchi, *Phys. Rev. A* **74**, 052320 (2006).  
 [21] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India* (IEEE, New York, 1984), p. 175; C. H. Bennett, *Phys. Rev. Lett.* **68**, 3121 (1992); N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Rev. Mod. Phys.* **74**, 145 (2002).  
 [22] A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991); A. K. Ekert, *Nature (London)* **358**, 14 (1992).  
 [23] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).  
 [24] S. Massar and S. Popescu, *Phys. Rev. Lett.* **74**, 1259 (1995); R. Derka, V. Buzek, and A. K. Ekert, *ibid.* **80**, 1571 (1998); J. I. Latorre, P. Pascual, and R. Tarrach, *ibid.* **81**, 1351 (1998); G. Vidal, J. I. Latorre, P. Pascual, and R. Tarrach, *Phys. Rev. A* **60**, 126 (1999); A. Acín, J. I. Latorre, and P. Pascual, *ibid.* **61**, 022113 (2000); G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *ibid.* **70**, 062105 (2004); G. Chiribella, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *Phys. Rev. Lett.* **93**, 180503 (2004); G. Chiribella, G. M. D'Ariano, and M. F. Sacchi, *Phys. Rev. A* **72**, 042338 (2005);  
 [25] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, *Phys. Rev. A* **70**, 032307 (2004).  
 [26] G. Chiribella, L. Maccone, and P. Perinotti, e-print quant-ph/0608042.  
 [27] A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).  
 [28] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, *Phys. Rev. A* **6**, 2211 (1972).  
 [29] More precisely, a spin coherent state corresponds to a point of the two-dimensional sphere  $S^2 = \text{SU}(2)/\text{U}(1)$ , where the quotient group  $\text{SU}(2)/\text{U}(1)$  takes into account the stability group  $\text{U}(1)$  corresponding to the rotations around the  $J_z$  axis, which leave the state  $| -j \rangle$  invariant, apart from an irrelevant phase factor.  
 [30] E. B. Davies and J. T. Lewis, *Commun. Math. Phys.* **17**, 239 (1970); K. Kraus, *States, Effects, and Operations* (Springer-Verlag, Berlin, 1983); M. Ozawa, *J. Math. Phys.* **25**, 79 (1984).  
 [31] A. Jamiolkowski, *Rep. Math. Phys.* **3**, 275 (1972).  
 [32] M. F. Sacchi, *Phys. Rev. A* **63**, 054104 (2001).  
 [33] D. P. Zhelobenko, *Compact Lie Groups and Their Representations* (American Mathematical Society, Providence, RI, 1973).  
 [34] Such a result can be checked, for example, by means of the power method [35].  
 [35] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods* (Dover, New York, 1994).