Minimum Bosonic Channel Output Entropies

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Abstract. Coherent-state inputs are conjectured to minimize the von Neumann entropies at the outputs of two Bosonic channels with thermal noise. Evidence in support of this conjecture is provided, including the fact that coherent-state inputs minimize the integer-order Rényi entropy and the Wehrl entropy at the outputs of these channels. A stronger conjecture—that output states resulting from coherent-state inputs majorize the output states from other inputs—is also discussed.

INTRODUCTION

A quantum channel can be characterized by a trace-preserving completely-positive (TPCP) map on the Hilbert space of the information carrier. In general, this evolution is not unitary, so that a pure state loses some coherence in its transit through the channel. Various measures of a channel's ability to preserve the coherence of its input state have been introduced. We shall focus on three output entropy measures: the von Neumann entropy, the integer-order Rényi entropies, and the Wehrl entropy. Our study will be restricted to Bosonic channels in which the electromagnetic field, used as the information carrier, interacts with a thermal-like noise source. We conjecture that the minimum von Neumann entropy at the channel output occurs when the input is a coherent state. Although we have yet to prove this conjecture, we shall present a physical rationale and mathematical evidence in support of its validity. The latter includes upper and lower bounds that are asymptotically tight in the limits of low and high noise. We also show that the integer-order Rényi entropies and the Wehrl entropy at the channel output are minimized when the channel's input is a coherent state. Finally, we will describe preliminary results in support of a stronger conjecture, viz., that the output state resulting from a coherent-state input majorizes the output states from all other inputs.

CHANNEL MODELS

We will consider two Bosonic channels with isotropic Gaussian noise—the thermal-noise channel and the classical-noise channel—and limit ourselves to single-mode propagation. The channel input is thus an electromagnetic-field mode with annihilation operator $\hat{a}$, and its output is another field mode with annihilation operator $\hat{a}'$.

The TPCP map $\mathscr{E}_\eta^N(\cdot)$ for the thermal-noise channel can be derived from the beam splitter relation $\hat{a}' = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{b}$, in which the annihilation operator $\hat{b}$ is associated with an environmental (noise) mode, and $0 < \eta \leq 1$ is the channel transmissivity. The $\hat{b}$ mode is in an isotropic-Gaussian mixture of coherent states with average photon number $N > 0$, viz., its density operator is the thermal state,

$$\hat{\rho}_b = \int \mathrm{d}^2\mu \, \frac{\exp(-|\mu|^2/N)}{\pi N} |\mu\rangle\langle\mu|. \tag{1}$$

The classical-noise channel can be viewed as the limiting form of a thermal-noise channel in which $\eta \to 1$ with $(1-\eta)N \to M > 0$. Its TPCP map can be shown to be

$$\hat{\rho}' = \mathscr{N}_M(\hat{\rho}) \equiv \int \mathrm{d}^2\mu \, \frac{\exp(-|\mu|^2/M)}{\pi M} \hat{D}(\mu)\hat{\rho}\hat{D}^\dagger(\mu), \tag{2}$$

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where $\hat{D}(\mu)$ is the displacement operator, i.e., $\hat{a}' = \hat{a} + m$ where m is a zero-mean, isotropic Gaussian noise with variance given by $\langle |m|^2 \rangle = M$. This channel is especially interesting, because it is the quantum version of the additive white Gaussian noise channel.

CHANNEL CAPACITY AND MINIMUM OUTPUT ENTROPY

A principal motivation for our quest for the minimum output entropies of the preceding channels comes from our recent work on the classical capacity of Bosonic channels with isotropic Gaussian noise [1, 2, 3]. In particular we have shown that the capacity of the thermal-noise channel satisfies,

$$g(\eta\bar{N} + (1-\eta)N) - g((1-\eta)N) \leq C \leq g(\eta\bar{N} + (1-\eta)N) - \inf_n [\mathbb{S}((\mathscr{E}_\eta^N)^{\otimes n})/n], \tag{3}$$

and that of the classical-noise channel obeys,

$$g(\bar{N} + M) - g(M) \leq C \leq g(\bar{N} + M) - \inf_n [\mathbb{S}(\mathscr{N}_M^{\otimes n})/n], \tag{4}$$

where $g(x) \equiv (x+1)\ln(x+1) - x\ln(x)$ is the Shannon entropy (in nats) of the Bose-Einstein distribution, $\bar{N}$ is the average photon number constraint on the transmitter, and

$$\mathbb{S}((\mathscr{E}_\eta^N)^{\otimes n}) \equiv \min_{\hat{\rho}} [S((\mathscr{E}_\eta^N)^{\otimes n}(\hat{\rho}))] \quad \text{and} \quad \mathbb{S}(\mathscr{N}_M^{\otimes n}) \equiv \min_{\hat{\rho}} [S(\mathscr{N}_M^{\otimes n}(\hat{\rho}))] \tag{5}$$

are the minimum von Neumann entropies for n uses of the thermal-noise and classical-noise channels. The lower bounds in Eqs. (3) and (4) are achieved by single-use random coding over coherent states with an isotropic Gaussian prior, for which $S(\mathscr{E}_\eta^N(|\alpha\rangle\langle\alpha|)) = g((1-\eta)N)$ and $S(\mathscr{N}_M(|\alpha\rangle\langle\alpha|)) = g(M)$. We conjecture that coherent-state inputs achieve $\mathbb{S}(\mathscr{E}_\eta^{N\otimes n})$ and $\mathbb{S}(\mathscr{N}_M^{\otimes n})$, implying that the channel capacities are equal to the lower bounds from (3) and (4).

Support for the Minimum von Neumann Entropy Conjecture

The von Neumann entropy is invariant to displacement $\hat{D}(\cdot)$, so we can focus our attention on showing that the vacuum state minimizes the output entropies for our isotropic Gaussian noise channels. Minimum output entropy corresponds to maximum output-state coherence. Physically, it does not seem possible that any non-vacuum state could extract coherence from the thermal-noise reservoirs associated with the thermal-noise and classical-noise channels so as to reduce their output entropies below the levels that are achieved by zero-photon (vacuum-state) inputs. Mathematically, it is easy to show that vacuum-state inputs yield *local* output-entropy minima for these channels; the argument for the classical-noise channel is as follows. Consider an input state $\hat{\sigma} = (1-t)\hat{\rho}_0 + t\hat{\rho}$, for $0 \leq t \leq 1$,that is a mixture of the multi-mode (n channel use) vacuum state $\hat{\rho}_0 = |\mathbf{0}\rangle\langle\mathbf{0}|$ and an arbitrary input state $\hat{\rho}$. We have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} S(\mathscr{N}_M^{\otimes n}(\hat{\sigma}))\Big|_{t=0} &= \mathrm{tr}\big[(\mathscr{N}_M^{\otimes n}(\hat{\rho}_0) - \mathscr{N}_M^{\otimes n}(\hat{\rho}))\ln(\mathscr{N}_M^{\otimes n}(\hat{\rho}_0))\big] \\ = \sum_{k=1}^{n} \mathrm{tr}[(\mathscr{N}_M^{\otimes n}(\hat{\rho}_0) - \mathscr{N}_M^{\otimes n}(\hat{\rho}))\hat{a}_k'^\dagger \hat{a}_k'] \ln\left(\frac{M}{M+1}\right) &= -\sum_{k=1}^{n} \mathrm{tr}(\hat{\rho}\hat{a}_k^\dagger \hat{a}_k) \ln\left(\frac{M}{M+1}\right) \geq 0, \end{aligned} \tag{6}$$

where the second equality follows from $\mathscr{N}_M(\hat{\rho}_0)$ being the product state—over n channel uses with input and output annihilation operators $\{\hat{a}_k, \hat{a}_k' : 1 \leq k \leq n\}$—of Bose-Einstein number-state mixtures, and the inequality demonstrates that $\hat{\sigma}|_{t=0} = \hat{\rho}_0$ is a local minimum. A similar local-minimum proof applies to the thermal-noise channel.

As yet we have been unable to complete the proof of our minimum von Neumann entropy conjecture. We have, however, obtained a suite of lower bounds on the minimum output entropy. These are illustrated in Fig. 1, for single uses ($n = 1$) of the thermal-noise and classical-noise channels. Our bounds provide fairly tight constraints on any possible gap between the minimum output entropies for these channels and their associated coherent-state upper bounds. Indeed these results imply that coherent-state inputs minimize the single-use output entropy at both low and high noise levels, as seen from the $\eta \to 0$ and $\eta \to 1$ limits for the thermal-noise channel in Fig. 1(a) and the $M \to 0$ and $M \gg 1$ limits for the classical-noise channel in Fig. 1(b). Derivations of these $n = 1$ bounds appear in [4]. Here we

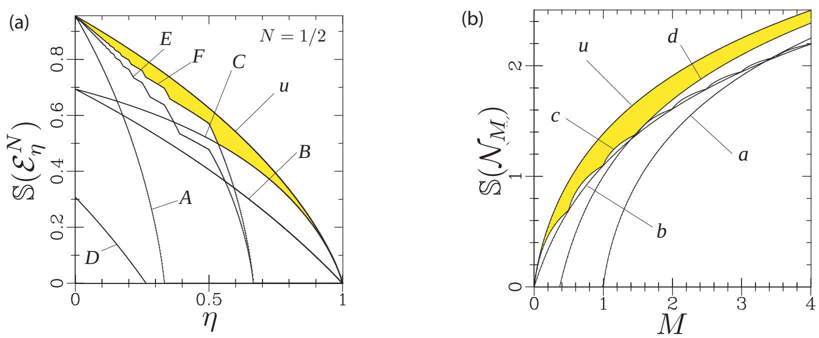

FIGURE 1. Bounds on the minimum output entropy for single-use Bosonic channels with isotropic Gaussian noise: (a) coherent-state upper bound $u = g((1-\eta)N)$ and lower bounds A, B, C, D, E, F on the minimum output entropy $\mathbb{S}(\mathscr{E}_\eta^N)$ of the thermal-noise channel plotted versus channel transmissivity η for the case $N = 1/2$; and (b) coherent-state upper bound $u = g(M)$ and lower bounds a, b, c, d on the minimum output entropy $\mathbb{S}(\mathscr{N}_M)$ of the classical-noise channel plotted versus the average photon number of the noise M. For details on the lower bounds see [4].

will content ourselves with deriving bound a for n uses of the classical-noise channel, viz., $S(\mathscr{N}_M^{\otimes n}(\hat\rho)) \geq ng(M-1)$, for $M > 1$. First, we represent $\hat\rho$ in terms of its Husimi function,

$$\hat\rho = \int \mathrm{d}^{2n}\alpha\, Q(\alpha)\hat\sigma(\alpha), \quad \text{where} \quad \hat\sigma \equiv \int \frac{\mathrm{d}^{2n}\lambda}{\pi^n} D(\lambda) e^{\lambda^\dagger \alpha - \lambda^T \alpha^* - \lambda^\dagger \lambda/2}. \tag{7}$$

Applying the TPCP map $\mathscr{N}_M^{\otimes n}(\cdot)$ then yields,

$$\mathscr{N}_M^{\otimes n}(\hat\rho) = \int \mathrm{d}^{2n}\alpha\, Q(\alpha) \mathscr{N}_M^{\otimes n}(\hat\sigma(\alpha)) \tag{8}$$

$$= \int \mathrm{d}^{2n}\alpha\, Q(\alpha) \left[D(\alpha) \int \frac{\mathrm{d}^{2n}\mu}{\pi^n} \frac{e^{-\mu^\dagger \mu/(M-1)}}{M-1} |\mu\rangle\langle\mu| D^\dagger(\alpha) \right] = D(\alpha) \left[\bigotimes_{k=1}^n \frac{1}{M} \left(\frac{M-1}{M} \right)^{\hat a_k^\dagger \hat a_k} \right] D^\dagger(\alpha), \tag{9}$$

where (9) assumes $M > 1$. The concavity of the von Neumann entropy gives us,

$$S(\mathscr{N}_M^{\otimes n}(\hat\rho)) \geq \int \mathrm{d}^{2n}\alpha\, Q(\alpha) S[\mathscr{N}_M^{\otimes n}(\hat\sigma(\alpha))], \quad \text{for} M > 1, \tag{10}$$

and $S[\mathscr{N}_M^{\otimes n}(\hat\sigma(\alpha))] = S[\mathscr{N}_M(\hat\sigma(\mathbf{0}))]$, whence

$$\mathbb{S}(\mathscr{N}_M^{\otimes n}) \geq S[\mathscr{N}_M^{\otimes n}(\hat\sigma(\mathbf{0}))] = S\left[\bigotimes_{k=1}^n \frac{1}{M} \left(\frac{M-1}{M} \right)^{\hat a_k^\dagger \hat a_k} \right] = \sum_{k=1}^n S\left[\frac{1}{M} \left(\frac{M-1}{M} \right)^{\hat a_k^\dagger \hat a_k} \right] = ng(M-1), \quad \text{for } M > 1, \tag{11}$$

which completes the proof.

Minimum Rényi and Wehrl Entropies

Additional support for our minimum entropy conjecture follows from our work on the Rényi and Wehrl output entropies of the thermal-noise and classical-noise channels. The Rényi output entropy of order z is $S_z(\hat\rho') \equiv -\ln(\mathrm{tr}(\hat\rho'^z))/(z-1)$, for $z > 1$. The von Neumann entropy of the channel output may be obtained from the Rényi entropy of that output state via $S(\hat\rho) = \lim_{z\to1} S_z(\hat\rho)$. We have shown [5] that coherent-state inputs minimize the integer-order Rényi entropy $S_k(\hat\rho')$ for $k = 2, 3, 4, \ldots,$ for single-channel uses. For the classical-noise channel this minimum is

$$\mathbb{S}_k(\mathscr{N}_M) \equiv \min_{\hat\rho} [S_k(\mathscr{N}_M(\hat\rho))] = \frac{\ln\left[(M+1)^k - M^k\right]}{k-1}. \tag{12}$$

Thus, assuming coherent states achieve

$$\mathbb{S}_z(\mathcal{N}_M) = \frac{\ln[(M+1)^z - M^z]}{z-1}, \quad \text{for } z > 1, \tag{13}$$

would then complete the minimum von Neumann entropy conjecture because Eq. (13) converges to $g(M)$ as $z \to 1$.

The Wehrl entropy is the Shannon entropy of the Husimi distribution, i.e.,

$$W(\hat{\rho}') \equiv -\int \frac{d^2\mu}{\pi} \langle\mu|\hat{\rho}'|\mu\rangle \ln(\langle\mu\hat{\rho}'|\mu\rangle), \tag{14}$$

gives the Wehrl entropy for the single-use ($n = 1$) channel output state $\hat{\rho}'$. For the thermal-noise and classical-noise channels we have shown [5] that this entropy is minimized by coherent-state inputs. Our primary interest is in minimizing the von Neumann entropy at the output of these channels, i.e., in showing that coherent-state inputs are optimum in this regard. Because Wehrl output entropy measures the phase-space uncertainty of the channel output, its minimum represents an output state of maximum coherence. Minimum von Neumann entropy at the channel output is another measure of maximum output-state coherence, so the fact that coherent-state inputs minimize the Wehrl output entropy lends credence to our von Neumann entropy conjecture.

Majorization Conjecture

Our research into minimizing the von Neumann entropy at the output of thermal-noise and classical-noise channels has suggested that an even stronger conjecture may be true, viz., that coherent-state inputs majorize the output states from other inputs. For the single-use ($n = 1$) case we have demonstrated that the vacuum state majorizes the output states from other number-state inputs [4]. Also, starting from randomly-chosen initial states we have used simulated annealing to minimize $S(\mathcal{N}_M)$ numerically; in all cases we found $\mathbb{S}(\mathcal{N}_M) \approx g(M)$ [4].

CONCLUSION

We have conjectured that coherent-state inputs minimize the von Neumann entropy at the output of two Bosonic channels with isotropic Gaussian noise: the thermal-noise channel and the classical-noise channel. We have shown that coherent states achieve *local* output-entropy minima, and we have obtained a suite of bounds that show the conjecture is asymptotically correct in the the low-noise and high-noise regimes. Additional support for our conjecture derives from related results that we have derived for the Rényi and Wehrl output entropies, and from evidence in favor of a stronger conjecture, that the outputs from coherent-state inputs majorize all other output states.

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REFERENCES

1. V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Phys. Rev. Lett. **92**, 027902 (2004).
2. V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, B. J. Yen, and H. P. Yuen, "Classical capacity of free-space optical communication," in O. Hirota, ed. *Quantum Information, Statistics, and Probability* (Rinton Press, Princeton, 2004).
3. J. H. Shapiro, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, and B. J. Yen, "Capacity of Bosonic communications," in S. M. Barnett, ed., *Proceedings of the Seventh International Conference on Quantum Communication, Measurement and Computing* (American Institute of Physics, New York, 2004).
4. V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, and J. H. Shapiro, "Minimum output entropy of Bosonic channels: a conjecture," to appear in Phys. Rev. A, e-print quant-ph/0404005.
5. V. Giovannetti, S. Lloyd, L. Maccone, J. H. Shapiro, and B. J. Yen, Phys. Rev. A **70**, 022328 (2004).