Characterizing a universal cloning machine by maximum-likelihood estimation

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We apply a general method for the estimation of completely positive maps to the one-to-two universal covariant cloning machine. The method is based on the maximum-likelihood principle, and makes use of random input states, along with random projective measurements on the output clones. The downhill simplex algorithm is applied for the maximization of the likelihood functional.

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I. INTRODUCTION

Perfect cloning of unknown quantum systems is forbidden by the laws of quantum mechanics [1]. However, universal covariant cloning has been proposed [2], and has been proved to be optimal in terms of fidelity [3,4]. Some relevant applications of cloning is eavesdropping in quantum cryptography [5], and the engineering of new kinds of joint measurements [6]. Now suppose that someone provides you with a physical device, telling you that it works as a universal cloning machine. How can you check, experimentally, this statement? You could perform different kinds of measurement on the output of the device, along with different preparations of the input states. Then, you could compare the correlations you have inferred from the outcomes with your theoretical predictions. However, the most reliable and general way to proceed could be entirely reconstructing the completely positive (CP) map that univocally characterizes a physical device.

Recently, a general method to solve this relevant issue has been proposed in Ref. [7]. The method is based on the maximum-likelihood (ML) principle, applied to the data obtained by random measurements on the output of the device. The ML method has been used in the context of phase measurement [8], and to estimate the density matrix [9], some parameters of interest in quantum optics [10], and the CP maps of quantum communication channels [7]. For the problem of CP map reconstruction [7], the constraints inherent to the maximization can be imposed by exploiting the isomorphism between the CP map from Hilbert spaces $\mathcal{H}$ to $\mathcal{K}$ and *non-negative* operators in the tensor-product space $\mathcal{K}\otimes\mathcal{H}$ [11–13]. Such isomorphism has already been useful for the study of positive maps [13] and to address the problem of separability of CP maps [14].

In this paper, we apply the general method of Ref. [7] to the one-to-two universal covariant cloning machine for spin-1/2 systems. In Sec. II we briefly review the ML principle and its use in measuring quantum devices. Section III presents the calculations needed to explicitly construct the likelihood functional to be maximized. Some numerical results obtained through a Monte Carlo simulation and the simplex searching algorithm are then shown to confirm the reliability of the method. Section IV is devoted to conclusions.

II. MEASURING QUANTUM DEVICES

The maximum-likelihood principle states that the best estimation of unknown parameters is given by the values that are most likely to produce the data one experimenter has observed. Hence, this principle involves the maximization of a function of the unknown parameters that is given by the theoretical probability of getting the collected data.

Consider a sequence of K independent measurements on the output of a physical device acting on quantum states. Each measurement is described by the element $F_l(x_l)$ of a POVM, where x_l denotes the outcome at the lth measurement, and $l=1,2,\dots,K$. Let us denote by ρ_l the state at the input at the lth run. The probability of getting the string of outcomes $\vec{x}=\{x_1,x_2,\dots,x_K\}$ is given by

$$p(\vec{x})=\Pi_{l=1}^{K}\,\mathrm{Tr}\,[\mathcal{E}(\rho_l)F_l(x_l)]. \tag{1}$$

The best estimate of the map $\mathcal{E}$ maximizes the logarithm of Eq. (1)

$$\mathcal{L}(\mathcal{E})=\sum_{l=1}^{K}\log\mathrm{Tr}[\mathcal{E}(\rho_l)F_l(x_l)] \tag{2}$$

over the set of completely positive maps. The likelihood function $\mathcal{L}(\mathcal{E})$ is concave, and in the present case it is defined on the convex set of CP maps. Its maximum is achieved by a single CP map if the data sample is sufficiently large, and the set of measurements is a *quorum* [15].

The constraints to be imposed in the maximization problem are the complete positivity and the trace-preserving property of the map $\mathcal{E}$. A trace-preserving CP map is a linear map from operators in Hilbert space $\mathcal{H}$ to operators in $\mathcal{K}$ that can be written in the Kraus form [16]

$$\mathcal{E}(\rho)=\sum_k A_k\,\rho\,A_k^\dagger, \tag{3}$$

where

$$\sum_k A_k^\dagger A_k=\mathbb{1}_{\mathcal{H}}. \tag{4}$$

Let $\dim(\mathcal{H})=N$ and $\dim(\mathcal{K})=M$, and consider an orthonormal basis $\{V_i\}$ for the space of linear operators on $\mathcal{H}$, namely

$$\mathrm{Tr}[V_i^\dagger V_j]=\delta_{ij}, \tag{5}$$

and for any operator O

$$O=\sum_{i=1}^{N^2} \mathrm{Tr}[V_i^\dagger O]\, V_i . \tag{6}$$

Upon defining the operator [11,12]

$$S=\sum_{i=1}^{N^2} \mathcal{E}(V_i)\otimes V_i^* , \tag{7}$$

where * denotes complex conjugation, one can write for linearity

$$\mathcal{E}(\rho)=\mathrm{Tr}_{\mathcal{H}}[(\mathbb{1}_{\mathcal{K}}\otimes\rho^T)\, S], \tag{8}$$

where T is the transposition. Notice that [17]

$$\sum_{i=1}^{N^2} V_i\otimes V_i^* = |\Psi\rangle\langle\Psi|, \tag{9}$$

where $|\Psi\rangle$ is given by the (unnormalized) maximally entangled state

$$|\Psi\rangle=\sum_{n=1}^{N} |n\rangle\otimes|n\rangle. \tag{10}$$

Hence, one has also

$$S=\mathcal{E}\otimes\mathbb{1}(|\Psi\rangle\langle\Psi|). \tag{11}$$

Equations (7) and (8) establish an isomorphism between linear maps from $\mathcal{H}$ to $\mathcal{K}$ and linear operators on the tensor-product space $\mathcal{K}\otimes\mathcal{H}$. Complete positivity and trace-preserving property of $\mathcal{E}$ imply [18]

$$S\geqslant 0 \quad \text{and} \quad \mathrm{Tr}_{\mathcal{K}}[S]=\mathbb{1}_{\mathcal{H}} . \tag{12}$$

For the construction of the likelihood function $\mathcal{L}(\mathcal{E})$ the condition $S\geqslant 0$ is crucial [7]. Actually, it allows us to write

$$S=C^\dagger C, \tag{13}$$

where C is an upper triangular matrix, with positive diagonal elements [19]. Similarly, one has for the density matrices ρ_l^T and the POVM's $F_l(x_l)$

$$\rho_l^T=R_l^\dagger R_l , \qquad F_l(x_l)=A_l^\dagger(x_l)A_l(x_l). \tag{14}$$

From Eqs. (8), (13), and (14), the likelihood functional in Eq. (2) rewrites

$$\mathcal{L}(\mathcal{E})\equiv\mathcal{L}(C)=\sum_{l=1}^{K} \ln \mathrm{Tr}\{C^\dagger C[R_l^\dagger R_l\otimes A_l^\dagger(x_l)A_l(x_l)]\}$$
$$=\sum_{l=1}^{K} \ln \sum_{n,m=1}^{NM} |\langle\langle n|C(R_l^\dagger\otimes A_l^\dagger(x_l))|m\rangle\rangle|^2, \tag{15}$$

where $\{|n\rangle\rangle\}$ denotes an orthonormal basis for $\mathcal{H}\otimes\mathcal{K}$. On one hand, the parametrization in Eq. (15) implicitly constrains the complete positivity of the map $\mathcal{E}$. On the other, the argument of the logarithm is explicitly positive, thus assuring the stability of numerical methods to evaluate $\mathcal{L}(C)$.

The trace-preserving condition is given in terms of the matrix S by $\mathrm{Tr}_{\mathcal{K}}[S]=\mathbb{1}_{\mathcal{H}}$. However, the constraint $\mathrm{Tr}[S]=N$ that follows from $\mathrm{Tr}_{\mathcal{K}}[S]=\mathbb{1}_{\mathcal{H}}$ isolates a closed convex subset of the set of positive matrices. Hence, the maximum of the concave likelihood functional still remains unique under this looser constraint, and one can check *a posteriori* that the condition $\mathrm{Tr}_{\mathcal{K}}[S]=\mathbb{1}_{\mathcal{H}}$ is fulfilled. Using the method of Lagrange multipliers, then one maximizes the effective functional

$$\tilde{\mathcal{L}}(C)=\mathcal{L}(C)-\mu\, \mathrm{Tr}[C^\dagger C], \tag{16}$$

where $\mathcal{L}(C)$ is given in Eq. (15), and the value of the multiplier μ can be obtained as follows. Writing S in terms of its eigenvectors as $S=\Sigma_i s_i^2|s_i\rangle\rangle\langle\langle s_i|$, the maximum likelihood condition $\partial\tilde{\mathcal{L}}(\mathcal{C})/\partial s_i=0$ implies

$$\sum_{l=1}^{K} \frac{\mathrm{Tr}\{[\rho_l^T\otimes F_l(x_l)]s_i\,|s_i\rangle\rangle\langle\langle s_i|\}}{\mathrm{Tr}\{[\rho_l^T\otimes F(x_l)]S\}}=\mu\, \mathrm{Tr}[s_i\,|s_i\rangle\langle\langle s_i|]. \tag{17}$$

Multiplying by s_i and summing over i gives $\mu=K/N$.

III. CHARACTERIZING THE UNIVERSAL CLONING MACHINE

We consider now the problem of estimating the CP map pertaining to the one-to-two universal covariant cloning machine. The map is given by [4]

$$\mathcal{E}(\rho)=\frac{2}{3}s_2(\rho\otimes\mathbb{1})s_2 , \tag{18}$$

where s_2 is the projection operator on the symmetric subspace, which is spanned by the set of vectors $\{|s_i\rangle\langle s_i|, i=0-2\}$, with $|s_0\rangle=|00\rangle$, $|s_1\rangle=1/\sqrt{2}(|01\rangle+|01\rangle)$, and $|s_2\rangle=|11\rangle$, where $\{|0\rangle,|1\rangle\}$ is a basis for each spin 1/2 system. Using the lexicographic ordering for the basis of the tensor-product Hilbert space [20] one has

$$s_2=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1/2 & 1/2 & 0\\ 0 & 1/2 & 1/2 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

In the following, we label with A and B,C the Hilbert spaces supporting the input state and the two output copies, respectively. One can apply Eq. (7) to obtain the corresponding matrix S. Upon using the operator basis $V_i=1/\sqrt{2}\sigma_i$, with $\sigma_0=\mathbb{1}$, and σ_i $(i=1,2,3)$ denoting the customary Pauli matrices, one has

$$S=\frac{2}{3}\frac{1}{2}\sum_{i=0}^{4}[(s_2)^{BC}(\sigma_i^B\otimes\mathbb{1}_C)(s_2)^{BC}]\otimes\sigma_i^{*A}$$
$$=\frac{1}{3}[\mathbb{1}_A\otimes(s_2)^{BC}]\left[\left(\sum_{i=0}^{4}\sigma_i^{*A}\otimes\sigma_i^B\right)\otimes\mathbb{1}_C\right][\mathbb{1}_A\otimes(s_2)^{BC}]$$
$$=\frac{2}{3}[\mathbb{1}_A\otimes(s_2)^{BC}][|\Psi\rangle_{AB\,AB}\langle\Psi|\otimes\mathbb{1}_C][\mathbb{1}_A\otimes(s_2)^{BC}]. \quad (20)$$

In the lexicographically ordered basis [20] of $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$ the matrix S writes

$$S=\frac{1}{6}\begin{pmatrix}4&0&0&0&0&2&2&0\\0&1&1&0&0&0&0&2\\0&1&1&0&0&0&0&2\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\2&0&0&0&0&1&1&0\\2&0&0&0&0&1&1&0\\0&2&2&0&0&0&0&4\end{pmatrix}. \quad (21)$$

Now we want to apply the method presented in the previous section to reconstruct the matrix S. We use random pure states at the input of the cloning machine

$$|\psi_l\rangle=\cos(\theta_l/2)|0\rangle_A+e^{i\phi_l}\sin(\theta_l/2)|1\rangle_A. \quad (22)$$

In matrix notation $\rho_l=|\psi_l\rangle\langle\psi_l|$ we write

$$\rho_l=\frac{1}{2}(\mathbb{1}_A+\vec{\sigma}^A\cdot\vec{n}_l), \quad (23)$$

with $\vec{n}_l=(\sin\theta_l\cos\phi_l,\sin\theta_l\sin\phi_l,\cos\theta_l)$. On the two output clones we perform independent projective measurements along random directions $\vec{r}_l=(\sin\alpha_l\cos\beta_l,\sin\alpha_l\sin\beta_l,\cos\alpha_l)$ and $\vec{t}_l=(\sin\gamma_l\cos\delta_l,\sin\gamma_l\sin\delta_l,\cos\gamma_l)$, then using the projector

$$F_l(a_l,b_l)=\frac{1}{4}(\mathbb{1}_B+a_l\vec{\sigma}^B\cdot\vec{r}_l)\otimes(\mathbb{1}_C+b_l\vec{\sigma}^C\cdot\vec{t}_l), \quad (24)$$

where a_l and b_l denote the outcomes one has indeed obtained (the possible values are ± 1). Defining

$$\tilde{\alpha}_l=\frac{\alpha_l}{2}+\pi\frac{a_l-1}{4}, \quad (25)$$

$$\tilde{\gamma}_l=\frac{\gamma_l}{2}+\pi\frac{a_l-1}{4}, \quad (26)$$

the Cholevsky decomposition for ρ_l^T and $F_l(a_l,b_l)$ writes as in Eqs. (14), with

$$R_l=\begin{pmatrix}\cos(\theta_l/2) & e^{i\phi_l}\sin(\theta_l/2)\\ 0 & 0\end{pmatrix} \quad (27)$$

and

$$A_l(a_l,b_l)=\begin{pmatrix}\cos\tilde{\alpha}_l & e^{-i\beta_l}\sin\tilde{\alpha}_l\\0&0\end{pmatrix}\otimes\begin{pmatrix}\cos\tilde{\gamma}_l & e^{-i\delta_l}\sin\tilde{\gamma}_l\\0&0\end{pmatrix}. \quad (28)$$

It follows that the matrix $Q_l(a_l,b_l)\equiv R_l^\dagger\otimes A_l^\dagger(a_l,b_l)$ that multiplies C in the likelihood function (15) is lower triangular with just the first column different from zero. One has explicitly

$$\begin{aligned}
[Q_l(a_l,b_l)]_{11}&=\cos(\theta_l/2)\cos\tilde{\alpha}_l\cos\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{21}&=e^{i\delta_l}\cos(\theta_l/2)\cos\tilde{\alpha}_l\sin\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{31}&=e^{i\beta_l}\cos(\theta_l/2)\sin\tilde{\alpha}_l\cos\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{41}&=e^{i(\beta_l+\delta_l)}\cos(\theta_l/2)\sin\tilde{\alpha}_l\sin\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{51}&=e^{-i\phi_l}\sin(\theta_l/2)\cos\tilde{\alpha}_l\cos\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{61}&=e^{i(\delta_l-\phi_l)}\sin(\theta_l/2)\cos\tilde{\alpha}_l\sin\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{71}&=e^{i(\beta_l-\phi_l)}\sin(\theta_l/2)\sin\tilde{\alpha}_l\cos\tilde{\gamma}_l,\\
[Q_l(a_l,b_l)]_{81}&=e^{i(\beta_l+\delta_l-\phi_l)}\sin(\theta_l/2)\sin\tilde{\alpha}_l\sin\tilde{\gamma}_l.
\end{aligned} \quad (29)$$

We now have all the ingredients to construct the likelihood function in Eq. (15), which will be a function of the 64 real parameters that specify the triangular matrix C. The problem of the maximization of $\mathcal{L}(C)$ enters the realm of programming and numerical algebra optimization, where various techniques are known [21].

In the following, we show the results of a simulation obtained by applying the method of downhill simplex [21,22] to find the maximum of the likelihood functional. This method is robust and efficient in case of a relatively small number of parameters. It has been reliably used in the reconstruction of the density matrix of radiation field and spin systems [9], and in the characterization of quantum communication channels for qubits [7].

The results are shown in Fig. 1. Pure states at the input of the cloning machine have been used, together with projective measurements over the two clones at the output. In both cases we adopted a uniform distribution on the Bloch sphere. The Monte Carlo method has been used to generate $K=10\,000$ data, by using the theoretical probability

$$p(a_l,b_l)=\frac{2}{3}\mathrm{Tr}[s_2(\rho_l\otimes\mathbb{1})s_2F_l(a_l,b_l)]. \quad (30)$$

A lengthy but straightforward calculation gives

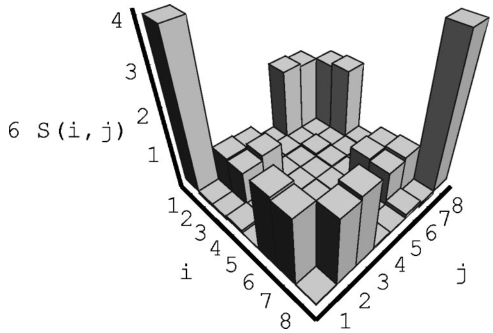

FIG. 1. Maximum-likelihood reconstruction of the CP map of the one-to-two universal covariant cloning. The picture represents the values of the (real part of the) elements of the matrix S. Random pure states at the input of the cloning machine, and projective measurements along random directions on the two clones at the output have been used, with $K=10\,000$ couple of measurements. The statistical error in the reconstruction is of the order 10^{-2}. The results compare very well with the theoretical values of Eq. (21).

$$p(a_l,b_l)=\frac{1}{4}+\frac{1}{6}(a_l\cos\alpha_l+b_l\cos\gamma_l)\cos\theta_l +\frac{1}{6}[a_l\sin\alpha_l\cos(\beta_l-\phi_l) +b_l\sin\gamma_l\cos(\delta_l-\phi_l)]\sin\theta_l +\frac{1}{12}a_l b_l[\cos\alpha_l\cos\gamma_l +\sin\alpha_l\sin\gamma_l\cos(\delta_l-\beta_l)]. \quad (31)$$

In Fig. 2 we reported the value of the average statistical error that affects the matrix elements of S versus the number K of simulated data. In accordance with the central limit theorem we find the asymptotic inverse-square-root dependence on K.

IV. CONCLUSIONS

In conclusion, we have applied a general method to reconstruct experimentally the completely positive map describing a physical device to the universal covariant cloning machine. The method, based on the maximum-likelihood principle, involves the maximization of a functional, which depends on the results of quantum measurements performed on the clones at the output. The maximization has to be made over all possible trace-preserving completely positive maps. A suitable parametrization is allowed by the isomorphism between the linear map from Hilbert spaces $\mathcal{H}$ to $\mathcal{K}$ and linear operators in $\mathcal{H}\otimes\mathcal{K}$, along with the Cholesky decomposition of positive matrices. The numerical results we showed here have been obtained by applying the method of the downhill simplex to search the maximum of the likelihood functional. In our example, a good characterization of the one-to-two universal cloning machine has been achieved, with a number of simulated data as low as 10^4. This is relevant, because some experiments are now feasible, but with low data rate or short stability time. In accordance with the central limit theorem, the statistical error of the characterization shows the inverse-square-root asymptotic dependence on the number of data.

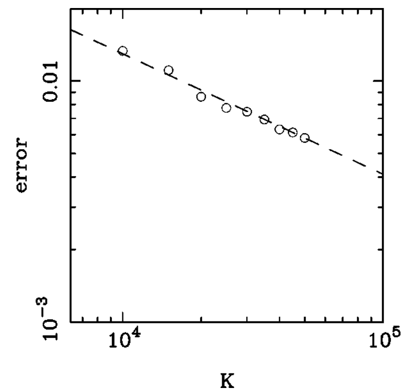

FIG. 2. Average statistical error in the characterization of the universal covariant cloning machine versus number of data K. The error affects the value of the reconstructed elements of the matrix S that is univocally related to the CP map of the cloning machine. The dotted line represents the asymptotic dependence on the inverse square root of K, in accordance with the central limit theorem.

The method is very general, can be implemented immediately in the lab, and can be adopted in many fields as quantum optics, spins, optical lattices, atoms, ion trap, etc.

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[1] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982); H. P. Yuen, Phys. Lett. A **113**, 405 (1986).
[2] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996); N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[3] D. Bruß, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, Phys. Rev. A **57**, 2368 (1998); D. Bruß, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).
[4] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
[5] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997); N. Gisin and B. Huttner, Phys. Lett. A **228**, 13 (1997).
[6] G. M. D'Ariano, C. Macchiavello, and M. F. Sacchi, J. Opt. B: Semiclassical Opt. **3**, 44 (2001); G. M. D'Ariano, F. De Martini, and M. F. Sacchi, Phys. Rev. Lett. **86**, 914 (2001); G. M. D'Ariano and M. F. Sacchi, e-print quant-ph/009104.
[7] M. F. Sacchi, Phys. Rev. A **63**, 054104 (2001).
[8] S. L. Braunstein, A. S. Lane, and C. M. Caves, Phys. Rev. Lett. **69**, 2153 (1992).

[9] K. Banaszek, G. M. D'Ariano, M. G. A. Paris, and M. F. Sacchi, Phys. Rev. A **61**, 010304(R) (2000).

[10] G. M. D'Ariano, M. G. A. Paris, and M. F. Sacchi, Phys. Rev. A **62**, 023815 (2000); G. M. D'Ariano, M. G. A. Paris, and M. F. Sacchi, quant-ph/0009081 (unpublished).

[11] J. de Pillis, Pac. J. Math. **23**, 129 (1967).

[12] A. Jamiołkowski, Rev. Mod. Phys. **3**, 275 (1972).

[13] S. Yu, Phys. Rev. A **62**, 024302 (2000).

[14] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).

[15] For the concept of *quorum*, see G. M. D'Ariano, L. Maccone, and M. G. A. Paris, Phys. Lett. A **276**, 25 (2000).

[16] See, for example, M. A. Nielsen and C. M. Caves, Phys. Rev. A **55**, 2547 (1997).

[17] G. M. D'Ariano, P. Lo Presti, and M. F. Sacchi, Phys. Lett. A **272**, 32 (2000).

[18] For a positive—but not completely positive—map the condition of positivity of the matrix S is relaxed, by only requiring positivity for tensor product of vectors, namely $\langle\phi|\otimes\langle\psi|S|\phi\rangle\otimes|\psi\rangle\geqslant 0$ (see Ref. [12]).

[19] Such decomposition is referred to as Cholesky decomposition, and is commonly used in linear programming (see, e.g., Ref. [21]). Moreover, if the matrix S is strictly positive the decomposition is unique.

[20] We mean that $|i\rangle\otimes|j\rangle$ precedes $|k\rangle\otimes|l\rangle$ if and only if either $i<k$ or $i=k$ and $j<l$.

[21] See, for example, P. G. Ciarlet, *Introduction to Numerical Linear Algebra and Optimization* (Cambridge University Press, Cambridge, 1989).

[22] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in Fortran: The Art of Scientific Computing* (Cambridge University Press, Cambridge, England, 1992).