

Nearly ideal binary communication in squeezed channels

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We analyze the effect of squeezing the channel in binary communication based on Gaussian states. We show that for coding on pure states, squeezing increases the detection probability at fixed size of the strategy, actually saturating the optimal bound already for moderate signal energy. Using Neyman-Pearson lemma for fuzzy hypothesis testing we are able to analyze also the case of mixed states, and to find the optimal amount of squeezing that can be effectively employed. It results that optimally squeezed channels are robust against signal mixing, and largely improve the strategy power by comparison with coherent ones.

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I. INTRODUCTION

The ultimate capacity of a communication network is essentially quantum limited, and the main concern of quantum communication is how to discriminate among quantum states that encode the relevant information [1]. Quantum coding states are generally nonorthogonal, such that they cannot be unambiguously discriminated. As a consequence, the detection strategy should be optimized at the receiving side in order to maximize the detection probability and/or minimize the transmission errors.

The scheme we have in mind is the following (see Fig. 1): a binary alphabet $\mathcal{A}=\{0,1\}$ with equal *a priori* probability symbols is being transmitted through a quantum communication channel. The information is encoded in two arbitrary Gaussian quantum states ϱ_0 and ϱ_1 . In the following we first consider the case of pure states $|\psi_0\rangle$ and $|\psi_1\rangle$, whereas, in the second part of this paper, the analysis will be extended to the mixed-state case. Information is amplitude-keyed encoded [2], such that the wave functions of the two states are given by

$$\begin{aligned}\psi_0(x) &= \langle x | \psi_0 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2} + if_0(x)\right], \\ \psi_1(x) &= \langle x | \psi_1 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-a)^2}{2\sigma^2} + if_1(x)\right],\end{aligned}\quad (1)$$

where $f_j(x), j=1,2$ are arbitrary phases, and $a \in \mathbb{R}^+$. Since the two states have the same *a priori* probability of being transmitted, the mean total energy traveling through the channel is given by $E_\tau = a^2/2 + (\sigma^2 - 1/2)^2/\sigma^2$ (measured in unit of $\hbar\tau$, τ being the characteristic time of the physical channel, e.g., the period for a bounded system, or the time length of the wave packet for a free system). The case $\sigma^2 = 1/2$ corresponds to customary on-off coherent modulation, whereas for $\sigma^2 < 1/2$ we are dealing with *squeezed* states [3]. Although squeezing increases the total energy introduced into the channel, we will show that it can be effectively employed to improve the communication scheme, and to approach the performances of an ideal channel.

At the receiver, we consider the standard detection of the signal observable $X, \hat{\mu}(x) = |x\rangle\langle x|$, such that the output probability densities are given by $p_0(x) = |\langle x | \psi_0 \rangle|^2 = G(x; 0, \sigma)$ and $p_1(x; a) = |\langle x | \psi_1 \rangle|^2 = G(x; a, \sigma)$, where $G(x; a, \sigma) = (2\pi\sigma^2)^{-1/2} \exp[-(x-a)^2/2\sigma^2]$ is a normalized Gaussian of mean a and variance σ^2 . On the basis of each measurement outcome we have to discriminate between two hypothesis: the *null hypothesis* \mathcal{H}_0 corresponding to the transmission of $|\psi_0\rangle$ (no signal), and the *alternative hypothesis* \mathcal{H}_1 , corresponding to the transmission of $|\psi_1\rangle$, i.e., to the presence of the signal. The process of measurement and inference is called a *decision strategy*. We denote by Q_1 the *power* of the strategy, that is the probability of inferring the alternative hypothesis when the signal is actually present (also called the detection probability), and by Q_0 the *size* of the strategy, i.e., the probability of inferring the alternative hypothesis when the null hypothesis is true (also called the false-alarm probability).

In the following we employ a threshold strategy, in which the alternative hypothesis is chosen if the outcome is greater than a threshold value x_0 . In order to determine the threshold value we should optimize the strategy, a goal that, in turn, requires to adopt an optimization criterion. Usually, one uses the criterion of minimizing the average cost of the decision, that is, in Bayesian terms, that of minimizing the probability of a wrong inference [4]. Alternatively, one may accept to occasionally obtain an inconclusive inference in order to achieve error-free discrimination [5]. Actually, these have been fruitful approaches in quantum-state recognition, especially in the M -ary decision problem [6]. However, the price of a small error probability is usually a small detection probability that too in turn, may imply the requirement of a high repetition rate. On the other hand, in the field of communi-

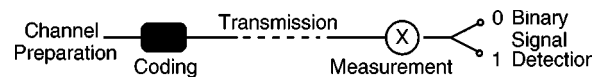


FIG. 1. Block diagram of the communication scheme. In the preparation stage the physical channel is squeezed, then the signal amplitude is either applied or not applied according to which symbol should be transmitted. At the end of the line, the signal observable X is measured and the outcome is compared with the threshold value x_0 in order to infer which state has been transmitted.

cation there exist several protocols that are robust [7], i.e., that may satisfactorily work also with a nonzero transmission-error rate. In this case, the main interest is that of maximizing the detection probability Q_1 , while maintaining the size Q_0 to a moderated tolerable level. A decision strategy that is optimized according to such a criterion, which we will employ throughout this paper, is said to be a Neyman-Pearson (NP) optimized strategy [8].

II. NEARLY IDEAL PERFORMANCE OF A SQUEEZED CHANNEL

The optimal NP threshold strategy for the present X measurement is given in terms of a density $\Pi(x)$, which represents the probability of choosing the alternative hypothesis after having observed the outcome x . We have (Neyman-Pearson Lemma)

$$\Pi(x) = \begin{cases} 1 & \text{if } \Lambda(x) \geq e^\kappa, \\ 0 & \text{if } \Lambda(x) < e^\kappa, \end{cases} \quad (2)$$

where $\Lambda(x) = p_1(x; a)/p_0(x)$ is the *likelihood ratio*, and κ is the *decision level*. By varying the decision level we obtain NP strategies with different sizes. The likelihood ratio is given by $\Lambda(x) = \exp[-(a^2 - 2ax)/2\sigma^2]$, and the NP strategy of Eq. (2) can be summarized as follows: the alternative hypothesis \mathcal{H}_1 is chosen if the outcome is greater than the threshold value $x_0 = (a^2 + 2\sigma^2\kappa)/2a$. The corresponding size and power are given by $Q_0 = \int_{x_0}^{\infty} dx \Pi(x) p_0(x)$ and $Q_1 = \int_{x_0}^{\infty} dx \Pi(x) p_1(x; a)$ i.e.,

$$Q_0 = \int_{x_0}^{\infty} dx p_0(x) = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{x_0}{\sigma\sqrt{2}} \right) \right], \quad (3)$$

$$Q_1 = \int_{x_0}^{\infty} dx p_1(x; a) = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{x_0 - a}{\sigma\sqrt{2}} \right) \right], \quad (4)$$

By eliminating the decision level κ between Eqs. (3) and (4) one obtains the characteristics $Q_1(Q_0)$

$$Q_1 = \frac{1}{2} \left\{ 1 - \operatorname{erf} \left[\operatorname{inverf}(1 - 2Q_0) - \frac{a}{\sqrt{2}\sigma} \right] \right\}. \quad (5)$$

Since the error function $\operatorname{erf}(x)$ and its inverse $\operatorname{inverf}(x)$ are monotonic, the power at fixed size increases with the term $a/\sqrt{2}\sigma$. As we will see, this quantity may be enhanced by squeezing, such that for any energy E_T -squeezed channels always show larger power than coherent ones.

The value of E_T is set by physical constraints, and a question arises about the optimal fraction of E_T that should be employed in squeezing the channel. In fact, the energy cannot be entirely spent in squeezing, since in this case no signal amplitude is left to be discriminated. Let us define the squeezing fraction γ as the fraction of the total energy E_T that is employed to squeeze the channel. In terms of γ and E_T the amplitude and the squeezing are given by $a = \sqrt{2E_T(1-\gamma)}$ and $\sigma = 1/2(\sqrt{\gamma E_T + 2} - \sqrt{\gamma E_T})$. Using these expressions we have

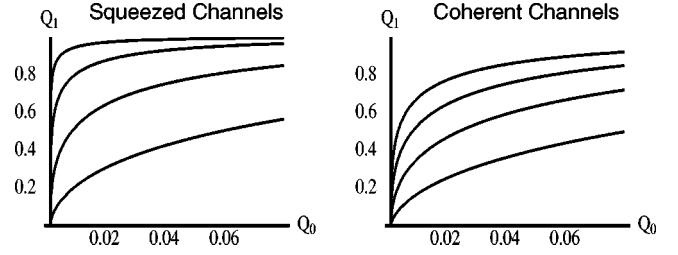


FIG. 2. Power-size characteristics $Q_1(Q_0)$ of the X -measurement NP strategy for different channel energies E_T . Left: optimally squeezed channels. Right: coherent channels. In both plots, from bottom to top $E_T = 0.5h\tau$, $h\tau$, $1.5h\tau$, and $2h\tau$.

$$\frac{a}{\sqrt{2}\sigma} = \frac{2\sqrt{E_T(1-\gamma)}}{\sqrt{\gamma E_T + 2} - \sqrt{\gamma E_T}}. \quad (6)$$

The maximum value is $(a/\sqrt{2}\sigma)_{\max} = \sqrt{E_T(E_T + 2)}$, which is reached for

$$\gamma_{OPT} = \frac{1}{2} \frac{E_T}{1 + E_T}. \quad (7)$$

γ_{OPT} thus represents the optimal squeezing fraction to discriminate, according to NP criterion, amplitude-keyed signals by X measurement. In Fig. 2 we show the characteristics $Q_1(Q_0)$ for optimally squeezed and coherent channels with different energies. The improvement due to squeezing is apparent.

We also notice that Q_1 is a smooth function of the squeezing fraction, which, in turn, should not be considered as a critical parameter. In fact, in order to obtain an enhancement of the strategy power, we do not need a fine tuning of γ . This is illustrated in Fig. 3, where a contour plot of Q_1 is shown as a function of Q_0 and γ . For fixed size Q_0 the power slowly varies with γ in a considerably large range of values.

For a given size Q_0 the bound $Q_1 = 1/2$ defines the minimum detectable signal. As it follows from Eq. (5), this corresponds to $a/\sqrt{2}\sigma = \operatorname{inverf}(1 - 2Q_0)$, and using Eq. (6) to

$$E_T^{\min} = \frac{1}{2} \frac{\operatorname{inverf}^2(1 - 2Q_0)}{1 - \gamma + \operatorname{inverf}^2(1 - 2Q_0) \sqrt{2\gamma(1-\gamma)}}. \quad (8)$$

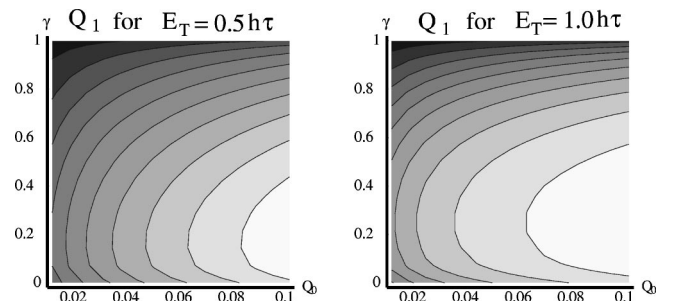


FIG. 3. Power Q_1 of the X -measurement NP strategy as a function of the size Q_0 and the squeezing fraction γ , for two different values of the total energy $E_T = 0.5h\tau$ and $E_T = 1.0h\tau$.

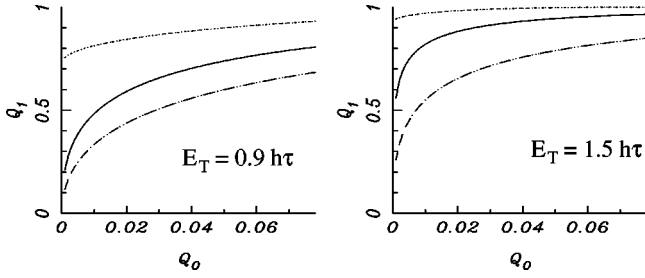


FIG. 4. Power-size characteristics for two different values of the energy E_T . The dotted line is the optimal NP strategy, the solid line the (optimally) squeezed channel for X strategy and the dashed line the coherent one.

E_T^{min} decreases with γ , i.e., squeezed channels allow one to discriminate weaker signals at given size. For small Q_0 , E_T^{min} increases quadratically for coherent channels $E_T^{min} = y^2/2$, and only linearly for optimally squeezed one $E_T^{min} = -1 + \sqrt{1+y^2}$, y being the principal solution of the equation $y\sqrt{\pi Q_0} = \exp(-y^2)$.

In order to appreciate better the benefit of squeezing, we compare the power Q_1 of the NP X -threshold strategy (2), with the optimal NP quantum measurement to discriminate between two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$. Such an optimal measurement has been found long ago [4,9], whereas a comprehensive approach for mixed states is still lacking [10]. For a pair of pure states the optimized measurement is given by

$$\hat{\mu}(x|\lambda) = |\psi_1\rangle\langle\psi_1| - \lambda|\psi_0\rangle\langle\psi_0|, \quad (9)$$

where λ is a Lagrange multiplier that determines the decision level. The decision strategy consists in choosing the alternative hypothesis \mathcal{H}_1 for positive outcomes, and the resulting detection probability reads as follows

$$Q_1 = \begin{cases} [\sqrt{Q_0\omega} + \sqrt{(1-Q_0)(1-\omega)}]^2, & 0 \leq Q_0 \leq \omega \\ 1, & \omega \leq Q_0 \leq 1, \end{cases} \quad (10)$$

where $\omega = |\langle\psi_0|\psi_1\rangle|^2$ is the overlap between the two states. Ideal NP strategies have been previously adopted to achieve the ultimate quantum limit to high-precision binary interferometry [11].

In Fig. 4 we show the power-size characteristics of the optimal strategy in comparison with that of the X strategy for coherent and optimally squeezed channels. For squeezed channels, the power increases and approaches the optimal value achieved already for moderate energy.

In order to summarize improvements due to squeezing we consider the mutual information between input and output $I = \sum_{ij} P_{ij} p_j \ln[P_{ij}/(\sum_j P_{ij} p_j)]$, where $p_0 = p_1 = 1/2$ are the *a priori* probabilities of the two symbols, and P_{ij} is the probability of choosing hypothesis \mathcal{H}_i when hypothesis \mathcal{H}_j is true. In our case, $P_{11} = Q_1$ and $P_{10} = Q_0$, such that $P_{01} = 1 - Q_1$ and $P_{00} = 1 - Q_0$. On the left panel of Fig. 5 we show the mutual information I_X of the X strategy as a function of the total energy E_T for an optimal choice of the squeezing fraction γ . I_X saturates to high value already for moderate energy, showing only a weak dependence on the size of the

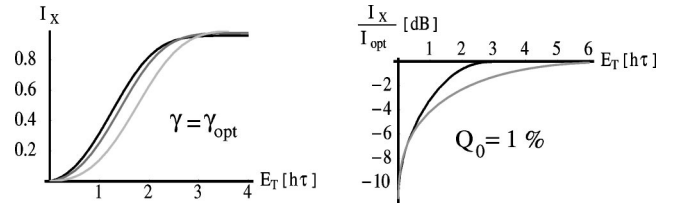


FIG. 5. Left: mutual information I_X as a function of the total energy E_T for $\gamma = \gamma_{opt}$ and for some values of the size $Q_0 = 1\%, 0.5\%, 0.1\%$ (lines in decreasing order of darkness). Right: ratio (in decibels) between the X strategy mutual information and the optimal one at fixed size as a function of the total energy for optimally squeezed (black line) and coherent (gray line) channels.

strategy. On the right panel we show the ratio (in decibels) between I_X and the ideal value I_{opt} , corresponding to the optimal NP strategy. It results that for a squeezed channel the mutual information is approaching the ideal value for much lower energy than a coherent one. Similar plots are obtained by varying the size of the strategy.

III. FUZZY HYPOTHESIS TESTING AND MIXED SIGNAL

So far we have considered information amplitude keyed on pure states. However, in practice, it is more likely to deal with mixture, either because the coding stage is imperfect, or as a result of noises in the transmitter and losses in the channel. For the sake of simplicity, we consider a situation in which the null hypothesis still corresponds to coding onto the vacuum (no amplitude) state, that is, $\varrho_0 = |\psi_0\rangle\langle\psi_0|$. On other hand, the alternative hypothesis now corresponds to coding the signal on the mixed state $\varrho_1 = \int db H_1(b) |\psi_b\rangle\langle\psi_b|$, where $|\psi_b\rangle$ coincides with $|\psi_1\rangle$ of Eq. (1) and $H_1(b)$ is a weight function, which will be taken of Gaussian form. The two hypotheses to be discriminated are no longer crisp, and the decision problem should be formulated in the framework of *fuzzy hypothesis testing* [12,13]. The fuzzy null and alternative hypotheses are formulated as follows: \mathcal{H}_j is true when a Gaussian state of amplitude b , distributed as $H_j(b)$, is transmitted. In our case the two membership density functions are given by $H_0(b) = \delta(b)$ and $H_1(b) = (2\pi\Sigma^2)^{-1/2} \exp[-(b-a)^2/2\Sigma^2]$.

In order to analyze the effect of squeezing with a mixed signal we need to find the best NP X strategy of its size. Recently, the Neyman-Pearson Lemma has been extended to fuzzy hypothesis testing [14], and this allows us to solve the decision problem. The NP strategy for mixed states is a density of the form (2) with the *fuzzy likelihood ratio* given by

$$\begin{aligned} \Lambda^F(x) &= \frac{\int db p_1(x;b) H_1(b)}{\int db p_0(x) H_0(b)} = \frac{\int db G(x;b,\sigma) H_1(b)}{G(x;0,\sigma)} \\ &= \sqrt{\frac{\beta^2}{1+\beta^2}} \exp\left[\frac{x^2 + 2ax\beta^2 - a^2\beta^2}{2\sigma^2(1+\beta^2)}\right], \end{aligned} \quad (11)$$

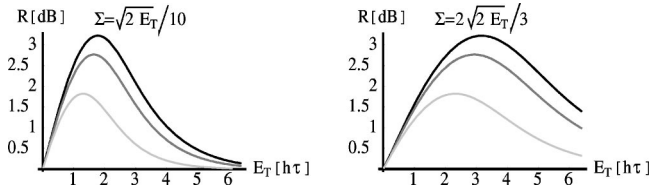


FIG. 6. Ratio $R = I_X^S/I_X^C$ between squeezed and coherent mutual information as a function of the total energy E_T for a weakly (left) and a strongly (right) mixed channel. In both plots curves for different values of the strategy size Q_0 are shown (black line, $Q_0 = 0.5\%$, dark gray, $Q_0 = 1\%$, light gray, $Q_0 = 5\%$).

where $\beta^2 = \sigma^2/\Sigma^2$. The pure-state case is obtained in the limit $\beta \rightarrow \infty$.

The power-size characteristics has the same functional form (5) as that of the pure-state case. However, for mixed signals part of the energy is degraded to noise, $E_T = (a^2 + \Sigma^2)/2 + (\sigma^2 - 1/2)^2/\sigma^2$, such that the amplitude reads as follows $a = \sqrt{2E_T(1 - \gamma) - \Sigma^2}$. After inserting this expression into the term $a/\sqrt{2}\sigma$, and maximizing over γ one obtains the optimal squeezing fraction for mixed channels

$$\gamma_{OPT}^M = \frac{(2E_T - \Sigma^2)^2}{8E_T(1 + E_T - \Sigma^2/2)}. \quad (12)$$

As it can be easily proved from Eq. (12) γ_{OPT}^M is always smaller than γ_{OPT} for any value of E_T , i.e., a smaller amount of squeezing can be employed in a mixed channel against a pure channel with the same energy. Correspondingly, also the power at fixed size decreases. However, squeezing a mixed channel is still extremely convenient to improve the X strategy by comparison with a mixed coherent channel of the same energy. In order to illustrate this behav-

ior, we define the ratio $R = I_X^S/I_X^C$ (at fixed energy and mixing parameter) between the mutual information of a squeezed and a coherent channel, respectively. In Fig. 6 we show R as a function of E_T for different values of the size for a weakly ($\Sigma = \sqrt{2E_T}/10$) and a strongly ($\Sigma = 2\sqrt{2E_T}/3$) mixed channel. Notice that $\Sigma = \sqrt{2E_T}$ is the limiting value, corresponding to a completely mixed signal with no amplitude and no squeezing. R linearly increases for small E_T and after a maximum of few decibels (the actual height depends on the size Q_0) decreases. In the (unrealistic) limit of very high energies, squeezing the channel is no longer convenient. We notice that for a strongly mixed channel such a decreasing is much slower, thus indicating that squeezing is effective in a wide range of energies. In other words, a squeezed channel is more robust against mixing of signals than a coherent one.

IV. CONCLUSIONS

In conclusion, we have shown that squeezing the channel in amplitude-keyed binary communication increases the detection probability at fixed size. We have found the optimal squeezing fraction and evaluated the mutual information for both pure and mixed signals. Optimally squeezed channels are robust against signal mixing, and largely improve the strategy power by comparison with coherent ones, approaching the performance of the ideal receiver, achieved already for moderate signal energy.

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