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MINIMUM OUTPUT ENTROPY OF A GAUSSIAN BOSONIC CHANNEL

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The entropy at the output of a Bosonic channel is analyzed when coherent fields are randomly added to the signal. Coherent-state inputs are conjectured to provide the minimal output entropy. Supporting physical and mathematical evidence is provided.

Keywords: quantum channels; entropy; additivity; capacity; bosonic channels.

A memoryless quantum channel[1] can be described with a completely positive (CP) linear super-operator $\mathcal{M}$ on the Hilbert space $\mathcal{H}$, that maps the input state ρ into the output state $\rho' \equiv \mathcal{M}(\rho)$. Here we analyze the minimal von Neumann entropy at the output of the channel[2], i.e.,

$$\mathbb{S} \equiv \min_{\rho\in\mathcal{H}} \{S(\rho')\} \equiv \min_{\rho\in\mathcal{H}} \{-\mathrm{Tr}\,[\rho' \ln \rho']\} \ , \tag{1}$$

where the minimization is performed over all possible input density matrices ρ. The quantity $\mathbb{S}$ defined in Eq. (1) provides a "measure" of the minimal amount of noise introduced by the channel $\mathcal{M}$ during the communication. It is also connected with the channel capacity[3] for the transmission of classical information. Here we focus on Bosonic channels[4] where the message is encoded onto electromagnetic field modes and where the map describes the noise encountered during the transmission. In particular, we consider a single mode channel with a Gaussian map[a] that randomly

[a] A Gaussian map transforms inputs with Gaussian characteristic function into outputs with the same property.

displaces the input signal according to a Gaussian probability distribution[5,6,7], i.e.

$$\rho' \equiv \mathcal{N}_n(\rho) = \int d^2\mu \; P_n(\mu) \; D(\mu)\rho D^\dagger(\mu) \tag{2}$$

where

$$P_n(\mu) = \frac{e^{-|\mu|^2/n}}{\pi n}, \tag{3}$$

and $D(\mu) \equiv \exp(\mu a^\dagger - \mu^* a)$ is the displacement operator associated with the input mode. This channel can be seen as a simplified version of a map where the signal photons interact linearly with a thermal environment[6,7]. In particular, $\mathcal{N}_n$ transforms the vacuum input $|0\rangle$ into the thermal output

$$\rho_0' \equiv \frac{1}{n+1} \sum_{m=0}^{\infty} \left(\frac{n}{n+1} \right)^n |m\rangle\langle m| \equiv \int d^2\mu \; P_n(\mu) \; |\mu\rangle\langle\mu| \; , \tag{4}$$

where $|m\rangle$ and $|\mu\rangle$ are Fock states and coherent states respectively. This output has von Neumann entropy equal to

$$S(\rho_0') = g(n) \equiv (1+n)\ln(1+n) - n \ln n \; . \tag{5}$$

Analogously one can show that the output associated with a coherent input $|\alpha\rangle$ is obtained by displacing ρ_0' of Eq. (4), i.e.

$$\rho_\alpha' = D(\alpha) \; \rho_0' \; D^\dagger(\alpha). \tag{6}$$

Thus, the invariance of the von Neumann entropy under unitary transformation guarantees that all coherent input states give origin to the same output entropy.

1. A conjecture

From the concavity of the von Neumann entropy we know that the minimum in Eq. (1) can be achieved with pure input states $\rho = |\psi\rangle\langle\psi|$. Recently, it has been conjectured[6] that the minimum output entropy of a wide class of Gaussian Bosonic channels is achieved by coherent input states. In the case of $\mathcal{N}_n$ this is equivalent to having

$$\mathbb{S} = g(n) \; . \tag{7}$$

The physical intuition behind this conjecture lies in the fact that the input state is contaminated by noise from a reservoir [characterized by the probability distribution of Eq. (2)] whose quantum phase is completely random. It is hence reasonable to expect that no coherence can be extracted from the reservoir to reduce the output entropy below the level when no photons are sent through the channel. Theoretical and numerical evidences[6] suggest that even a stronger version of this conjecture might hold, namely that the output states produced by coherent inputs *majorize*[8] all the other output states. Here we will focus only on the (weaker) version (7) of the conjecture.

Even though the relation (7) has not been proven yet, ample supporting evidence has been obtained[6,7,9]. In the following we discuss some of the main results.

1.1. *Local minimum*

Coherent states produce local minima in the output von Neumann entropy[6]. In fact, an output state σ'_0 is a local minimum of the output entropy if the following directional derivative is non-negative, i.e.

$$\frac{\partial}{\partial t} S(\sigma'_0(1-t)+\sigma' t)\Big|_{t=0^+} = \mathrm{Tr}[(\sigma'_0-\sigma')\ln\sigma'_0] \geq 0, \tag{8}$$

for any output σ'. In the case of vacuum input $\sigma_0 = |0\rangle\langle 0|$, the output state (4) can be written as $\sigma'_0 \propto \exp[-\zeta (a')^\dagger a']$ with $\zeta > 0$. Since σ'_0 is the output state with minimum average photon number, we have $\mathrm{Tr}[(\sigma'_0-\sigma')(a')^\dagger a'] \geq 0$ and Eq. (8) is satisfied for vacuum input (and, hence, for any other coherent state input). If one could show that the inequality (8) is true only for input coherent states, then the conjecture would be proved.

1.2. *Lower bounds*

The output entropy for coherent inputs gives a trivial upper bound to the minimal output entropy of a channel. Lower bounds for $\mathbb{S}$ have been derived in Ref. 6. Here we only analyze a couple of them, i.e.,

$$\mathbb{S} \geqslant \begin{cases} g(n-1) \\ \ln(2n+1)\ . \end{cases} \tag{9}$$

The first one gives a good bound for $n \gg 1$. It is obtained by considering the

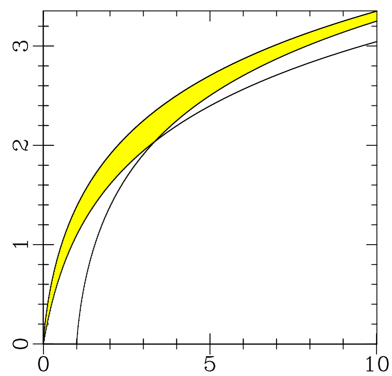

Fig. 1. Bounds for the minimal output entropy of the channel $\mathcal{N}_n$ as a function of the noise parameter n. The curve u is the upper bound obtained feeding the channel with coherent inputs; a and b are the first and second lower bounds of Eq. (9) respectively. The minimal output entropy $\mathbb{S}$ must reside between the upper and lower bounds, i.e. in the shaded area.

Husimi expansion[10,11] of the input state ρ,

$$\rho = \int d^2\alpha\ Q(\alpha)\ \sigma(\alpha)\ , \tag{10}$$

where $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle/\pi$ and $\sigma(\alpha)$ is a convolution of displacement operators of the field. It implies

$$\rho' = \int d^2\alpha \; Q(\alpha) \; \sigma'(\alpha) \; , \tag{11}$$

where $\sigma'(\alpha)$ is the evolution of $\sigma(\alpha)$ through the channel. The first inequality of (9) then follows from the convexity of the von Neumann entropy. The second inequality of (9), on the other hand, gives a good bound for $n \sim 0$. It is derived by observing that for any state ρ, the von Neumann entropy satisfies

$$S(\rho) \geq -\ln(\mathrm{Tr}[\rho^2]), \tag{12}$$

and by using the bounds on $\mathrm{Tr}[\rho'^2]$ described in the next section.

1.3. *Rényi entropy*

Generalizing an argument by Caves[12] we proved[7] that the output Rényi entropies S_k of integer order k greater than or equal to 2 are minimized by input coherent states, i.e.

$$S_k(\rho') \equiv \frac{\ln(\mathrm{Tr}[(\rho')^k])}{1-k} \geq \frac{\ln[(n+1)^k - n^k]}{k-1} \qquad k = 2, 3, \ldots \tag{13}$$

To derive this inequality we express $\mathrm{Tr}[(\rho')^k]$ as an expectation value of a Gaussian operator A acting on an extended Hilbert space of k modes, i.e.

$$\mathrm{Tr}[(\rho')^k] \equiv \mathrm{Tr}[\underbrace{\rho \otimes \rho \otimes \cdots \otimes \rho}_{k \text{ times}} A]. \tag{14}$$

The maximum eigenvalue of A is an upper bound for $\mathrm{Tr}[(\rho')^k]$ (i.e. a lower bound for $S_k(\rho')$) which is achieved by input coherent states. Notice that in the limit of $k \to 1$, the Rényi entropy tends to the von Neumann entropy. If we could generalize Eq. (13) to any $k \in]1, 2[$ then the conjecture would follow from the continuity of Rényi entropy.

For the Gaussian Bosonic channel, the above technique was also successful[9] in analyzing the additivity properties[13] of the minimum Rényi entropies of integer order for successive uses of the channel.

1.4. *Wehrl entropy*

The Wehrl entropy of a state ρ is the Shannon entropy of its Husimi function, i.e.

$$W(\rho) \equiv -\int \frac{d^2\alpha}{\pi} \; \langle\alpha|\rho|\alpha\rangle \; \ln\langle\alpha|\rho|\alpha\rangle. \tag{15}$$

This quantity measures the localization of a state in phase space and is minimized by coherent states[10,11], as Lieb proved. Using Young's inequality and Hausdorff-Young's inequality one can show[7] that the Wehrl entropy at the output of $\mathcal{N}_N$ achieves its minimum over coherent inputs, i.e.

$$W(\rho') \geq 1 + \ln(1+n). \tag{16}$$

In the same way, one can prove[7] that the Rényi-Wehrl entropies[14] W_k with order k greater than 1 are minimized by coherent inputs, namely[b]

$$W_k(\rho') \equiv \frac{\ln[\int \frac{d^2\alpha}{\pi} \langle\alpha|\rho'|\alpha\rangle^k]}{1-k} \geq \frac{\ln[k(n+1)^{k-1}]}{k-1}. \tag{17}$$

Notice that in the limit of $k \to 1$, Eq. (17) gives Eq. (16).

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[b] N.B.: here it is not necessary to restrict k to integer values, as this property is valid for any $k > 1$.