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On the analytical convergence of the QPA procedure

Chiara Macchiavello¹

Dipartimento di Fisica “A. Volta” and INFN–Unità di Pavia, Via Bassi 6, 27100 Pavia, Italy

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Abstract

We present an analytical proof of the convergence of the “quantum privacy amplification” procedure proposed by Deutsch et al. [Phys. Rev. Lett. 77 (1996) 2818]. The proof specifies the range of states which can be purified by this method. © 1998 Elsevier Science B.V.

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Purification schemes of two-particle entangled states have been recently proposed [1,2]. The purpose of such schemes is to distill a subset of states with enhanced purity from a larger set of non-pure entangled states of two particles. They have proved essential to perform various tasks in quantum information theory, such as teleportation of quantum states over noisy channels [1], secure quantum cryptography in the presence of noise [2], quantum error correction [3] and distributed quantum computation [4]. They have also been related to some fundamental problems in quantum mechanics, such as the separability of quantum states and the nature of entanglement [5]. The most efficient scheme known so far for a wide range of the initial states is the so-called “quantum privacy amplification” (QPA). It was originally designed for cryptographic purposes [2]. In this paper we analyse in detail the QPA algorithm and present an analytical proof of its convergence. This part was missing in the original paper, where some conclusions were based on numerical analysis.

An elementary step of the QPA protocol is described by the following map,

$$A = (a^2 + b^2)/p, \tag{1}$$

$$B = 2cd/p, \tag{2}$$

$$C = (c^2 + d^2)/p, \tag{3}$$

$$D = 2ab/p, \tag{4}$$

where $p = (a + b)^2 + (c + d)^2$, while $\{a, b, c, d\}$ and $\{A, B, C, D\}$ are the diagonal elements of the density operator describing the state of a “noisy” EPR pair in the Bell basis representation $\{|\phi^+\rangle, |\psi^-\rangle, |\psi^+\rangle, |\phi^-\rangle\}$,

$$\begin{aligned} |\phi^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \\ |\psi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \end{aligned} \tag{5}$$

The two states $\{|0\rangle, |1\rangle\}$ form the basis of the two-dimensional systems belonging to the EPR pair. The lower case letters a, b, c, d correspond to the density operator of the pair before a QPA step, while the capi-

¹ E-mail: macchiavello@pv.infn.it.

tal letters A, B, C, D correspond to the surviving state at the output. It is understood that normalisation of the density operator requires $a + b + c + d = 1$, and therefore the map involves only three independent parameters.

In the following we want to prove analytically that the QPA map converges to the value $\{1, 0, 0, 0\}$ (corresponding to the pure state $|\phi^+\rangle$) for any initial value $a > 0.5$. In other words, if the initial fraction of the $|\phi^+\rangle$ component of the density operator at the beginning of the procedure is larger than 0.5, then the iteration of the QPA algorithm will asymptotically lead to the final state $\{1, 0, 0, 0\}$.

The proof is based on showing the following assertions:

- (i) There exists a monotonic function in the region $\mathcal{R} = \{a \in (0.5, 1], b, c, d \in [0, 0.5); a + b + c + d = 1\}$, i.e. it increases under iterations of the QPA map.
- (ii) The extremal value of this monotonic function in the abovementioned region corresponds to the fixed point of the QPA map in this region, which is $\{1, 0, 0, 0\}$.

The above properties of the map lead to the conclusion that the fixed point $\{1, 0, 0, 0\}$ is also an attractor (and the only one) in the region of interest, thus proving the convergence of the map for any initial value $a > 0.5$.

The most natural candidate function one could think of is the first component a in the diagonal of the density operator, which asymptotically (for a large number of iterations of the procedure) reaches its maximum value 1. This is the case for example in the first proposed purification scheme of Ref. [1], where it was easily shown that the first component always increases at each iteration, which immediately proved the convergence of the method. In the present QPA algorithm this is not the case, as a is not always smaller than the corresponding A .

The function we will consider in this paper has the quadratic form

$$f(a, b) = (2a - 1)(1 - 2b). \quad (6)$$

Let us point out in passing that such a function depends only on two of the three independent parameters involved in the map. In order to prove the monotonicity of function f we will first prove that when the parameters of the input density operator $\{a, b, c, d\}$

belong to the region \mathcal{R} it increases in one iteration of the map, i.e.

$$f(A, B) > f(a, b) \quad (7)$$

and then show that if the initial diagonal values of the density operator belong to the region of interest \mathcal{R} , they stay in this region at all subsequent iterations of the map.

After some straightforward algebra, condition (7) can be more conveniently expressed only in terms of parameters c and d as

$$g(c, d) = 2(c + d)^4 - 4(c + d)^3 + 4(c + d)^2 - (c + d) - (c^2 + d^2) < 0. \quad (8)$$

Notice that also this condition involves only two of the three independent parameters which characterise the map. In order to prove Eq. (8) we just have to prove that function g is always negative in the region $\{c, d \in [0, 0.5), c + d < 0.5\}$.

We can easily see that the function is negative on the boundaries of the region of interest, namely on the axis $c = 0$ and $d = 0$ with $0 < c, d < 0.5$ and on the line $c + d = 0.5$ (apart from the points $c = d = 0$ and $c = d = 0.25$ where $g = 0$). Moreover, the function is decreasing when we depart from the boundaries towards the inner part of the interested region. We can also show that g has only one extremal point in the region, which is a minimum. This can be seen by introducing the variables $x = c - d$ and $y = c + d$: the vanishing conditions for the partial derivatives $\partial g / \partial x$ and $\partial g / \partial y$ lead to the solution $y = 0$ (i.e. $c = d$) and the following equation for y ,

$$8y^3 - 12y^2 + 7y - 1 = 0. \quad (9)$$

We can see analytically that the above equation has only one solution, which lies between 0 and 0.5: the exact value has been found numerically to be $y_0 = 0.205122$. As the function g is continuous, the above features guarantee the negativity of g in the region of interest. We have thus proved that $f(A, B) > f(a, b)$ for $a > 0.5$ in one iteration of the QPA map.

Moreover, for one iteration of the map we have

$$1 - 2A = \frac{(2a - 1)(2b - 1)}{p}. \quad (10)$$

The above equation shows that $A > 0.5$ if $a > 0.5$, i.e. if the initial value of a is larger than 0.5 then it will be

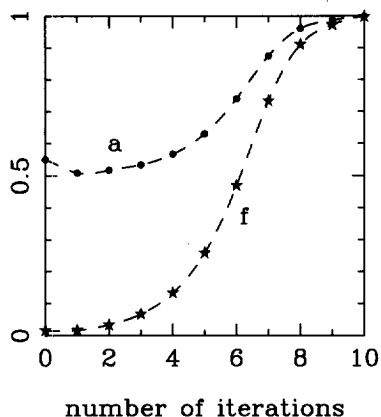


Fig. 1. Plot of the first component a in the diagonal of the density operators (circles) and function f (stars) as functions of the number of iterations of the QPA map for the initial state $\{0.57, 0.41, 0.01, 0.01\}$. The zeroth iteration corresponds to the initial values. Please note that a decreases in the first step of the iteration.

always larger than 0.5 at all subsequent iterations of the map and will never cross the $a = 0.5$ boundary line. This means that the evolutions of parameters a, b, c and d under the QPA map will never leave the region \mathcal{R} . Therefore, the function f is also always increasing for all iterations of the map if a is initially larger than 0.5. In this way we have proved point (i).

Regarding point (ii), it is easy to see that the function $f(a, b)$ takes its maximum value for $f(1, 0) = 1$ in the region of interest $\{a \in (0.5, 1], b \in [0, 0.5], a + b \leq 1\}$: $f(1, 0)$ corresponds to the maximum value along the boundaries of the region and no local extremal points are present inside. Therefore, by iterating the QPA map, the function f is bound to reach its maximum value and the surviving states of the pairs asymptotically approach the pure state $|\phi^+\rangle$. This allows to conclude that whenever the initial value of a is bigger than 0.5, the density operator of the pairs is driven to the fixed point of the map $\{1, 0, 0, 0\}$. Such a point is therefore a global attractor for any initial value $a > 0.5$.

As an example, in Fig. 1 we plot the behaviour of f as a function of the number of iterations for the initial state $\{0.57, 0.41, 0.01, 0.01\}$, in contrast to the non-monotonic behaviour of the first component a of the density operator.

Let us now analyse what happens when the diagonal elements of the initial density operator do not be-

long to the region \mathcal{R} . From Eq. (10) we can see that after the first iteration of the map we get $A > 0.5$ for any initial value $b > 0.5$. Thus, for an initial value of b bigger than 0.5 after one iteration the parameters a, b, c and d belong to the region \mathcal{R} and the map is therefore converging to the state $\{1, 0, 0, 0\}$. Notice also that the QPA map is symmetric under the following exchange of parameters: $a \leftrightarrow c, b \leftrightarrow d$. This implies that the proof we have presented here is valid also for any initial value $c, d > 0.5$ and in this case the map converges to the state $\{0, 0, 1, 0\}$. We can then conclude that this kind of map leads to a perfect state purification whenever one of the initial coefficients a, b, c or d is larger than 0.5. The final pure state is either $|\phi^+\rangle$ if a or b are initially larger than 0.5, or $|\psi^+\rangle$ if c or d are initially larger than 0.5.

When the initial diagonal components of the density operator are all smaller than 0.5 it is not possible to purify the state by means of the present QPA map. Actually, we can see from Eq. (10) that A cannot be greater than 0.5 if both a and b are smaller. Moreover, for one iteration of the map we have

$$1 - 2B = \frac{2(c^2 + d^2) - 2(c + d) + 1}{p}, \quad (11)$$

and this expression is always positive for $c, d < 0.5$, meaning that also B is smaller than 0.5. Because of the symmetry of the map under exchange $a \leftrightarrow c$ and $b \leftrightarrow d$, the same conclusions hold for C and D . Therefore, when we initially have $a, b, c, d < 0.5$, then $A, B, C, D < 0.5$ and none of the diagonal components of the density operator will ever cross the 0.5 boundary line at any iteration of the map. The procedure in this case is not successful.

At the moment the physical meaning of the function f is still unclear. Notice that other monotonic functions can be obtained from f by any monotony preserving transformation, but still we could not give any obvious physical explanation for the convergence of the map. Let us then leave this as a challenge to our colleagues in the field.

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