



# Optimal state estimation for $d$ -dimensional quantum systems

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## Abstract

We establish a connection between optimal quantum cloning and optimal state estimation for  $d$ -dimensional quantum systems. In this way we derive an upper limit on the fidelity of state estimation for  $d$ -dimensional pure quantum states and, furthermore, for generalized inputs supported on the symmetric subspace. © 1999 Published by Elsevier Science B.V.

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One of the fundamental problems in quantum physics is the question of how well one can estimate the state  $|\psi\rangle$  of a quantum system, given that only a finite number of identical copies is available. An appropriate figure of merit in this context is the fidelity which will be defined below. The optimal fidelity for state estimation of two-level systems has been derived in Ref. [1], and an algorithm for constructing an optimal positive operator valued measurement (POVM) for a general quantum system has been given in Ref. [2]. The purpose of this Letter is to derive the optimal fidelity for state estimation of an ensemble of  $N$  identical pure  $d$ -dimensional quantum systems by establishing a connection to optimal quantum cloning.

In the following we will prove the link between optimal quantum cloning and optimal state estimation, using a similar line of argument as in Ref. [3]. We consider both processes to be universal, in the sense that the corresponding fidelity does not depend on the

input state  $|\psi\rangle$ . The link we want to show is given by the equality

$$F_{d,\text{est}}^{\text{opt}}(N) = F_{d,\text{QCM}}^{\text{opt}}(N, \infty). \quad (1)$$

Here  $F_{d,\text{QCM}}^{\text{opt}}(N, M)$  is the fidelity of the optimal quantum cloner for  $d$ -dimensional systems, taking  $N$  identical pure inputs and creating  $M$  outputs, which was derived in Ref. [4] to be

$$F_{d,\text{QCM}}^{\text{opt}}(N, M) = \frac{M - N + N(M + d)}{M(N + d)}. \quad (2)$$

(This formula refers to the fidelity between an output one-particle reduced density operator and one of the identical inputs.) In Eq. (1)  $F_{d,\text{est}}^{\text{opt}}(N)$  is the optimal average fidelity of state estimation for  $N$  identical  $d$ -dimensional inputs, defined as

$$F_{\text{est}} = \sum_{\mu} p_{\mu}(\psi) |\langle \psi | \psi_{\mu} \rangle|^2, \quad (3)$$

where  $p_{\mu}(\psi)$  is the probability of finding outcome  $\mu$  (to which we associate candidate  $|\psi_{\mu}\rangle$ ), given that the inputs were in state  $|\psi\rangle$ .

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Let us introduce the generalized Bloch vector  $\lambda$  by expanding a  $d$ -dimensional density matrix in the following way [5],

$$\rho_d = \frac{1}{d} \mathbf{1} + \frac{1}{2} \sum_{i=1}^{d^2-1} \lambda_i \tau_i, \tag{4}$$

where  $\tau_i$  are the generators of the group  $SU(d)$  with

$$\text{Tr} \tau_i = 0, \quad \text{Tr}(\tau_i \tau_j) = 2\delta_{ij}. \tag{5}$$

Note that the length of the generalized Bloch vector for pure states is

$$|\lambda| = \sqrt{2(1 - 1/d)}, \tag{6}$$

which reduces to the familiar case  $|\lambda| = 1$  for qubits, i.e.  $d = 2$ .

It has been shown in Ref. [6] that, as far as optimality of the fidelity for a universal map is concerned, one can restrict oneself to covariant transformations. Furthermore, a covariant map, acting on pure  $d$ -dimensional input states, can only shrink the generalized Bloch vector, namely it transforms Eq. (4) into the output density operator

$$\rho_d = \frac{1}{d} \mathbf{1} + \frac{1}{2} \eta_d \sum_{i=1}^{d^2-1} \lambda_i \tau_i, \tag{7}$$

where we call  $\eta_d$  the shrinking factor. Note that for pure input states the fidelity is related to  $\eta_d$  as follows,

$$F_d = \frac{1}{d} [1 + (d - 1) \eta_d]. \tag{8}$$

Remember that, as mentioned above, in this paper we consider universal quantum cloning and universal state estimation and therefore the above considerations apply. In order to clarify the role of the shrinking factor in quantum state estimation we notice that Eq. (3) can also be interpreted as

$$F_{d,\text{est}}(N) = \langle \psi | \rho_{d,\text{est}} | \psi \rangle, \tag{9}$$

where the density operator  $\rho_{d,\text{est}}$ , due to universality, is the shrunk version of the input  $|\psi\rangle\langle\psi|$ , namely

$$\rho_{d,\text{est}} = \eta_{d,\text{est}}(N) |\psi\rangle\langle\psi| + (1 - \eta_{d,\text{est}}(N)) \frac{1}{d} \mathbf{1}. \tag{10}$$

We now start proving the equality (1) by noticing that after performing a universal measurement procedure

on  $N$  identically prepared input copies  $|\psi\rangle$ , we can prepare a state of  $L$  systems, supported on the symmetric subspace of  $\mathcal{H}_d^{\otimes L}$ , where each system has the same reduced density operator, given by  $\rho_{d,\text{est}}$ . The symmetric subspace is defined as the space spanned by all states which are invariant under any permutation of the constituent subsystems.

As shown in Ref. [4], a universal cloning process generates outputs that are supported on the symmetric subspace. Therefore, the above method of performing state estimation followed by preparation of a symmetric state can be viewed as a universal cloning process and thus it cannot lead to a higher fidelity than the optimal  $N \rightarrow L$  cloning transformation. Therefore we find the inequality

$$F_{d,\text{est}}^{\text{opt}}(N) \leq F_{d,\text{QCM}}^{\text{opt}}(N, L). \tag{11}$$

The above inequality must hold for any value of  $L$ , in particular for  $L \rightarrow \infty$ .

In order to derive the opposite inequality, we consider a measurement procedure on  $N$  copies which is composed of an optimal  $N \rightarrow L$  cloning process and a subsequent universal measurement on the  $L$  output copies. This total procedure is also a possible state estimation method. As mentioned above the output  $\rho_L$  of the optimal universal  $d$ -dimensional cloner is supported on the symmetric subspace and therefore can be decomposed as [4]

$$\rho_L = \sum_i \alpha_i |\psi_i\rangle\langle\psi_i|^{\otimes L}, \quad \sum_i \alpha_i = 1, \tag{12}$$

where the coefficients  $\alpha_i$  are not necessarily positive. After performing the optimal universal measurement on the  $L$  cloner outputs we can calculate the average fidelity of the total process, due to linearity of the measurement procedure, as follows,

$$F_{d,\text{total}}(N, L) = \sum_i \alpha_i \langle \psi | \left[ \eta_{d,\text{est}}^{\text{opt}}(L) |\psi_i\rangle\langle\psi_i| + (1 - \eta_{d,\text{est}}^{\text{opt}}(L)) \frac{1}{d} \mathbf{1} \right] | \psi \rangle. \tag{13}$$

(Remember that  $\sum_i \alpha_i |\psi_i\rangle\langle\psi_i|$  is the one particle reduced density matrix at the output of the  $N \rightarrow L$  cloner and thus depends on  $N$  and  $L$ .) In the limit  $L \rightarrow \infty$  we have  $\eta_{d,\text{est}}^{\text{opt}}(\infty) = 1$  and the average fidelity can be written as

$$\begin{aligned}
\lim_{L \rightarrow \infty} F_{d,\text{total}}(N, L) &= \langle \psi | \left[ \sum_i \alpha_i |\psi_i\rangle \langle \psi_i| \right] | \psi \rangle \\
&= \langle \psi | \left[ \eta_{d,\text{QCM}}^{\text{opt}}(N, \infty) |\psi\rangle \langle \psi| \right. \\
&\quad \left. + (1 - \eta_{d,\text{QCM}}^{\text{opt}}(N, \infty)) \frac{1}{d} \mathbf{1} \right] | \psi \rangle \\
&= \frac{1}{d} [1 + (d-1) \eta_{d,\text{QCM}}^{\text{opt}}(N, \infty)], \quad (14)
\end{aligned}$$

where in the second and third lines we have explicitly written down the output of the cloning stage for clarity. This fidelity cannot be higher than the one for the optimal state estimation performed directly on  $N$  pure inputs, thus we conclude

$$F_{d,\text{QCM}}^{\text{opt}}(N, \infty) \leq F_{d,\text{est}}^{\text{opt}}(N). \quad (15)$$

The above inequality, together with Eq. (11), leads to the equality (1). Thus we have derived the optimal fidelity for state estimation of  $N$  copies of a  $d$ -dimensional quantum system to be

$$F_{d,\text{est}}^{\text{opt}}(N) = \frac{N+1}{N+d}. \quad (16)$$

Note that we can extend this argument for optimal state estimation to more general inputs, namely to inputs supported on the symmetric subspace. Using the decomposition (12), we see immediately that we can always reach at least the same shrinking factor as for pure inputs, due to linearity of the measurement procedure. Moreover, we can prove by contradiction that the shrinking factor cannot be larger than for pure states: let us assume we could perform better on such a described entangled input. We can think of arranging the following procedure. We concatenate an  $N \rightarrow M$  cloning transformation taking  $N$  pure inputs and creating  $M$  outputs with a subsequent state estimation.

Notice that, generalizing the result of Ref. [3], the shrinking factors of two concatenated universal operations multiply, given that the output of the first is supported by the symmetric subspace. If we could perform better than in the pure case at the second stage of this concatenation, we could, by reconstructing the output state according to the state estimation result, create an  $N \rightarrow \infty$  cloner that is better than the optimal one, thus arriving at a contradiction.

In conclusion, we have derived the optimal fidelity for state estimation of an ensemble of identical  $d$ -dimensional quantum states, pointing out the connection to optimal quantum cloning. We have also extended the possible inputs for state estimation in  $d$  dimensions to those supported on the symmetric subspace. Note that an algorithm to construct the according POVM consisting of a finite set of operators has been given in Ref. [2].

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