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Bounds on the efficiency of cloning for two-state quantum systems

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Abstract. We analyse the efficiency of quantum cloning for qubits. We investigate, in particular, the role played by the form of allowed input states and show how the fidelity of cloning changes by restricting the class of inputs.

Keywords: Quantum information, quantum cloning

1. Introduction

Perfect cloning of quantum states that are *a priori* unknown is forbidden by the laws of quantum mechanics [1]. Perfect cloning is only possible when the input states belong to a known set of orthogonal states. For example, the Controlled-Not quantum gate [2], which operates as follows on two qubits (two-state systems):

$$|x_1\rangle|x_2\rangle \to |x_1\rangle|x_1 \oplus x_2\rangle, \tag{1}$$

where $\oplus$ denotes addition modulo two and $|x_i\rangle = \{|0\rangle, |1\rangle\}$ represent basis states for each qubit, implements a perfect cloning transformation for qubits (the first qubit is the one to be cloned and is initially in one of the two orthogonal states $|0\rangle$ or $|1\rangle$, while the second one is initially prepared in state $|0\rangle$). The requirement that the input state belongs to a known class of orthogonal states is quite restrictive. It is intuitive to expect that by relaxing the conditions on the class of allowed input states perfect cloning can be approximated with a decreasing efficiency.

In this paper we describe approximate cloning transformations for different classes of input states and analyse the corresponding qualities. We consider the case of general $N \to M$ cloning transformations for qubits. In section 2 we describe the least restrictive case, where the input states of the qubits are completely unknown. In section 3 we start restricting the class of inputs and take all possible states lying on the equator of the Bloch sphere. In section 4 we analyse the smallest nontrivial class of input states, namely the set of two nonorthogonal states. We derive bounds on the efficiencies of the various processes and show how the efficiency increases by restricting the class of inputs.

2. Universal cloning

We consider here the least restrictive class of input states, namely the one corresponding to the whole two-dimensional Hilbert space of a qubit. We will investigate universal cloning transformations, namely tranformations whose efficiency does not depend on the form of the input state. We study the efficiency of a cloning transformation in terms of the fidelity F of the density operator $\rho_{\rm out}$ describing the state of each output copy with respect to the original pure input state $|\psi\rangle$, namely

$$F = \langle\psi|\rho_{\rm out}|\psi\rangle. \tag{2}$$

Universal $N \to M$ quantum cloning is a unitary transformation acting on an extended input which contains N original qubits all in the same unknown pure state $|\psi\rangle$, $M - N$ 'blank' qubits and K auxiliary systems, and giving M output clones together with the auxiliary K systems. Both 'blanks' and the auxiliary systems are initially in some prescribed quantum state. In order to guarantee that the M output qubits are described by the same reduced density operator $\rho_{\rm out}$ we require that the output state describing the M copies is supported on the symmetric subspace (the symmetric subspace is defined as the space spanned by all states which are invariant under any permutation of the constituent subsystems). We describe the state of each qubit in terms of its Bloch vector representation

$$\rho = \tfrac{1}{2}(\mathbb{I} + \vec{s}\cdot\vec{\sigma}), \tag{3}$$

where $\mathbb{I}$ is the 2×2 identity matrix, $\vec{s}$ is the Bloch vector (with unit length for pure states) and σ_i are the Pauli matrices. By requiring that all input states must be treated in the same way (universality condition) it has been shown [3] that the reduced density operator describing the state of each of the M output qubits of a universal cloning transformation is related to the input one by the following shrinking transformation:

$$\rho_{\rm out} = \tfrac{1}{2}(\mathbb{I} + \eta_{\rm u}(N, M)\vec{s}\cdot\vec{\sigma}), \tag{4}$$

namely the Bloch vector is just shrunk by a shrinking factor $\eta_{\rm u}(N, M)$. Notice that the shrinking factor is simply related to the fidelity (2) as $F_{\rm u}(N, M) = (1 + \eta_{\rm u}(N, M))/2$.

In order to optimize the fidelity $F_{\rm u}(N, M)$, or equivalently the shrinking factor $\eta_{\rm u}(N, M)$, of an $N \to M$ universal cloning transformation we follow the approach of [4], relating universal cloning to state estimation. The

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aim of state estimation is to find a measurement procedure which leads to the best possible estimation of the *a priori* unknown quantum state $|\psi\rangle$. This is described in general in terms of positive-valued operator measures (POVM), namely a set of positive operators $\{P_\mu\}$, such that $\sum_\mu P_\mu = \mathbb{1}$ [5,6]. Suppose that we have at our disposal N copies of state $|\psi\rangle$. The outcome of each instance of the measurement provides, with probability $p_\mu(\psi) = \mathrm{Tr}(P_\mu|\psi\rangle\langle\psi|^{\otimes N})$, the 'candidate' $|\psi_\mu\rangle$ for $|\psi\rangle$. We can calculate the fidelity $F_{\mathrm{est}}(N)$ of state estimation by averaging over the outcomes of the measurement as follows:

$$F_{\mathrm{est}}(N) = \sum_\mu p_\mu(\psi)|\langle\psi|\psi_\mu\rangle|^2 = \langle\psi|\rho_{\mathrm{est}}|\psi\rangle, \qquad (5)$$

where $\rho_{\mathrm{est}} = \sum_\mu p_\mu(\psi)|\psi_\mu\rangle\langle\psi_\mu|$ represents the reconstructed density operator corresponding to state $|\psi\rangle$. For a universal state estimating procedure the fidelity must not depend on $|\psi\rangle$, thus the reconstructed density operator ρ_{est} can also be written as in equation (4), with shrinking factor $\eta_{\mathrm{est}}(N)$. It has been shown in [5] that the optimal fidelity $F_{\mathrm{est}}^{\mathrm{opt}}(N)$ for state estimation of N pure qubits has the form

$$F_{\mathrm{est}}^{\mathrm{opt}}(N) = \frac{N+1}{N+2}, \qquad (6)$$

corresponding to the optimal shrinking factor

$$\eta_{\mathrm{est}}^{\mathrm{opt}}(N) = N/(N+2). \qquad (7)$$

The connection we want to show between optimal universal cloning and optimal universal state estimation is given by the following equality:

$$F_{\mathrm{est}}^{\mathrm{opt}}(N) = F_{\mathrm{u}}^{\mathrm{opt}}(N,\infty). \qquad (8)$$

To prove it, we first consider a measurement procedure performed on N copies which is composed of an optimal $N \to L$ cloning process and a subsequent universal measurement on the L output copies. This total procedure can be regarded as a possible state estimation method. Since the output ϱ_L of the optimal universal cloner is supported on the symmetric subspace, it can be conveniently decomposed as [7]

$$\rho_L = \sum_i \gamma_i|\psi_i\rangle\langle\psi_i|^{\otimes L}, \qquad (9)$$

where the coefficients γ_i add up to one ($\sum_i \gamma_i = 1$) but are not necessarily positive. After performing the optimal universal measurement on the L outputs of the cloner we can calculate the average fidelity of the total process, due to linearity of the measurement procedure, as follows:

$$F_{\mathrm{total}}(N,L) = \sum_i \gamma_i \sum_\mu p_\mu(\psi_i)|\langle\psi_i|\psi_\mu\rangle|^2 \qquad (10)$$

$$= \sum_i \gamma_i \,\mathrm{Tr}\{[\eta_{\mathrm{est}}^{\mathrm{opt}}(L)|\psi_i\rangle\langle\psi_i| + (1-\eta_{\mathrm{est}}^{\mathrm{opt}}(L))\tfrac{1}{2}\mathbb{1}]|\psi\rangle\langle\psi|\}, \qquad (11)$$

where we explicitly exploited the universality of state estimation from equations (10) to (11). In the limit $L \to \infty$ we have $\eta_{\mathrm{est}}^{\mathrm{opt}}(\infty) = 1$ and remembering that at the output of the $N \to L$ cloner $\sum_i \gamma_i|\psi_i\rangle\langle\psi_i| = \eta_{\mathrm{u}}^{\mathrm{opt}}(N,L)|\psi\rangle\langle\psi| + \mathbb{1}(1-\eta_{\mathrm{u}}^{\mathrm{opt}}(N,L))/2$, the average fidelity can be written as

$$\lim_{L\to\infty} F_{\mathrm{total}}(N,L) = \mathrm{Tr}\{[\eta_{\mathrm{u}}^{\mathrm{opt}}(N,\infty)|\psi\rangle\langle\psi| + (1-\eta_{\mathrm{u}}^{\mathrm{opt}}(N,\infty))\tfrac{1}{2}\mathbb{1}]|\psi\rangle\langle\psi|\} = \tfrac{1}{2}[1+\eta_{\mathrm{u}}^{\mathrm{opt}}(N,\infty)]. \qquad (12)$$

This fidelity cannot be higher than the one for the optimal state estimation performed directly on N pure inputs, thus we conclude

$$F_{\mathrm{u}}^{\mathrm{opt}}(N,\infty) \leqslant F_{\mathrm{est}}^{\mathrm{opt}}(N). \qquad (13)$$

We can derive the opposite inequality by noticing that after performing a universal measurement procedure on N identically prepared input copies $|\psi\rangle$, we can prepare a state of L systems, supported on the symmetric subspace, where each system has the same reduced density operator, given by ρ_{est}. As mentioned above, a universal cloning process generates outputs that are supported on the symmetric subspace. Therefore, the above method of performing state estimation followed by preparation of a symmetric state can be viewed as a universal cloning process and thus it cannot lead to a higher fidelity than the optimal $N \to L$ cloning transformation. Therefore we find the inequality

$$F_{\mathrm{est}}^{\mathrm{opt}}(N) \leqslant F_{\mathrm{u}}^{\mathrm{opt}}(N,L), \qquad (14)$$

which holds for any value of L, in particular for $L \to \infty$. The above inequality, together with equation (13), leads to the equality (8).

An interesting property of universal cloning transformations is that the shrinking factors of universal cloning machines multiply [4], namely the shrinking factor of a universal $N \to L$ cloner composed of a sequence of an $N \to M$ cloner followed by an $M \to L$ cloner is the product of the two shrinking factors: $\eta_{\mathrm{u}}(N,M)\cdot\eta_{\mathrm{u}}(M,L)$. Moreover, since a sequence of an $N \to M$ and an $M \to \infty$ universal cloner cannot perform better than the optimal $N \to \infty$ universal cloner, we can write the following upper bound for an $N \to M$ cloner:

$$\eta_{\mathrm{u}}(N,M) \leqslant \frac{\eta_{\mathrm{u}}^{\mathrm{opt}}(N,\infty)}{\eta_{\mathrm{u}}^{\mathrm{opt}}(M,\infty)} = \frac{N(M+2)}{M(N+2)}, \qquad (15)$$

where we have used equation (7) on the right-hand side. The above bound is achieved by the cloning transformations proposed in [8].

3. Phase covariant cloning

In this section we start restricting the class of input states and consider only states of the form

$$|\psi_\phi\rangle = \tfrac{1}{\sqrt{2}}[|0\rangle + \mathrm{e}^{\mathrm{i}\phi}|1\rangle], \qquad (16)$$

where $\phi \in [0, 2\pi)$. Notice that this class of states corresponds to a Bloch vector lying on the x–y plane in the Bloch sphere representation. We are interested in cloning transformations that treat each input state belonging to this class in the same way, namely whose efficiency does not depend on the value of the phase ϕ. This requirement corresponds to imposing the following phase covariant condition on the operation of the cloning map C_{NM}:

$$U_\varphi \rho^{\rm out} U_\varphi^\dagger = R[C_{NM}(U_\varphi^N |\psi_\phi\rangle\langle\psi_\phi|^{\otimes N} U^{\dagger N}{}_\varphi)] \qquad (17)$$

for all input states $|\psi_\phi\rangle$ and for all unitary phase shift operators $U_\varphi = \exp[-\frac{\rm i}{2}(\sigma_z - 1)\varphi]$, where $\varphi \in [0, 2\pi)$ and σ_z is one of the Pauli operators (in the above equation R denotes the trace operation over all the output copies except one). Cloning transformations satisfying the above condition will be called phase covariant.

It can be shown [9] that phase covariant cloning transformations for input states $|\psi_\phi\rangle$ correspond to a shrinking of the Bloch vector by a factor $\eta_{\rm pc}(N, M)$ and that the sequence of two covariant cloners, the first taking N states $|\psi_\phi\rangle$ and giving M output copies, and the second taking the M output copies as input and generating L output copies, can be viewed as an $N \to L$ phase covariant cloner with shrinking factor $\eta_{\rm pc}(N, M) \cdot \eta_{\rm pc}(M, L)$. As in the case of universal cloning, the fidelity for phase covariant cloning $F_{\rm pc}(N, M)$ is related to the shrinking factor as $F_{\rm pc}(N, M) = (1 + \eta_{\rm pc}(N, M))/2$.

Moreover, similarly to the case of universal cloning, the following link between phase covariant cloning and phase estimation on qubits of the form $|\psi_\phi\rangle$ can be proved [9]:

$$F_{\rm pc}^{\rm opt}(N, \infty) = F_{\rm ph}^{\rm opt}(N), \qquad (18)$$

where $F_{\rm ph}^{\rm opt}(N)$ is fidelity of the reconstructed reduced density operator after performing optimal covariant phase estimation on N qubits in the pure state $|\psi_\phi\rangle$.

The fidelity for optimal covariant phase estimation for qubits in the state $|\psi_\phi\rangle$ takes the form [6]

$$F_{\rm ph}^{\rm opt}(N) = \frac{1}{2} + \frac{1}{2^{N+1}} \sum_{l=0}^{N-1} \sqrt{\binom{N}{l}\binom{N}{l+1}}. \qquad (19)$$

As for phase covariant cloners, the fidelity of covariant phase estimation is simply related to the shrinking factor as $F_{\rm ph}(N) = (1 + \eta_{\rm ph}(N))/2$ and, therefore, equality (18) also holds for the corresponding shrinking factors.

By concatenating an $N \to M$ and an $N \to \infty$ phase covariant cloner and exploiting equation (18) we can find the following upper bound on the fidelity of an $N \to M$ phase covariant cloner [9]:

$$\eta_{\rm pc}^{\rm opt}(N, M) \leqslant 2^{(M-N)} \frac{\sum_{l=0}^{N-1} \sqrt{\binom{N}{l}\binom{N}{l+1}}}{\sum_{j=0}^{M-1} \sqrt{\binom{M}{j}\binom{M}{j+1}}}. \qquad (20)$$

Note that while in the case of the universal cloning we know the explicit form of the cloning map which achieves bound (15), in the case of phase covariant cloners on qubits of the form $|\psi_\phi\rangle$ we know that the above bound can be achieved for the simplest case of the $1 \to 2$ cloner [9], but we do not know yet whether it can be achieved for any values of N and M. Notice that bound (20) is always greater than the optimal shrinking factor for the universal cloner (15), as expected.

4. Cloning of two nonorthogonal states

In this section we further restrict the class of input states and consider the smallest nontrivial one, namely a set of two nonorthogonal pure states. We parametrize the two states in the following way:

$$\begin{aligned} |a\rangle &= \cos\theta|0\rangle + \sin\theta|1\rangle \\ |b\rangle &= \sin\theta|0\rangle + \cos\theta|1\rangle, \end{aligned} \qquad (21)$$

where $\theta \in [0, \pi/4]$. The set of the two input states can equivalently be specified by means of their scalar product $S = \langle a|b\rangle = \sin 2\theta$.

We will present here a lower bound for the fidelity of an optimal $N \to M$ cloning transformation that operates on N input states of the form $|x\rangle^{\otimes N}$, with $x = a, b$, generalizing the results presented in [3] for the $1 \to 2$ case. The resulting transformation is also called a state-dependent cloner, because its form depends explicitly on the form of the initial states, namely on the parameter θ.

We will consider a unitary operator V_{NM} acting on the Hilbert space of M qubits and define the final states $|\alpha_{NM}\rangle$ and $|\beta_{NM}\rangle$ as

$$|\alpha_{NM}\rangle = V_{NM}(|a\rangle^{\otimes N} \otimes |0\rangle^{\otimes M-N}) \qquad (22)$$

$$|\beta_{NM}\rangle = V_{NM}(|b\rangle^{\otimes N} \otimes |0\rangle^{\otimes M-N}). \qquad (23)$$

Unitarity gives the following constraint on the scalar product of the final states:

$$\langle\alpha_{NM}|\beta_{NM}\rangle = (\langle a|b\rangle)^N = S^N. \qquad (24)$$

As a convenient criterion for optimality of the cloning transformation, we maximize the global fidelity $F_g(N, M)$ of both final states $|\alpha_{NM}\rangle$ and $|\beta_{NM}\rangle$ with respect to the perfect cloned states $|a^M\rangle \equiv |a\rangle^{\otimes M}$ and $|b^M\rangle \equiv |b\rangle^{\otimes M}$. The global fidelity is defined formally as

$$F_g(N, M) = \tfrac{1}{2}(|\langle\alpha_{NM}|a^M\rangle|^2 + |\langle\beta_{NM}|b^M\rangle|^2). \qquad (25)$$

It can be easily shown† that the above fidelity is maximized when the states $|\alpha_{NM}\rangle$ and $|\beta_{NM}\rangle$ lie in the two-dimensional space $\mathcal{H}_{a^M,b^M}$ which is spanned by vectors $\{|a^M\rangle, |b^M\rangle\}$.

We will now maximize explicitly the value of the global fidelity (25). We can think about it in a geometrical way and define χ, δ and γ as the 'angles' between vectors $|a^M\rangle$ and $|b^M\rangle$, $|a^M\rangle$ and $|\alpha_{NM}\rangle$, $|\alpha_{NM}\rangle$ and $|\beta_{NM}\rangle$, respectively. The global fidelity (25) then takes the form

$$F_g(N, M) = \tfrac{1}{2}(\cos^2\delta + \cos^2(\chi - \gamma - \delta)) \qquad (26)$$

and is thus maximized when the angle between $|a^M\rangle$ and $|\alpha_{NM}\rangle$ is equal to the angle between $|b^M\rangle$ and $|\beta_{NM}\rangle$, i.e. $\delta = \frac{1}{2}(\chi - \gamma)$. The optimal situation thus corresponds to the maximal symmetry in the disposition of the vectors. This disposition guarantees that the two possible input states $|a^N\rangle$ and $|b^N\rangle$ are treated in the same way, namely the

† The proof is analogous to the one reported in appendix B of [3] by simply replacing $|\alpha\rangle$ and $|\beta\rangle$ with $|\alpha_{NM}\rangle$ and $|\beta_{NM}\rangle$, $|aa\rangle$ and $|bb\rangle$ with $|a^M\rangle$ and $|b^M\rangle$, S^2 with S^M.

fidelity of the cloner is the same for both. By inserting the explicit definitions of the angles $\chi = \arccos(S^M)$ and $\gamma = \arccos(S^N)$, the optimal global fidelity then takes the form

$$F_g(N, M) = \tfrac{1}{2}\left(1 + S^{N+M} + \sqrt{1 - S^{2N}}\sqrt{1 - S^{2M}}\right). \quad (27)$$

In order to compare the efficiency of the state-dependent cloner with the one of universal and phase covariant cloners shown in the previous sections we will now derive the explicit expression of the fidelity of each output copy with respect to the initial state. We will first write the output states as

$$\begin{aligned} |\alpha_{NM}\rangle &= (A + B)|a^M\rangle + (A - B)|b^M\rangle \\ |\beta_{NM}\rangle &= (A - B)|a^M\rangle + (A + B)|b^M\rangle, \end{aligned} \quad (28)$$

where

$$A = \frac{1}{2}\sqrt{\frac{1 + S^N}{1 + S^M}} \qquad B = \frac{1}{2}\sqrt{\frac{1 - S^N}{1 - S^M}}. \quad (29)$$

From the above equations the reduced density operator ρ_α corresponding to one of the M output copies can be easily derived (notice that the global states of the M copies $|\alpha_{NM}\rangle$ and $|\beta_{NM}\rangle$ belong to the symmetric subspace, therefore each output copy is described by the same reduced density operator)

$$\begin{aligned} \rho_\alpha = {} & (A + B)^2|a\rangle\langle a| + (A - B)^2|b\rangle\langle b| \\ & + (A^2 - B^2)S^{M-1}(|a\rangle\langle b| + |b\rangle\langle a|). \end{aligned} \quad (30)$$

The fidelity is then calculated as

$$\begin{aligned} F_{\rm sd}(N, M) = {} & \langle a|\rho_\alpha|a\rangle = A^2(1 + S^2 + 2S^M) \\ & + B^2(1 + S^2 - 2S^M) + 2AB(1 - S^2). \end{aligned} \quad (31)$$

As mentioned above, notice that by the symmetry of the transformation the fidelity of the output state ρ_β with respect to the input $|b\rangle$ leads to the same result.

Notice that the fidelities for the cloner of nonorthogonal states (31) are just a lower bound. Actually, in order to find the optimal state-dependent cloner to be compared with the universal and phase covariant ones, the fidelity $F_{\rm sd}(N, M)$ should be maximized explicitly, and additional auxiliary systems interacting with the M qubits should, in general, be considered in the definition of the cloning transformation V_{NM}. In [3] it was shown that for the $1 \to 2$ case the maximization of $F_{\rm sd}(1, 2)$ leads to a different cloning transformation than the one considered here, where the global fidelity is maximized. However, the value of the resulting optimal fidelity is only slightly different from the fidelity reported in equation (32).

For the $1 \to 2$ cloner the fidelity $F_{\rm sd}(1, 2)$ was first derived in [3] and is given by

$$F_{\rm sd}(1, 2) = \frac{1}{2}\left[1 + \frac{1 - S^2}{\sqrt{1 + S^2}} + \frac{S^2(1 + S)}{1 + S^2}\right]. \quad (32)$$

As an illustration, we report here the explicit form of the bound (31) for the fidelity corresponding to the case of the $1 \to 3$ cloner

$$\begin{aligned} F_{\rm sd}(1, 3) = {} & \frac{1}{4}\Bigg[2S^3\left(\frac{1 + S}{1 + S^3} - \frac{1 - S}{1 - S^3}\right) \\ & + (1 + S^2)\left(\frac{1 + S}{1 + S^3} + \frac{1 - S}{1 - S^3}\right) + 2(1 - S^2)\sqrt{\frac{1 - S^2}{1 - S^6}}\Bigg]. \end{aligned} \quad (33)$$

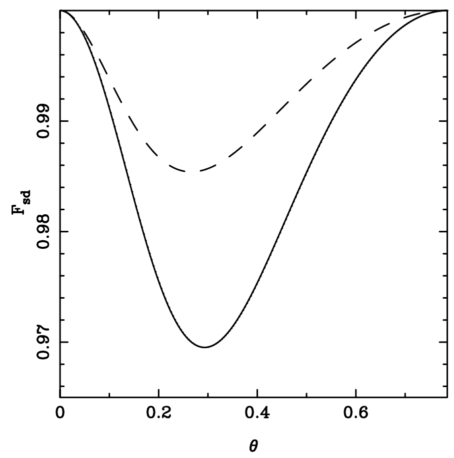

Figure 1. Fidelity for each output copy of the state-dependent cloner as a function of the parameter θ. The broken curve refers to the $1 \to 2$ cloner (equation (32)), while the full curve corresponds to the $1 \to 3$ cloner (equation (33)).

In figure 1 we show the fidelities for the $1 \to 2$ and the $1 \to 3$ cloners as functions of the parameter θ. The broken curve corresponds to $F_{\rm sd}(1, 2)$, the full curve to $F_{\rm sd}(1, 3)$. As expected, the values of the fidelity are always much higher than $F_{\rm u}(1, 2) = \frac{5}{6}$ and $F_{\rm pc}(1, 2) = (\sqrt{2} + 1)/2\sqrt{2}$ for the $1 \to 2$ optimal universal and phase covariant cloners respectively, and higher than $F_{\rm u}(1, 3) = \frac{7}{9}$ and $F_{\rm pc}(1, 3) = (7 + 2\sqrt{3})/[2(2\sqrt{3} + 3)]$.

We can also analyse the modulus of the Bloch vector of the output state, to see how much it is shrunk with respect to the initial one. For the $1 \to 2$ case it takes the form

$$|\vec{s}(1, 2)| = \sqrt{\frac{S^2(1 + S)^2}{(1 + S^2)^2} + \frac{1 - S^2}{1 + S^2}}, \quad (34)$$

while for the $1 \to 3$ case it is given by

$$\begin{aligned} |\vec{s}(1, 3)| = \Bigg\{ & \Bigg[2(A^2 + B^2) + 4AB\sqrt{1 - S^2} \\ & + 2S^3(A^2 - B^2) - 1\Bigg]^2 \\ & + [2(A^2 + B^2)S + 2S^2(A^2 - B^2)]^2\Bigg\}^{1/2}, \end{aligned} \quad (35)$$

where A and B are given in equation (29) with $N = 1$ and $M = 3$. These values are plotted in figure 2 as functions of the parameter θ. As in the previous figure, the broken curve corresponds to the $1 \to 2$ cloner while the full curve corresponds to the $1 \to 3$ cloner. Also, in this case, the values are much higher than the corresponding shrinking factors of the universal and the phase covariant cloners for all values of θ.

Notice, however, that differently from the universal and phase covariant cases in state-dependent cloning the Bloch vector of the input states is not simply shrunk along the direction of the input Bloch vector but is also rotated in the Bloch sphere.

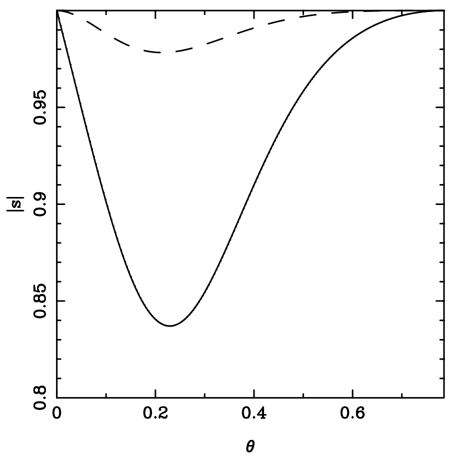

Figure 2. Modulus of the Bloch vector for each output copy of the state-dependent cloner as a function of the parameter θ. The broken curve refers to the $1 \rightarrow 2$ cloner (equation (34)), while the full curve corresponds to the $1 \rightarrow 3$ cloner (equation (35)).

5. Conclusions

We have analysed the operation of quantum cloning for two-state quantum systems†. We have shown that, as can be expected intuitively, its efficiency depends crucially on the class of allowed input states: the lowest optimal fidelity is achieved by universal cloners, which operate in the same way on the input state independently of its form, while higher values of the fidelity can be reached when the input state is known to belong to a particular set of states, such as, for example, states whose Bloch vector lies on a circle on the Bloch sphere or states which belong to a set of two known pure nonorthogonal states. For these two cases we have given, respectively, a lower and an upper bound on the fidelity of the cloning transformation. The ultimate limits that can be really achieved by explicit cloning transformations in these restricted cases are still an open problem.

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† Notice that this analysis does not include the type of cloners considered in [10], where the cloning transformation is exact but probabilistic.