

## Optimal cloning for two pairs of orthogonal states

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### Abstract

We study the optimal cloning transformation for two pairs of orthogonal states of two-dimensional quantum systems, and derive the corresponding optimal fidelities.

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The possibility of cloning quantum states approximately has attracted much attention in recent years. Limits for the efficiency of cloning transformations have been derived in various cases. For two-dimensional quantum systems the optimal fidelity has been reported for universal cloning [1–4], where the cloning transformation is optimized for the case that all possible states from the Bloch sphere are treated in the same way; for phase-covariant cloning [5], where the fidelity is optimized for states lying on a great circle of the Bloch sphere, and for the case of just two non-orthogonal states [6].

In this paper we study the optimal cloning for an ensemble of input states that consists of two pairs of orthogonal states for a two-dimensional quantum system. These four states can be parametrized in the Bloch sphere representation with a single parameter in the following way. The four Bloch vectors  $\vec{m}_i$  for the states  $|\psi_i\rangle$  with

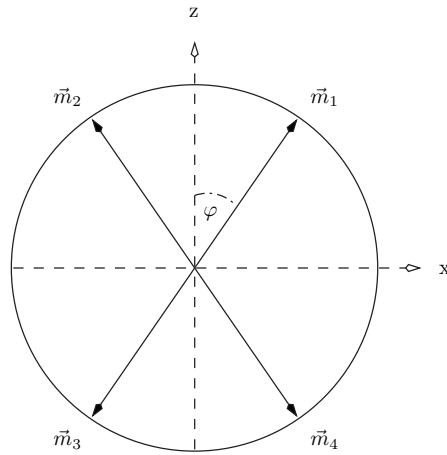
$$|\psi_i\rangle\langle\psi_i| = \frac{1}{2}(\mathbb{1} + \vec{m}_i \cdot \vec{\sigma}) \quad i = 1, \dots, 4 \quad (1)$$

where  $\mathbb{1}$  is the identity operator and  $\sigma_i$  with  $i = x, y, z$  are the Pauli matrices, are given by

$$\begin{aligned} \vec{m}_1 &= \begin{pmatrix} \sin \varphi \\ 0 \\ \cos \varphi \end{pmatrix} & \vec{m}_2 &= \begin{pmatrix} -\sin \varphi \\ 0 \\ \cos \varphi \end{pmatrix} \\ \vec{m}_3 &= \begin{pmatrix} -\sin \varphi \\ 0 \\ -\cos \varphi \end{pmatrix} & \vec{m}_4 &= \begin{pmatrix} \sin \varphi \\ 0 \\ -\cos \varphi \end{pmatrix}. \end{aligned} \quad (2)$$

In this representation the four vectors are lying in the  $x, z$ -plane, and each of them includes an angle  $\pm\varphi$  or  $\pm(\pi - \varphi)$  with the  $z$ -axis, see figure 1.

The two pairs of orthogonal states are given by  $\{|\psi_1\rangle, |\psi_3\rangle\}$  and  $\{|\psi_2\rangle, |\psi_4\rangle\}$ .



**Figure 1.** Geometrical disposition of two pairs of orthogonal states.

We could also parametrize the states  $|\psi_i\rangle$  with the real parameters  $\alpha$  and  $\beta$  with  $\alpha^2 + \beta^2 = 1$ :

$$\begin{aligned} |\psi_1\rangle &= \alpha|0\rangle + \beta|1\rangle & |\psi_2\rangle &= \alpha|0\rangle - \beta|1\rangle \\ |\psi_3\rangle &= \beta|0\rangle - \alpha|1\rangle & |\psi_4\rangle &= \beta|0\rangle + \alpha|1\rangle \end{aligned} \quad (3)$$

where the relation between the parameters  $\alpha$  and  $\varphi$  is given by

$$\alpha = \cos \frac{\varphi}{2}. \quad (4)$$

We study the case of  $1 \rightarrow 2$  cloning, namely two output copies are produced from a single input. In this case a cloning transformation is generally described by a unitary operation acting on the input, a prescribed blank qubit, and an auxiliary system, initially in an arbitrary state  $|X\rangle$ . In order to derive the optimal cloning transformation, it is sufficient to define its action on the basis states of the input, namely

$$\begin{aligned} U|0\rangle|0\rangle|X\rangle &= a|00\rangle|A\rangle + b(|01\rangle + |10\rangle)|B\rangle + c|11\rangle|C\rangle \\ U|1\rangle|0\rangle|X\rangle &= \tilde{a}|11\rangle|\tilde{A}\rangle + \tilde{b}(|10\rangle + |01\rangle)|\tilde{B}\rangle + \tilde{c}|00\rangle|\tilde{C}\rangle \end{aligned} \quad (5)$$

where the coefficients  $a, b, c$  can be taken real and positive by including possible phases into the ancilla states. The above form for the cloning transformation guarantees that the two output copies are described by the same reduced density operator. We study cloning transformations that lead to the same efficiency for the four states  $|\psi_i\rangle$ . Since the four states are transformed into one another by renaming the basis states, i.e.  $|0\rangle \leftrightarrow |1\rangle$ , the cloning transformation will be invariant under the exchange of  $|0\rangle$  and  $|1\rangle$ . This condition leads to  $a = \tilde{a}, b = \tilde{b}, c = \tilde{c}$ . Moreover, unitarity of the cloning transformation  $U$  dictates the condition

$$a^2 + 2b^2 + c^2 = 1. \quad (6)$$

We describe the efficiency of the cloning transformation in terms of the fidelity  $F$  of each output copy with respect to the input state, namely

$$F = \langle \psi_i | \rho_i | \psi_i \rangle \quad (7)$$

where  $\rho_i = \text{Tr}[U|\psi_i\rangle\langle\psi_i|U^\dagger]$  and the trace is performed over the auxiliary system and one of the output copies. With our symmetric way to parametrize the states we can easily derive

the fidelity for the four input states, as we just have to calculate the fidelity once and can then use symmetry arguments in order to find the explicit form of the other three cases, e.g. we can replace  $\beta$  by  $-\beta$  to go from the fidelity for  $|\psi_1\rangle$  to the fidelity for  $|\psi_2\rangle$ .

As mentioned above, we require the four fidelities to be equal. This condition leads to

$$F = a^2(\alpha^4 + \beta^4) + 2c^2\alpha^2\beta^2 + b^2 + \alpha^2\beta^2[ab \cdot 2 \operatorname{Re}(\langle A | \tilde{B} \rangle + \langle B | \tilde{A} \rangle) + bc \cdot 2 \operatorname{Re}(\langle B | \tilde{C} \rangle + \langle C | \tilde{B} \rangle)]. \quad (8)$$

Independently of the coefficients  $a, b, c$  the fidelity will be maximal for the following choice of scalar products between the auxiliary states:

$$\begin{aligned} \langle A | \tilde{B} \rangle &= 1 = \langle B | \tilde{A} \rangle \\ \langle B | \tilde{C} \rangle &= 1 = \langle C | \tilde{B} \rangle \end{aligned} \quad (9)$$

which can be reached with a two-dimensional ancilla and, e.g., the choice

$$\begin{aligned} |A\rangle &= |0\rangle & |B\rangle &= |1\rangle & |C\rangle &= |0\rangle \\ |\tilde{A}\rangle &= |1\rangle & |\tilde{B}\rangle &= |0\rangle & |\tilde{C}\rangle &= |1\rangle. \end{aligned} \quad (10)$$

Inserting this into equation (8) we arrive at

$$F = \frac{1}{2} + \frac{1}{2}(a^2 - c^2) \cos^2 \varphi + b(a + c) \sin^2 \varphi. \quad (11)$$

The optimal cloning transformation corresponds to the maximum value of the fidelity (11), together with the constraint (6) due to unitarity.

Using the method of Lagrange multipliers we thus have to solve the system of equations

$$\begin{aligned} a \cos^2 \varphi + b \sin^2 \varphi &= 2a\lambda \\ (a + c) \sin^2 \varphi &= 4b\lambda \\ -c \cos^2 \varphi + b \sin^2 \varphi &= 2c\lambda \\ a^2 + 2b^2 + c^2 &= 1 \end{aligned} \quad (12)$$

where  $\lambda$  is the Lagrange multiplier. The solution for the coefficients  $a, b$  and  $c$  turns out to be

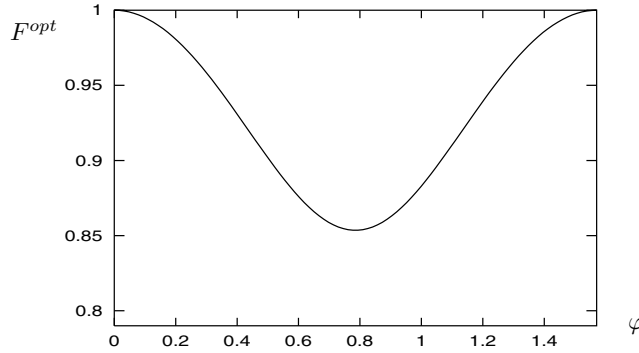
$$\begin{aligned} a &= \frac{1}{2} \left( 1 + \cos^2 \varphi \sqrt{\frac{1}{\sin^4 \varphi + \cos^4 \varphi}} \right) \\ b &= \frac{1}{2} \sin^2 \varphi \sqrt{\frac{1}{\sin^4 \varphi + \cos^4 \varphi}} \\ c &= \frac{1}{2} \left( 1 - \cos^2 \varphi \sqrt{\frac{1}{\sin^4 \varphi + \cos^4 \varphi}} \right). \end{aligned} \quad (13)$$

Inserting this into equation (11) leads to the optimum fidelity

$$F^{\text{opt}} = \frac{1}{2} \left( 1 + \sqrt{\sin^4 \varphi + \cos^4 \varphi} \right). \quad (14)$$

The explicit form of the resulting optimal cloning transformation is found immediately by inserting equations (13) and (10) into equation (5).

In figure 2 we plot  $F^{\text{opt}}$  as a function of the angle  $\varphi$ . The figure demonstrates that the cloning task is performed in the worst way for the two pairs being maximally spread, i.e. in the



**Figure 2.** Optimal fidelity for cloning two pairs of orthogonal states, as a function of  $\varphi$ .

case  $\varphi = \pi/4$ . This is the well known setting for quantum cryptography in the BB84 scheme [7]. Notice that the value for the optimal fidelity of cloning in the BB84 scheme, derived in [5], is recovered here. As the ability to make approximate clones of a state is related to the security of a cryptographic protocol, our calculations indicate that the BB84 scheme is the most favourable setting when using four states for cryptography.

We point out the following geometrical description of the cloning transformation. For states with a Bloch vector lying on the  $x$ - $z$  plane of the Bloch sphere, namely states given by the density operator  $\rho = \frac{1}{2}(\mathbb{1} + m_x\sigma_x + m_z\sigma_z)$ , we can describe the cloning transformation (5) in terms of two shrinking factors  $\eta_x$  for the  $x$ -component of the Bloch vector, and  $\eta_z$  for its  $z$ -component, such that the output state of each copy takes the form  $\rho_{\text{out}} = \frac{1}{2}(\mathbb{1} + \eta_x m_x \sigma_x + \eta_z m_z \sigma_z)$ . The explicit expression for the two shrinking factors with our choice of ancillas (10) is given by

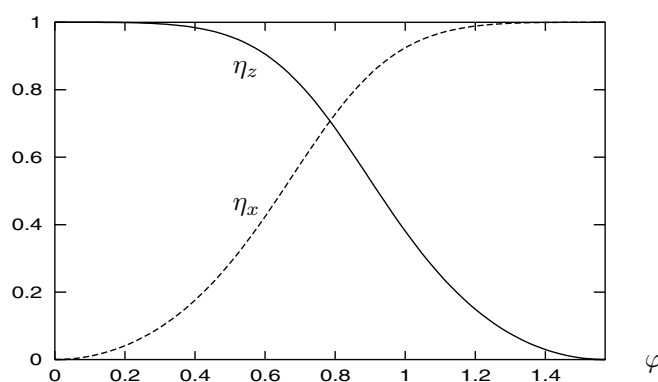
$$\begin{aligned}\eta_x &= 2b(a+c) \\ \eta_z &= a^2 - c^2.\end{aligned}\tag{15}$$

In the case of the optimal transformation, according to equation (13), the shrinking factors depend only on the value of  $\varphi$ :

$$\begin{aligned}\eta_x &= \sin^2 \varphi \sqrt{\frac{1}{\sin^4 \varphi + \cos^4 \varphi}} \\ \eta_z &= \cos^2 \varphi \sqrt{\frac{1}{\sin^4 \varphi + \cos^4 \varphi}}.\end{aligned}\tag{16}$$

These shrinking factors are reported in figure 3. Notice that they become equal for  $\varphi = \pi/4$ , namely  $\eta_x(\pi/4) = \eta_z(\pi/4) = 1/\sqrt{2}$ , and according to the symmetry of the input ensemble (2) that we used to perform the optimization they are related as  $\eta_x(\varphi) = \eta_z(\pi/2 - \varphi)$ . Furthermore, the identity  $\eta_x^2 + \eta_z^2 = 1$  holds.

In summary, we have found the best cloning transformation and the corresponding optimal fidelity for cloning two pairs of orthogonal states, and mentioned the implications for choosing such an ensemble of states for quantum cryptography. A further application of our results could be the connection to state estimation for this set of inputs, as optimal quantum cloning can be part of an optimal state estimation process.



**Figure 3.** Shrinking factors  $\eta_x$  and  $\eta_z$  for cloning two pairs of orthogonal states, plotted as a function of  $\varphi$ .

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