

Detection of entanglement with few local measurements

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(Received 15 May 2002; published 9 December 2002)

We introduce a general method for the experimental detection of entanglement by performing only few local measurements, assuming some prior knowledge of the density matrix. The idea is based on the minimal decomposition of witness operators into a pseudomixture of local operators. We discuss an experimentally relevant case of two qubits, and show an example how bound entanglement can be detected with few local measurements.

DOI: 10.1103/PhysRevA.66.062305

PACS number(s): 03.67.Dd, 03.65.Ud, 03.67.Hk

A central aim in the physics of quantum information is to create and detect entanglement—the resource that allows to realize various quantum protocols. Recently, much progress has been achieved experimentally in creating entangled states [1]. In every real experiment noise and imperfections are present so that the generated states, although intended to be entangled, may in fact be separable. Therefore, it is important to find efficient experimental methods to test whether a given imperfect state ρ is indeed entangled.

Obviously, the ultimate goal of entanglement detection is to characterize entanglement quantitatively, and identify regions in the parameter space which allow to maximize entanglement for a particular quantum information processing task. The first step towards this ambitious goal is to detect whether a given state is entangled or not.

The question of direct detection of quantum entanglement has been recently addressed in Refs. [2–4]. In [3,4] the authors study the case of mixed states and find efficient ways to estimate the entanglement of an unknown state. Their method is based on structural approximations of some linear maps followed by a spectrum estimation. Although experimentally viable the method is not very easy to implement and it requires further modifications in order to be performed by local measurements [5]. Here, we approach the same problem from a different perspective. We use special observables, the so-called witness operators [6,7] and their optimal decomposition into a sum of *local* projectors. Note that in this way we answer an open question posed recently in Ref. [8], where *nonlocal* measurements of entanglement witnesses were studied.

The construction of a witness for a given arbitrary state is, in general, a formidable task. It can, however, be accomplished in typical experimental situations where one has some *a priori* information about the density matrix. This is always the case when the experiment is *aimed* at producing a certain state, rather than checking properties of an *a priori* unknown state. We discuss two experimentally relevant situations in this paper, namely, the generation of a definite pure entangled state of two parties, and the generation of a specific bound entangled edge state. In both cases our method can be applied in arbitrary dimensions.

Having constructed a witness, its measurement can be performed *locally*, since every observable can be decomposed in terms of a product basis in the operator space. Here, we propose two ways of optimizing such local measurements. The first one consists in looking for the optimal number of local projectors (ONP). The second one consists in searching for the optimal number of settings of detecting devices (ONS). By a setting of the device of a single observer we understand here the choice of the local orthonormal basis in the corresponding Hilbert space. The device then measures simultaneously projections onto the vectors belonging to the basis; the set of these projectors forms a complete set of commuting observables [9]. A setting of the devices for a pair of observers corresponds then to a correlated choice of individual settings.

Both optimization methods are formulated as the problem of decomposing a given operator into a sum of projectors on product states with an optimal number of terms. No general solutions for this kind of problems are known so far.

Before describing the details of our method, let us briefly discuss other methods of entanglement detection with local measurements. For systems of two qubits or of one qubit and one qutrit, a necessary and sufficient criterion for entanglement, namely, the nonpositivity of the partial transpose [10], is known. Thus, using tomography of the state ρ , which can be achieved with local measurements [11], one can fully determine ρ , calculate the partial transpose, and check its positivity. However, for two qubits this approach requires nine different settings of the measuring devices in order to determine 15 parameters describing the state in general. In our example below, only three settings suffice to detect entanglement if we have certain knowledge about ρ and we optimize the local decomposition of the corresponding witness. Our approach has similar advantages with respect to the detection of *entanglement visibility* [12], which in principle requires a continuous family of devices' settings.

Another way of detecting entanglement by local measurements consists in a test of a generalized Bell inequality [13,14], with which our approach has a formal similarity. Nevertheless, there exist many entangled states which do not violate any known Bell inequality [15]. However, for any given state one can *always* find a witness operator and its

local decomposition, such that the entanglement of this state can be detected locally. For the situations, in which a previous knowledge of ρ allows to construct a suitable witness, our method is more powerful than a test of a Bell inequality.

We introduce and illustrate our method in the scenario of creating and measuring the entanglement of two qubits, and then discuss shortly the generalization to higher-dimensional states, including bound entangled states.

Let us consider an experiment that produces the following convex combination of a desired pure entangled state $|\psi\rangle\langle\psi|$ and a mixed state σ representing some noise,

$$\varrho = p|\psi\rangle\langle\psi| + (1-p)\sigma, \quad 0 \leq p \leq 1, \quad (1)$$

where $|\psi\rangle$ can be written in the Schmidt decomposition as $|\psi\rangle = a|01\rangle + b|10\rangle$ with $a, b > 0$ and $a^2 + b^2 = 1$. The noise σ is assumed to be within a ball of radius d around the totally mixed state, i.e., $\|\sigma - 1/4\| \leq d$. Here, $\|A\| := \sqrt{\text{Tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm for operators A on the Hilbert space. One neither knows the probability p , nor the exact shape of σ . The task is to determine whether ϱ is entangled or not.

Let us briefly summarize the well-known concept of entanglement witnesses [6,7]: a density matrix ϱ is entangled iff there exists a Hermitian operator W such that $\text{Tr}(W\varrho) < 0$, but for all separable states $\text{Tr}(W\varrho_{sep}) \geq 0$ holds. In this sense W “detects” the entanglement of ϱ . Note that W has at least one negative eigenvalue. Methods to construct entanglement witnesses have been presented in Refs. [6,7]. For states with a nonpositive partial transpose there is a simple and straightforward construction of W : let $|e_-\rangle$ be the eigenvector of ϱ^{T_A} that corresponds to its minimal (negative) eigenvalue, namely, $\varrho^{T_A}|e_-\rangle = \lambda_{min}|e_-\rangle$, with $\lambda_{min} < 0$. Here T_A refers to partial transposition with respect to the first subsystem. Thus, $W = (|e_-\rangle\langle e_-|)^{T_A}$ detects the entanglement of ϱ , as $\text{Tr}(|e_-\rangle\langle e_-|^{T_A}\varrho) = \text{Tr}(|e_-\rangle\langle e_-|\varrho^{T_A}) = \lambda_{min} < 0$.

This witness is tangent to the set of separable states and already optimal [7], i.e., there is no witness that detects other states in addition to the ones detected by W .

For ϱ given in Eq. (1) and the case $d=0$ one finds $\lambda_{min} = (1-p)/4 - pab$, and $|e_-\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$, i.e., the witness is given by

$$W = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

Note that this witness neither depends on p , nor on a . Hence, as long as the experimental apparatus produces *any* superposition of $|01\rangle$ and $|10\rangle$ plus white noise, W will be a suitable operator. For $d \neq 0$ it still provides the possibility of entanglement detection.

Let us point out that this witness is also suitable for other physical scenarios. For example, consider the case where the noise mechanisms in the experimental setup are characterized by memory effects, or the case where the entangled state $|\psi\rangle$, [defined via Eq. (1)], is generated perfectly and then sent through a transmission channel with correlated noise. If

the noise mechanisms acting on the state can be described as a depolarising channel with some correlations of strength μ [16], then the resulting state will be of the form:

$$\varrho = \{1 \otimes 1 + \eta(a^2 - b^2)[\sigma_z \otimes 1 - 1 \otimes \sigma_z] + [\mu + (1 - \mu)\eta^2] \times [-\sigma_z \otimes \sigma_z + 2ab(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)]\}/4. \quad (3)$$

Here η and μ describe the depolarization and the degree of memory introduced by the noise process. This family of states is now characterized by three independent parameters: a , η , and μ . Remarkably, W turns out to detect the entanglement of this whole family of states. This is proven by calculating the range of the family’s three parameters, where ϱ^{T_A} has a negative eigenvalue, and showing that for this range $\text{Tr}(W\varrho) < 0$ holds.

We look now for a decomposition of the witness into a sum of projectors onto product vectors, i.e.,

$$W = \sum_i c_i |a_i, b_i\rangle\langle a_i, b_i| = \sum_i c_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad (4)$$

where the coefficients c_i are real and fulfill $\sum_i c_i = 1$. Note that at least one coefficient has to be negative—this characterizes a so-called *pseudomixture*. Any bipartite Hermitian operator can be decomposed in projectors onto product states, like in Eq. (4), in many different ways. However, we are interested in finding the *optimal* decompositions in the two ways described above. Optimal pseudomixtures in the sense of minimizing the number of nonvanishing c_i represent an ONP. They have been studied for two qubits in Ref. [17] where it was shown that any general vector $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ with α, β real and different from zero, and $\alpha^2 + \beta^2 = 1$, can be decomposed minimally with five terms

$$(|\phi\rangle\langle\phi|)^{T_A} = \frac{(\alpha + \beta)^2}{3} \sum_{i=1}^3 |f_i f_i\rangle\langle f_i f_i| - \alpha\beta(|01\rangle\langle 01| + |10\rangle\langle 10|), \quad (5)$$

where we have used the definitions

$$\begin{aligned} |f_1\rangle &= e^{-i(\pi/3)} \cos \theta |0\rangle + e^{i(\pi/3)} \sin \theta |1\rangle = |f_2^*\rangle, \\ |f_3\rangle &= \cos \theta |0\rangle + \sin \theta |1\rangle = |f_1\rangle + |f_2\rangle, \\ \cos \theta &= \sqrt{\alpha/(\alpha + \beta)}, \quad \sin \theta = \sqrt{\beta/(\alpha + \beta)}. \end{aligned} \quad (6)$$

Here $*$ denotes complex conjugation. The case $\alpha = 1/\sqrt{2} = -\beta$ corresponds exactly to the decomposition of W in Eq. (2) that we are looking for. Such a decomposition into five terms requires four different settings of the measuring device (in the sense described above and in the footnote [9]). It is therefore optimal in the number of projectors (ONP), but *not* optimal with respect to the number of correlated devices’ settings (ONS). Indeed, the witness W considered in Eq. (2) can be optimally implemented by using only three settings. This can be shown as follows: defining the eigenstates of the

Pauli matrices as $|z^+\rangle=|0\rangle$, $|z^-\rangle=|1\rangle$, $|x^\pm\rangle=(1/\sqrt{2})\times(|0\rangle\pm|1\rangle)$, and $|y^\pm\rangle=(1/\sqrt{2})(|0\rangle\pm i|1\rangle)$, we find

$$\begin{aligned} (|\phi\rangle\langle\phi|)^{TA} &= \alpha^2|z^+z^+\rangle\langle z^+z^+| + \beta^2|z^-z^-\rangle\langle z^-z^-| \\ &+ \alpha\beta(|x^+x^+\rangle\langle x^+x^+| + |x^-x^-\rangle\langle x^-x^-| \\ &- |y^+y^-\rangle\langle y^+y^-| - |y^-y^+\rangle\langle y^-y^+|). \end{aligned} \quad (7)$$

Note that this decomposition contains six terms, but only three correlated devices' settings. Alice and Bob have to classically correlate their measurements in their respective x, y and z directions as indicated in Eq. (7), and add the resulting expectation values with the according positive or negative weight in order to determine $\text{Tr}(W\rho)$. Let us note in passing that, in general, an entanglement witness does not provide an entanglement measure. However, for fixed ab and $d=0$ the probability p in Eq. (1) can be found from the measurement outcome via the relation $p=[1-4\text{Tr}(W\rho)]/(1+4ab)$.

The pseudomixture (7) provides an ONS, since it is impossible to decompose $(|\phi\rangle\langle\phi|)^{TA}$ with less than three devices' settings. Let us assume the contrary, i.e.,

$$\begin{aligned} (|\phi\rangle\langle\phi|)^{TA} &= \sum_{i,j=1}^2 U_{ij}|u_i^A\rangle\langle u_i^A| \otimes |u_j^B\rangle\langle u_j^B| \\ &+ \sum_{i,j=1}^2 V_{ij}|v_i^A\rangle\langle v_i^A| \otimes |v_j^B\rangle\langle v_j^B|, \end{aligned} \quad (8)$$

where $\langle u_i^A|u_i^A\rangle = \delta_{ii} = \langle v_i^A|v_i^A\rangle$ and the same holds for B . We expand $(|\phi\rangle\langle\phi|)^{TA} = \sum_{i,j=0}^3 \lambda_{ij} \sigma_i \otimes \sigma_j$, where we denote $\sigma_0 = \mathbb{1}$, and

$$(\lambda_{ij}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{\alpha^2 - \beta^2}{4} \\ 0 & \frac{\alpha\beta}{2} & 0 & 0 \\ 0 & 0 & \frac{\alpha\beta}{2} & 0 \\ \frac{\alpha^2 - \beta^2}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \quad (9)$$

Note that the 3×3 submatrix in the right bottom corner is of rank three. Now we write any projector in the right hand side (rhs) of Eq. (8) with Bloch vectors: the projector $|u_1^A\rangle\langle u_1^A| = \sum_{i=0}^3 s_i^A \sigma_i$ is represented by the vector $\vec{s}^A = (1/2)(1, s_1^A, s_2^A, s_3^A)$, and $|u_2^A\rangle\langle u_2^A|$ by $\vec{s}_1^A = (1/2)(1, -s_1^A, -s_2^A, -s_3^A)$. Expanding the first sum on the rhs of Eq. (8) in the $(\sigma_i \otimes \sigma_j)$ basis leads to a 3×3 submatrix in the right bottom corner which is proportional to $(U_{11} - U_{12} - U_{21} + U_{22})(s_1^A, s_2^A, s_3^A)^T (s_1^B, s_2^B, s_3^B)$, and thus of rank one. The corresponding matrix from the second sum on the rhs of Eq. (8) is also of rank one, and we arrive at a contradiction: no matrix of rank three can be written as a sum of two matrices of rank one. ■

We now emphasize the power of witness operators as a tool for the detection of entanglement by discussing the noise in Eq. (1) in some more detail. When $d \neq 0$, the state ρ lies

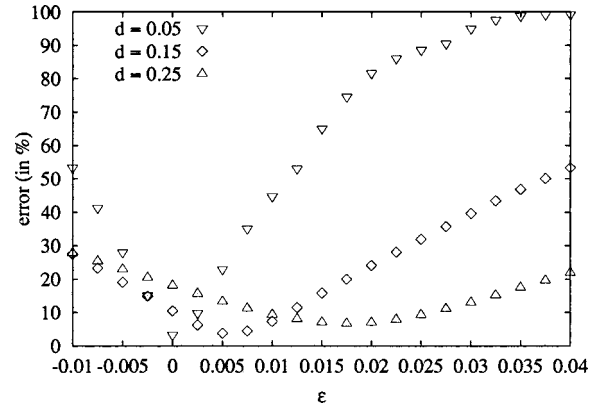


FIG. 1. Error (maximised over all p) as a function of ε , for three different values of d .

within a ball $B_{p,d}$ with radius $(1-p)d$. If p is such that this ball is either included in the set of separable states, or in the set of entangled states, then the given W is optimal and the sign of $\text{Tr}(W\rho)$ provides a signature of entanglement versus separability. If, however, the ball lies across the boundary between those two sets, errors may occur. Different questions can then be addressed: first, one may want to be sure that a given state is separable. For the case $a=b=1/\sqrt{2}$ (which we assume here and in the following), we can estimate a lower bound τ such that if $\text{Tr}(W\rho) \geq \tau$, then $\rho(p,d)$ is necessarily separable. This bound, which depends on d , is given by:

$$\tau(d) = \frac{1}{4} - d^2 - \sqrt{\left(\frac{1}{12} - d^2\right)\left(\frac{3}{4} - d^2\right)}. \quad (10)$$

For any τ' with $0 \leq \tau' < \tau$, there exists an entangled state $\rho(p,d)$ with $\text{Tr}(W\rho) = \tau'$. To derive Eq. (10), one uses the fact that there is a ball \mathcal{B} of separable states of maximal radius $1/\sqrt{12}$ around $1/4$. Since we do not know p we can only say that $\rho \in \cup_{p \in [0,1]} B_{p,d}$, which is a kind of a convex cone originating in $|\psi\rangle\langle\psi|$ and terminating in the ball $B_{0,d}$ of radius d around the totally mixed state. The bigger $\text{Tr}(W\rho)$, the closer to the ball $B_{0,d} \subset \mathcal{B}$ is ρ . Obviously, if $\text{Tr}(W\rho)$ is big enough we have $\rho \in \mathcal{B}$. In this manner, one can determine the value of τ . ■

Second, one may be interested in minimizing the probability of making an error, either by mistaking a separable state for an entangled one, or vice versa. If we assign $\text{Tr}(W\rho) > 0 \Leftrightarrow \rho$ separable, it turns out that—depending on the value of d —in order to minimize this error, it is more favorable to use $W_\varepsilon := W - \varepsilon \mathbb{1}$. This operator is not a “witness” in the original sense, because it yields negative expectation values for some separable states. Note that it requires the same measurement settings as W . To estimate the error inherent in this detection scheme, we have used the method from Ref. [18] to randomly generate a sample of 50 000 density matrices of the form (1), and then checked their separability using both the partial transposition criterion and applying W_ε . The percentage of errors when using W_ε is plotted in Fig. 1. For large d the operators W_ε are in fact less

erroneous in detecting entanglement. Further, numerical analysis suggests that the optimal ϵ increases quadratically with d .

Generalizing the above results to higher dimensions for states with nonpositive partial transpose states is possible, although not straightforward. In an $N \times M$ dimensional Hilbert space with $N \leq M$, we first have to identify the vector $|\phi\rangle$ that corresponds to the minimal (negative) eigenvalue of Q^{TA} . Without losing generality, we may assume that it has maximal Schmidt rank, and is given by $|\phi\rangle = \sum_{i=0}^{N-1} \alpha_i |ii\rangle$. Obviously, the ONP corresponding to $(|\phi\rangle\langle\phi|)^{TA}$ must contain at least N^2 terms since the rank of $(|\phi\rangle\langle\phi|)^{TA}$ is: N^2 . For $N=2$, i.e., for $2 \times M$ dimensional systems the results obtained for $M=2$ are also valid for $M>2$, since the maximal Schmidt rank in such spaces is 2. This implies that the ONP must contain five terms, whereas the ONS can be realized with three settings. For $N \times M$ systems with $N \geq 3$, $N \leq M$, we can easily construct a pseudomixture with $2N^2 - N$ terms using the same method as in the case of 2×2 systems. This gives the upper bound for the number of terms in ONP, and corresponds to an upper bound for ONS of $2N-1$ ($2N$) for even (odd) N . By generalizing the method used to demonstrate that two settings are not enough in 2×2 to the $N \times M$ case one can prove that any ONS must contain at least $N+1$ settings. It is not clear, however, whether this bound can be reached in general.

In higher dimensions there also exist entangled states with positive partial transpose, namely bound entangled states [19,20]. For this type of states no general operational entanglement criterion is known, and thus even the full knowledge of the density matrix may not suffice to decide whether the density matrix is entangled or not. There exists, however, an important class of bound entangled states, the so-called ‘‘edge’’ states, for which optimal witness operators can be constructed explicitly. A state δ is called an edge state iff it cannot be represented as $\delta = q\delta' + (1-q)\sigma$, where σ is a separable state, δ' is a state with positive partial transpose, and $0 \leq q < 1$. Edge states are, in a certain sense, the bound entangled analogs of pure entangled states. In the situation where an experiment is aimed at the generation of an edge state, our method of local decomposition of a witness provides a genuine experimental test.

Let us illustrate this with the example of unextendible product basis (UPB) states [21] in a 3×3 dimensional space. The states

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), & |\psi_2\rangle &= \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle, & |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle, \\ |\psi_4\rangle &= \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle) \end{aligned} \quad (11)$$

form a UPB, i.e., they are orthogonal to each other and there exists no product vector which is orthogonal to all of them.

The state $\rho_{BE} = (1/4)(1 - \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i|)$ constructed from this UPB is an entangled state with positive partial transpose. The generic form of an entanglement witness for such a state is [6,7]

$$W = (P + Q^{TA})/2 - \epsilon \mathbb{1}, \quad (12)$$

$$\epsilon = \inf_{|e,f\rangle} \langle e,f | P + Q^{TA} | e,f \rangle / 2,$$

where P and Q denote the projectors onto the kernel of ρ_{BE} and the kernel of ρ_{BE}^{TA} , respectively. For the given UPB state, we have $P = Q^{TA} = \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i|$. The main problem for the construction of W is to find ϵ . An analytical bound obtained by Terhal [6] gives $\epsilon \geq 0.0013$. Numerical analysis leads, however, to the much bigger value $\epsilon \approx 0.0284$. Once ϵ is found, the decomposition of W is straightforward: the explicit form of the witness fixes five elementary measurements, and the identity can be decomposed into nine orthogonal projectors onto product vectors such that four of them coincide with four of the UPB states. This follows from the fact that the vectors $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ can be extended to an orthonormal basis by defining

$$\begin{aligned} |\bar{\psi}_4\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle), & |\bar{\psi}_5\rangle &= \frac{1}{\sqrt{2}}|2\rangle(|1\rangle + |2\rangle), \\ |\bar{\psi}_6\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|2\rangle, & |\bar{\psi}_7\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)|0\rangle, \\ |\bar{\psi}_8\rangle &= |1\rangle|1\rangle. \end{aligned} \quad (13)$$

Altogether we are left with a pseudomixture that contains ten projectors. Denoting by $B_1 = \{|0\rangle, |1\rangle, |2\rangle\}$, $B_2 = \{(|0\rangle - |1\rangle)/\sqrt{2}, |2\rangle, (|0\rangle + |1\rangle)/\sqrt{2}\}$, $B_3 = \{(|1\rangle - |2\rangle)/\sqrt{2}, |0\rangle, (|1\rangle + |2\rangle)/\sqrt{2}\}$, and $B_4 = \{(|0\rangle - |1\rangle)/\sqrt{2}, (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}, (|0\rangle + |1\rangle - 2|2\rangle)/2\}$, we easily see that measurement of W for this decomposition requires six correlated settings for Alice and Bob: $B_1B_2, B_2B_1, B_1B_3, B_3B_1, B_4B_4$, and B_1B_1 . This result implies that the ONS must be ≤ 6 . By subtracting in Eq. (12) some positive operator I of full rank instead of the identity $\mathbb{1}$, one can reduce the number of projectors in the decomposition of W to nine, and this gives an ONP, since the number of terms in any ONP must be larger or equal than the rank of the witness, which is equal to nine. The idea is to form I as a convex sum of projectors onto $|\psi_i\rangle_{i=0,\dots,4}$ and onto four other product vectors that are obviously not orthogonal to $|\psi_i\rangle_{i=0,\dots,4}$, but such that the set of the nine vectors forms a basis. Here the bound for ϵ has to be adapted to the choice of I , such that positivity of W on all separable states is guaranteed, i.e.,

$$\epsilon' = \inf_{|e,f\rangle} \frac{\langle e,f | (P + Q^{TA})/2 | e,f \rangle}{\langle e,f | I | e,f \rangle}. \quad (14)$$

If we choose as additional vectors $|\bar{\psi}_i\rangle_{i=4,\dots,7}$ the decomposition contains nine projectors in only five settings. Numerical analysis leads to $\epsilon' \approx 0.0311$ for this choice of I . Note that when the bound entangled state is affected by

white noise, namely, $\rho_p = p\rho_{BE} + (1-p)\mathbb{1}/9$, the witness given above is still suitable for the detection of entanglement, provided $\text{Tr}(W\rho_p) < 0$. For the witness in Eq. (12) this is the case when $p > (1-9\epsilon/5)$. Let us mention that for experimental purposes it is not necessary to decompose the identity in Eq. (12) since this term will only add a constant $(-\epsilon)$ to the outcome of the measurement of the “prewitness” $\bar{W} = (P + Q^{TA})/2$. This implies that indeed one only needs to find the corresponding ONP and ONS for \bar{W} . In this way one obtains an ONP with only five terms, and that the ONS has to be less or equal to five.

In summary, we have introduced optimal decompositions of witness operators into local projectors for the detection of entanglement. This method can be used with present experimental techniques. It is a very powerful method to detect entanglement in the cases where one has a certain knowledge about the state that one wants to create, e.g., when the aim is to produce a specific pure entangled state, but this state is corrupted by noise. At the present stage of quantum information processing, several experiments strive at creating such

entangled states, and thus it is important to show that the produced state is indeed entangled. The more knowledge about the state is given, the less knowledge about the underlying noise is necessary for unambiguous classification of the state. For the situation in which little knowledge about the created state exists it will be favorable and maybe necessary to utilize more than one witness operator. A detailed analysis of the trade-off between the initial knowledge of the state and the witnesses needed is left for further research.

Note added. Recently, we became aware of a recent work by A. Pittenger and M. Rubin [22], where the ideas of this paper have been further developed. In particular it is shown there that if N is prime, a projector onto a maximally entangled state with full Schmidt rank can be measured with $N+1$ local measurements, so our bound can be reached for this case.

We wish to thank I. Cirac, S. Haroche, S. Huelga, B. Kraus, H. Weinfurter, and K. Życzkowski for discussions. This work has been supported by DFG (Graduiertenkolleg 282 and Schwerpunkt “Quanteninformationsverarbeitung”), the ESF-Program PESC, and the EU IST-Program EQUIP.

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- [1] W. Tittel *et al.*, Phys. Rev. Lett. **81**, 3563 (1998); G. Weihs *et al.*, *ibid.* **81**, 5039 (1998); E. Hagley *et al.*, *ibid.* **79**, 1 (1997); C.A. Sackett *et al.*, Phys. Rev. Lett. **404**, 256 (2000); A. Furusawa *et al.*, Science **282**, 706 (1998); J.C. Howell, A. Lamas-Linares, and D. Bouwmeester, Phys. Rev. Lett. **88**, 030401 (2002); H. Weinfurter (private communication).
- [2] J.M.G. Sancho and S.F. Huelga, Phys. Rev. A **61**, 042303 (2000).
- [3] P. Horodecki and A. Ekert, e-print quant-ph/0111064.
- [4] P. Horodecki e-print quant-ph/0111082.
- [5] A. Ekert and P. Horodecki (private communication).
- [6] B. Terhal, Linear Algebr. Appl. **323**, 61 (2001); see also Ref. [10](b).
- [7] M. Lewenstein *et al.*, Phys. Rev. A **62**, 052310 (2000); *ibid.* **63**, 044304 (2001); D. Bruß *et al.*, J. Mod. Opt. **49**, 1399 (2002).
- [8] R. Filip, Phys. Rev. A **65**, 062320 (2002).
- [9] For detection of spin S particles a Stern-Gerlach device can be used, for which a setting corresponds to the choice of the direction of the magnetic field and detection of each of the possible $2S+1$ outputs.
- [10] (a) A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); (b) M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
- [11] A.G. White *et al.*, Phys. Rev. Lett. **83**, 3138 (1998).
- [12] G. Jaeger, M.A. Horne, and A. Shimony, Phys. Rev. A **48**, 1023 (1993); J. Volz, C. Kurtsiefer, and H. Weinfurter, Appl. Phys. Lett. **79**, 869 (2001).
- [13] J. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1965); J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969); for a review of recent results see R. F. Werner; and M. M. Wolf; e-print quant-ph/0107093.
- [14] B. Terhal, Phys. Lett. A **271**, 319 (2000).
- [15] For example, all bound entangled states with positive partial transposes with respect to all possible groupings of the involved observers into two parties do not violate any Bell inequality involving two dichotomic observables for each of the observers. It has been even conjectured that such states admit a local hidden variable model; A. Peres, Found. Phys. **29**, 589 (1999); R.F. Werner and M.M. Wolf, e-print quant-ph/9910063; see also R.F. Werner and M.M. Wolf, e-print quant-ph/0102024; M. Żukowski and C. Brukner, e-print quant-ph/0102039.
- [16] C. Macchiavello and G.M. Palma, Phys. Rev. A **65**, 050301 (2002).
- [17] A. Sanpera, R. Tarrach, and G. Vidal, Phys. Rev. A **58**, 826 (1998).
- [18] K. Życzkowski and H.-J. Sommers, J. Phys. A **34**, 7111 (2001).
- [19] P. Horodecki, Phys. Lett. A **232**, 233 (1997).
- [20] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
- [21] C.H. Bennett *et al.*, Phys. Rev. Lett. **82**, 5385 (1999).
- [22] A. Pittenger and H. Rubin, e-print quant-ph/0207024.