



Experimental detection of entanglement via witness operators and local measurements

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Abstract. In this paper we address the problem of detection of entanglement using only few local measurements when some knowledge about the state is given. The idea is based on an optimized decomposition of witness operators into local operators. We discuss two possible ways of optimizing this local decomposition. We present several analytical results and estimates for optimized detection strategies for NPT states of 2×2 and $N \times M$ systems, entangled states in 3 qubit systems, and bound entangled states in 3×3 and 2×4 systems.

1. Introduction

One of the key problems in experiments in quantum information theory is the generation and detection of entanglement [1]. The generation of entangled states is practically always aimed at particular states that can then be used for various applications in quantum information processing. Very often one wants to produce certain pure states. Generation process in a laboratory is, however, never free of imperfections and noise. The states that one produces may, but not necessarily have the desired properties. In particular they may, but do not have to be entangled. For this reason it is important to develop efficient and easy to apply experimental procedures to detect entanglement.

Obviously, the ultimate goal of detection is to characterize the entanglement quantitatively, and to identify the regions in the parameter space which would allow one to maximize entanglement useful for the particular application. Before this ambitious goal is realized, however, it is important to know whether the state one is dealing with is entangled at all, or not. The reasons for that are at least threefold: (i) it is interesting from the fundamental point of view; (ii) it is important to know before applying more complex experimental tools to quantify entanglement; (iii) it is essential, if one wants to use the generated states for any of the distillation or purification protocols [2].

For the detection of entanglement several strategies are known: first of all there is the possibility of quantum state tomography [3], and then direct application of known necessary or sufficient entanglement criteria [4]. Determining the density matrix via quantum state tomography requires, however, typically a lot of measurements. Note also that only for 2×2 and 2×3 systems is a necessary and sufficient criterion known, the famous Peres–Horodecki criterion of positivity of partial transpose [5]. In general, it might be difficult to find a sufficient criterion for a given state. One could also look for a violation of some Bell inequalities, although there exist entangled states (bound entangled states) that do not violate any known Bell-like inequality [6]. In fact, there is even a conjecture that these states admit a local hidden variable model [7]. Looking for a violation of Bell inequalities may thus not be sufficient.

Recently, there have been several proposals for the detection of entanglement without estimating the whole density matrix [8–10]. Although being attractive from both experimental and theoretical point of view, these possibilities have some disadvantages. The recent proposals require collective measurements on several qubits or the construction of quantum gates and networks. These requirements are possible, but not easy to fulfil with the present experimental techniques of local measurements.

On the other hand, the methods for the detection of entanglement mentioned above are in some sense too general for experiments. They assume that no *a priori* knowledge about the density matrix is given. But, as we already mentioned, in an experimental situation one usually tries to prepare some particular state. Although one fights with the problems of noise and imperfections, in any case the produced state cannot be considered completely arbitrary.

In this paper, which expands an earlier study [11], we address the problem of checking whether a state ρ is entangled or not, when some knowledge about the density matrix is given. We want to solve this problem using only local von Neumann measurements which can be implemented in a laboratory using the present techniques. We also aim to use as less measurement as possible.

The scheme we use for the detection of entanglement relies on the well-known concept of witness operators [12, 13]. Let us briefly recall what these operators are. A Hermitian operator W is called an entanglement witness detecting the entangled state ρ_e if $\text{Tr}(W\rho_e) < 0$ and $\text{Tr}(W\rho_s) \geq 0$ for all separable states ρ_s . So, if we have a state ρ and we measure $\text{Tr}(W\rho) < 0$, we can be sure that ρ is entangled. For every entangled state there exists an entanglement witness. There has also been an enormous progress in constructing witnesses for different classes of states [13].

After having constructed a witness operator we decompose it into a weighted sum of projectors onto product vectors. In this way the expectation value of the witness can be measured locally: Alice and Bob measure the expectation value of the projector and add their results with the weights to receive the expectation value of the witness. Knowing the value of $\text{Tr}(W\rho)$ they can decide if the state is detected to be entangled, or not. The decomposition of the witness into projectors onto product vectors should be optimal, i.e. contain in a certain sense a possibly small number of terms.

The paper is divided into four sections. In the first section we illustrate our proposal and notation with a simple example: we consider a pure two-qubit state affected by white noise. The main parts of this paper are the extensions of this example: in the second section we investigate the two-qubit scenario if the noise is

not white. We also investigate the limits of our scheme and the errors that may occur. The third section deals with extensions of the decomposition of the witness from the first section. We derive decompositions for witnesses in three-qubit case. With these operators GHZ-states and W-states can be detected. Eventually, we discuss how one can decompose witnesses for bipartite systems in higher dimensions. In the last section we apply our method to bound entangled states. We mainly discuss how some types of bound entanglement in 3×3 and 2×4 systems can be detected.

2. Two qubits with white noise

We consider a setup that is intended to produce a particular pure state $|\psi\rangle\langle\psi|$, but due to the imperfections some noise is added. So, it produces a mixed state ϱ of the form

$$\varrho(p, d) := p|\psi\rangle\langle\psi| + (1 - p)\sigma, \tag{1}$$

where we know that the noise added to the state $|\psi\rangle$ is close to the totally mixed state, i.e.

$$\|\sigma - \frac{1}{4}\mathbb{1}\| \leq d. \tag{2}$$

If $d = 0$ the noise would be white, but in general this does not have to be the case. In the beginning, we do not make any assumption on $|\psi\rangle$, but when we discuss the case $d > 0$ we restrict ourselves to the important case that $|\psi\rangle$ is a Bell state: $|\psi^+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$. Further we do not make any restriction on p , i.e. we assume that it is unknown to us. As a byproduct the witness will later donate us a possibility to determine p . Our aim is to give a scheme to check whether $\varrho(p, d)$ is entangled or not.

Now we have to clarify our notation. We will denote the set of all density matrices by M and the set of separable matrices by S . We can endow the space of all matrices with the scalar product $\langle A|B\rangle_{HS} := \text{Tr}(A(B^\dagger))$ and the corresponding Hilbert–Schmidt norm $\|A\| := \sqrt{\text{Tr}(A(A^\dagger))}$. For any $\varrho \in M$ and $r \geq 0$ we can define by

$$B(\varrho, r) := \{\varrho' \in M, \|\varrho' - \varrho\| \leq r\} \tag{3}$$

the ball around ϱ with radius r . We denote by $B_{p,d}$ the ball which must include $\varrho(p, d)$ for a given p and d . It is given by $B_{p,d} = B((1 - p/4)\mathbb{1} + p|\psi^+\rangle\langle\psi^+|, (1 - p)d)$. Finally we define L as the line between $1/4$ and $|\psi\rangle\langle\psi|$; we have: $L = \{\varrho(p, 0), p \in [0, 1]\}$.

2.1. Construction of the witness

The optimal witness for an entangled $\varrho(p, 0)$ is easy to construct [13]. First, one has to compute the eigenvector corresponding to the negative eigenvalue of $\varrho(p, 0)^{T_B}$, then the witness is given by the partially transposed projector onto this eigenvector.

If the Schmidt decomposition of $|\psi\rangle$ is $|\psi\rangle = a|01\rangle + b|10\rangle$ with $a, b \geq 0$, the spectrum of $\varrho(p, 0)^{T_B}$ is given by $\{(1 - p)/4 + pa^2; (1 - p)/4 + pb^2; (1 - p)/4 +$

$pab; (1 - p)/4 - pab\}$. Therefore $\varrho(p, 0)$ is entangled iff $p > 1/(1 + 4ab)$. The eigenvector corresponding to the minimal eigenvalue λ_- is given by

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{4}$$

and thus the witness W_0 is given by

$$W_0 = |\phi_-\rangle\langle\phi_-|^{T_B} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Note that this witness does neither depend on p , nor on the Schmidt coefficients a, b . It detects $\varrho(p, 0)$ iff it is entangled, since we have $Tr(|\phi_-\rangle\langle\phi_-|^{T_B}\varrho(p, 0)) = Tr(|\phi_-\rangle\langle\phi_-|\varrho(p, 0)^{T_B}) = \lambda_-$. For our special case of white noise W_0 has also the advantage that from $Tr(W_0\varrho(p, 0)) \geq 0$ it follows that $\varrho(p, 0)$ is separable. This is not a general property of witnesses, and indeed if the noise is not white this is not true anymore.

2.2. Decomposition of the witness

For an experimental setup it is necessary to decompose the witness into operators which can be measured locally. Thus we need a decomposition into projectors onto product vectors of the form

$$W = \sum_{i=1}^k c_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|. \tag{6}$$

Such a decomposition can be measured locally: Alice and Bob measure the expectation value of the $|e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$ and add their results with the weights c_i . One can construct such a decomposition in many ways, but it is reasonable to do it in a way which corresponds to expenses of Alice and Bob which are as small as possible. There are several possibilities to define an optimal decomposition.

One possibility is to look for the optimal number of product vectors (ONP), i.e. one can try to minimize k in (6). This optimization strategy looks very natural and has already been considered in the literature. For 2×2 systems it was proven in [14] that in general the ONP, which we denote as k_- , is five. Also a constructive way for computing this optimal decomposition was given.

What is the ‘cost’ Alice and Bob have to pay when measuring W via such a decomposition? It is the number of measurements they have to perform. When we talk about measurements here, we consider only von Neumann measurements. We do not look at POVMs since their implementation would require additional ancilla systems. One measurement on Alice’s side in this sense consists of a choice of one orthonormal basis for Alice’s Hilbert space. For an particle with spin s one may interpret this as the choice of a direction for a Stern–Gerlach-like apparatus. Alice sets up her device in the desired direction and is able to distinguish between $2s + 1$ different states.

For one local measurement Bob also has to choose an orthonormal basis in his Hilbert space; all together this yields one orthonormal product basis for both. Thus, if we are in a $N \times N$ system a term of the form

$$\sum_{k,l=1}^N c_{kl} |A_k\rangle\langle A_k| \otimes |B_l\rangle\langle B_l| \tag{7}$$

with $\langle A_s|A_t\rangle = \langle B_s|B_t\rangle = \delta_{st}$ can be measured with one collective setting of measurement devices of Alice and Bob. Alice and Bob can discriminate between the states $|A_k B_l\rangle$, measure the probabilities of these states and add their results with the weights c_{kl} using one collective setting and some classical communication. We call such a collective setting of measurement devices a local von Neumann measurement (LvNM).

It is therefore reasonable to find a decomposition of the form

$$W = \sum_{i=1}^m \sum_{k,l=1}^N c_{kl}^i |A_k^i\rangle\langle A_k^i| \otimes |B_l^i\rangle\langle B_l^i| \tag{8}$$

with $\langle A_s^i|A_t^i\rangle = \langle B_s^i|B_t^i\rangle = \delta_{st}$ and an optimal number of devices' settings (ONS), i.e. a minimal m . In this sense m is the minimal number of measurements Alice and Bob have to perform. The construction of a decomposition of the form (8), and the determination of the minimal m is the problem we want to solve in this section.

Please note that a decomposition like (6) with the minimal k_- (ONP) requires in general k_- LvNMs because the out-coming vectors on Alice's side $|e_i\rangle\langle e_i|$ do not have to be orthogonal.

We also would like to emphasize that a decomposition of the form (8) is more general than a decomposition into a sum of tensor products of operators:

$$W = \sum_{i=1}^m \gamma_i A_i \otimes B_i. \tag{9}$$

The decomposition (9) has the advantage that Alice and Bob do not have to distinguish between some states, they only have to measure locally some expectation values of Hermitean operators. A decomposition like (8) can be written in the form of (9) if for all i the matrices (c_{kl}^i) are of rank one. In the following we will see that for qubit systems there is not a big difference between (8) and (9). From the optimal decomposition in the sense of (8) we can derive a decomposition of the form (9) where some of the operators are the identity (1), so they do not require new measurement settings. For $N \times N$ -systems we will see that it is straightforward to derive the optimal decomposition in the sense of (9).

Our witness W_0 has a special form: it is a partially transposed projector $|\psi\rangle\langle\psi|^{T_B}$. It is therefore natural to look at the Schmidt decomposition of $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. For our W_0 we have the special case $\alpha = 1/\sqrt{2} = -\beta$, but we want to deal with the most general $|\psi\rangle$.

When we compute the ONP-decomposition with the minimal k_- according to [14] we arrive at

$$|\psi\rangle\langle\psi|^{T_B} = \frac{(\alpha + \beta)^2}{3} \sum_{i=1}^3 |A_i' B_i'\rangle\langle A_i' B_i'| - \alpha\beta(|01\rangle\langle 01| + |10\rangle\langle 10|), \tag{10}$$

where we have used the definitions

$$\begin{aligned}
 |A'_1\rangle &= e^{i(\pi/3)} \cos(\theta)|0\rangle + e^{-i(\pi/3)} \sin(\theta)|1\rangle = |B'_1\rangle \\
 |A'_2\rangle &= e^{-i(\pi/3)} \cos(\theta)|0\rangle + e^{i(\pi/3)} \sin(\theta)|1\rangle = |B'_2\rangle \\
 |A'_3\rangle &= |A'_1\rangle + |A'_2\rangle = |B'_3\rangle \\
 \cos(\theta) &= \sqrt{\alpha/(\alpha + \beta)} \\
 \sin(\theta) &= \sqrt{\beta/(\alpha + \beta)}.
 \end{aligned}
 \tag{11}$$

This decomposition into five product vectors requires four correlated settings for Alice and Bob. But we can measure W_0 with less settings: if we define the spin directions by $|z^+\rangle = |0\rangle, |z^-\rangle = |1\rangle, |x^\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle), |y^\pm\rangle = (1/\sqrt{2})(|0\rangle \pm i|1\rangle)$ we have the decomposition

$$\begin{aligned}
 |\psi\rangle\langle\psi|^{T_B} &= \alpha^2|z^+z^+\rangle\langle z^+z^+| + \beta^2|z^-z^-\rangle\langle z^-z^-| + \alpha\beta(|x^+x^+\rangle\langle x^+x^+| \\
 &\quad + |x^-x^-\rangle\langle x^-x^-| - |y^+y^-\rangle\langle y^+y^-| - |y^-y^+\rangle\langle y^-y^+|) \\
 &= \frac{1}{4}(1 \otimes 1 + \sigma_z \otimes \sigma_z + (\alpha^2 - \beta^2)(\sigma_z \otimes 1 + 1 \otimes \sigma_z) \\
 &\quad + 2\alpha\beta(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)).
 \end{aligned}
 \tag{12}$$

This decomposition into six product vectors requires only a measurement of three settings: Alice and Bob have only to set up their Stern–Gerlach devices in the x -, y - and z -direction to measure $|\psi\rangle\langle\psi|^{T_B}$.

Now we want to prove that three LvNM are really necessary. Our proof is a special case of a theorem about $N \times N$ systems we will show later. But in the two-qubit case the proof is particularly simple, and therefore we present it here separately.

Proposition 1: *In a two qubit system a decomposition of $|\psi\rangle\langle\psi|^{T_B}$ of the form (8) requires at least three measurements.*

Proof: Consider a decomposition requiring two measurements:

$$|\psi\rangle\langle\psi|^{T_B} = \sum_{i,j=1}^2 c_{ij}^1 |A_i^1\rangle\langle A_i^1| \otimes |B_j^1\rangle\langle B_j^1| + \sum_{i,j=1}^2 c_{ij}^2 |A_i^2\rangle\langle A_i^2| \otimes |B_j^2\rangle\langle B_j^2|. \tag{13}$$

With the help of a Schmidt decomposition as above we can write $|\psi\rangle\langle\psi|^{T_B} = \sum_{i,j=0}^3 \lambda_{ij} \sigma_i \otimes \sigma_j$ with

$$(\lambda_{ij}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{\alpha^2 - \beta^2}{4} \\ 0 & \frac{\alpha\beta}{2} & 0 & 0 \\ 0 & 0 & \frac{\alpha\beta}{2} & 0 \\ \frac{\alpha^2 - \beta^2}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}. \tag{14}$$

Note that the 3×3 submatrix in the right bottom corner is of rank 3. Now we write any projector on the rhs of (20) as a vector in the Bloch sphere: $|A_1^1\rangle\langle A_1^1| = \sum_{i=0}^3 s_i^A \sigma_i$ is represented by the vector $\vec{s}_{A_1^1} = (1/2, s_1^A, s_2^A, s_3^A)$ and $|A_2^1\rangle\langle A_2^1|$ by

$\vec{s}_{A_2^1} = (1/2, -s_1^A, -s_2^A, -s_3^A)$; $|B_1^1\rangle\langle B_1^1|$ can be written similarly. If we expand the first sum on the rhs of (20) in the $(\sigma_i \otimes \sigma_j)$ basis, the 3×3 submatrix in the right bottom corner is given by $(c_{11}^1 - c_{12}^1 - c_{21}^1 + c_{22}^1)(s_1^A, s_2^A, s_3^A)^T (s_1^B, s_2^B, s_3^B)$. This matrix is of rank one. The corresponding submatrix from the second sum on the rhs of (13) is also of rank one and we arrive at a contradiction: no matrix of rank 3 can be written as a sum of two matrices of rank one. \square

The idea of a generalization of this proposition to $N \times N$ systems is straightforward: one expands both sides of an equation of the type (13) in some product basis of the space of all operators. Then one tries to reach lower bounds for the rank of some submatrix of the coefficient matrix (14) and upper bounds for the rank of the matrix corresponding to one LvNM. This gives a lower bound for the number of LvNMs.

3. The case of non-white noise

Our investigation of this case proceeds in two steps. First, we argue why the witness W_0 should also be used in this instance. We also show that under some circumstances, i.e. if $Tr(W_0 \varrho(p, d))$ is large enough, we can make a sure decision that $\varrho(p, d)$ is separable. For the case that we cannot make a sure decision we derive analytical bounds and show numerical estimates for the error. This is done in the second step.

We would like to remind the reader that if $d > 0$, we only consider maximally entangled states, so everywhere in this section we set $a = b = 1/\sqrt{2}$.

3.1. Properties of W_0

Let us note some typical distances in M :

Remark 1: (a) *The states on ∂M , the boundary of M , closest to $1/4$ have a distance of $1/\sqrt{12} \approx 0.29$. The states with the largest distance to $1/4$ are just the pure states, for which the distance is $\sqrt{3}/2 \approx 0.87$.*

(b) *The states on ∂S closest to $1/4$ have also distance of $1/\sqrt{12}$.*

Proof: These distances can be calculated simply by maximizing or minimizing $\|1/4 - \varrho\|$ under some conditions, for instance for (b) under the condition that $\det(\varrho^{T_B}) = 0$. \square

From part (a) it follows that we have to assume that $d \leq 1/\sqrt{12}$. Please note also that the point where L crosses the border of separability, ∂S , is given by $\varrho(1/3, 0)$ and this point has just the distance $1/\sqrt{12}$ from $1/4$, this means it is as close as possible.

Remark 2: *For all p the point on ∂S closest to $\varrho(p, 0)$ is given by $\varrho(1/3, 0)$.*

Proof: For $p < 1/3$ we have mentioned it already. For $p > 1/3$ we have a look at the set

$$\mathcal{N}(W_0) := \{\varrho \in M; Tr(W_0 \varrho) = 0\}. \tag{15}$$

We have $\varrho(1/3, 0) \in \mathcal{N}(W_0)$, and one can directly compute that for an arbitrary $N \in \mathcal{N}(W_0)$ and $\varrho(p, 0)$ the relation $\langle \varrho(1/3, 0) - \varrho(p, 0) | \varrho(1/3, 0) - N \rangle_{HS} = 0$ holds. Therefore $\mathcal{N}(W_0)$ and L are orthogonal. From this and the properties of a witness operator the claim for $p > 1/3$ follows. \square

Remark 3: *The witness W_0 is the best possible witness in the following sense: for all p and d there is no other witness that detects a subset of $B_{p,d}$ as entangled which has a bigger volume than the subset of $B_{p,d}$ which is detected as entangled by W_0 .*

Proof: From the second remark it follows that if $B_{p,d} \cap S = \emptyset$ the witness W_0 detects the whole ball, independent of d .

Thus, we can assume that p has a value such that $B_{p,d}$ contains separable and entangled states and that there exists a witness W' , which detects a bigger volume of $B_{p,d}$. Then we look at the set $\mathcal{N}(W')$, defined analogous to (15). If $p < 1/3$ the minimal distance between matrices of $\mathcal{N}(W')$ and $\varrho(p, 0)$ must be smaller than the minimal distance between $\mathcal{N}(W_0)$ and $\varrho(p, 0)$, which is $\|\varrho(p, 0) - \varrho(1/3, 0)\|$. This means that there exist entangled states (and therefore states on ∂S) which are closer to $\varrho(p, 0)$ than $\varrho(1/3, 0)$ and we have a contradiction to Remark 2. If $p \geq 1/3$ the minimal distance between matrices of $\mathcal{N}(W')$ and $\varrho(p, 0)$ must be bigger than the distance between $\mathcal{N}(W_0)$ and $\varrho(p, 0)$, but then W' must ‘detect’ the separable state $\varrho(1/3, 0)$, which is a contradiction. \square

The critical reader may ask at this point why we are so innocent and use the term of a ‘volume’ in M . It seems that it is difficult to say something about volumes in our norm, since we are not in the \mathbb{R}^n with the Euclidean norm [15, 16]. But we are not too far away from the \mathbb{R}^n : We can write any density matrix as $\varrho = \sum_{i=0}^{15} \mu_i G_i$ where $\mu_i \in \mathbb{R}$, $G_0 \sim 1$ and the G_1, \dots, G_{15} are the traceless generators of the $SU(4)$. The G_i can be normalized in a way that they form an orthonormal basis: $\langle G_i | G_j \rangle_{HS} = \delta_{ij}$. We will explain and use this decomposition in greater detail later. Here we only point out that $\|\varrho\|^2 = \sum_{i=0}^{15} \mu_i^2$ and so our norm in the space of Hermitean operators is just the Euclidean norm in \mathbb{R}^{16} . M corresponds to a subset of a 15-dimensional hyperplane since μ_0 is fixed, and the usual formulas for volumes of balls can be applied.

Now having proven that it is reasonable to use W_0 , one may ask what the expectation value of W_0 tells us for the case $d > 0$. From the definition of a witness operator it follows that if the expectation value is negative we can be sure that $\varrho(p, d)$ is entangled. On the first view it seems that if it is positive we can not make a sure decision if $\varrho(p, d)$ is entangled or not. But if $Tr(W_0 \varrho(p, d)) \gg 0$ this means that $\varrho(p, d)$ is far away from $\mathcal{N}(W_0)$ and since we know that $\varrho(p, d) \in B_{p,d}$ for some p , the state $\varrho(p, d)$ must be separable. So there must exist a $\tau(d)$ such that from $Tr(\varrho(p, d)W_0) \geq \tau$ it follows that $\varrho(p, d)$ is separable. We can directly compute τ , but first we introduce some new definitions.

If we measure $Tr(\varrho W_0) = \alpha$ the expectation value α tells us that ϱ is in some hyperplane cutting M . This hyperplane is orthogonal to L , as we have shown in the proof of Remark 2. It intersects this line at some $\varrho(q, 0)$ and so we denote this plane by $\mathcal{P}(q)$. For instance an essential part of the proof of Remark 2 was the statement that $\mathcal{P}(1/3) = \mathcal{N}(W_0)$. The connection between the expectation values of W_0 and the planes \mathcal{P} is given by

$$Tr(\varrho W_0) = \alpha \Leftrightarrow \varrho \in \mathcal{P}(q) \quad \text{with} \quad q = \frac{1}{3} - \frac{4}{3}\alpha. \tag{16}$$

Since we do not deal with general ϱ , but with some $\varrho(p, d)$, it will be useful to consider the set of all possible $\varrho(p, d)$. This is

$$K(d) := \bigcup_{p \in [0,1]} B_{p,d}. \tag{17}$$

We can now define the intersections

$$\begin{aligned}
 SP(q) &:= S \cap \mathcal{P}(q), \\
 KP(q, d) &:= K(d) \cap \mathcal{P}(q), \\
 BP(q, p, d) &:= B_{p,d} \cap \mathcal{P}(q), \\
 XP(q) &:= B\left(\frac{1}{4}\mathbb{1}, \frac{1}{\sqrt{12}}\right) \cap \mathcal{P}(q).
 \end{aligned}
 \tag{18}$$

SP is the set of all separable states which belong to one possible expectation value. KP is the set of all possible $\varrho(p, d)$ yielding the same expectation value. It is clear that $BP(q, p, d) \subseteq KP(q, d)$. Since we know already from Remark 1 that the states in $B(1/4, 1/\sqrt{12})$ are separable, we can conclude that $XP(q) \subseteq SP(q)$. For the sake of notational simplicity we often suppress the parameters q, p, d .

The strategy of computing τ is now clear: We have to find the values of q for which $KP(q) \subseteq SP(q)$ holds. For the corresponding expectation values of W_0 our knowledge that $\varrho(p, d) \in K$ guarantees us that $\varrho(p, d)$ is separable. Since it is difficult to characterize SP we replace it by XP . This replacement is justified later.

Proposition 2: *Let*

$$\tau = \frac{1}{4} - d^2 - \sqrt{\left(\frac{1}{12} - d^2\right)\left(\frac{3}{4} - d^2\right)}.
 \tag{19}$$

Then we have:

$$\text{Tr}(\varrho(p, d)W_0) \geq \tau \Rightarrow \varrho(p, d) \text{ is separable.}
 \tag{20}$$

Furthermore: the τ defined in (19) is the minimal τ with the property (20), i.e. for all $0 \leq \tau' < \tau$ there exists an entangled state $\varrho(p, d)$ with $\text{Tr}(\varrho(p, d)W_0) = \tau'$.

Proof: The idea of the proof is as described above: first, we compute the parameter q_- , such that for $q \leq q_-$ we have $KP(q) \subset SP(q)$. Via (16) we arrive at τ . Finally we show that for $q > q_-$ we have $KP(q) \not\subset SP(q)$.

KP is just like BP a ball in the hyperplane \mathcal{P} . The radius of BP is determined by $r^2(BP) = (1 - p)^2 d^2 - \|\varrho(q, 0) - \varrho(p, 0)\|^2 = (1 - p)^2 d^2 - (3/4)(q - p)^2$. Maximizing this over all p yields

$$r^2(KP) = \frac{3d^2}{3 - 4d^2}(1 - q)^2.
 \tag{21}$$

The ball XP has the squared radius

$$r^2(XP) = \frac{1}{12} - \frac{3}{4}(q)^2.
 \tag{22}$$

If we choose q small enough, it follows from $XP(q) \subset SP(q)$ that $KP(q) \subset XP(q) \subset SP(q)$. So we can determine q_- by the equation $r^2(KP(q_-)) = r^2(XP(q_-))$, apply (16) and arrive at τ .

Why can we not use a smaller τ ? The reason is that for any $q \in [0, 1/3]$ the ball $XP(q)$ contains at least one state of ∂S . This means that from $KP(q) \not\subset XP(q)$ it follows that $KP(q) \not\subset SP(q)$. To see the existence of a state $\in \partial S$ in $XP(q)$ first note that if we take our first Bell state $|\psi^+\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle) =: (1/\sqrt{2})(|0'0'\rangle + |1'1'\rangle)$ and another Bell state $|\phi^+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ we have:

$$\langle (\frac{1}{4}\mathbb{1} - |\psi^+\rangle\langle\psi^+|) | (\frac{1}{4}\mathbb{1} - |\phi^+\rangle\langle\phi^+|) \rangle_{HS} = -\frac{1}{4} < 0.
 \tag{23}$$

We can transform the bases $|0'0'\rangle \rightarrow |00\rangle, |1'1'\rangle \rightarrow |11\rangle$, in a continuous way and thereby construct a continuous map

$$\gamma : [0, 2] \rightarrow M; \quad t \mapsto \gamma(t) \tag{24}$$

with $\gamma(0) = |\psi^+\rangle\langle\psi^+|$ and $\gamma(2) = |\phi^+\rangle\langle\phi^+|$ and for all t , $\gamma(t)$ is a maximally entangled state. There must exist a t_0 (and without losing generality we can set $t_0 = 1$) with

$$\langle (\frac{1}{4}\mathbb{1} - |\psi^+\rangle\langle\psi^+|) | (\frac{1}{4}\mathbb{1} - \gamma(1)) \rangle_{HS} = 0. \tag{25}$$

This equation implies that $\gamma(1) \in \mathcal{P}(q = 0)$. The map γ induces another continuous map by

$$\eta : [0; 1] \rightarrow S; \quad t \mapsto \eta(t) = \frac{2}{3}\frac{1}{4}\mathbb{1} + \frac{1}{3}\gamma(t). \tag{26}$$

Since $\gamma(t)$ is a maximally entangled state $\eta(t)$ is a point on ∂S . Furthermore all $\eta(t)$ have a distance of $1/\sqrt{12}$ to the state $1/4$, i.e. $\eta(t) \in B(1/4, 1/\sqrt{12})$. Since $\eta(0) \in \mathcal{P}(1/3)$ and $\eta(1) \in \mathcal{P}(0)$, for any $q \in [0, 1/3]$, $\mathcal{P}(q)$ contains one $\eta(t_q)$ and therefore $XP(q)$ contains a state on ∂S . □

If one knows the value of p one can sharpen the statement of the last proposition.

Proposition 3. *Let p be fixed, $\varrho(p, d) \in B_{p,d}$ and*

$$\vartheta := \frac{1}{4} - \frac{1}{24p} - \frac{3p}{8} + \frac{(1-p)^2 d^2}{2p}. \tag{27}$$

Then we have:

$$\text{Tr}(\varrho(p, d)W_0) \geq \vartheta \Rightarrow \varrho(p, d) \text{ is separable.} \tag{28}$$

Furthermore: the ϑ defined in (27) is the minimal ϑ with the property (28) i.e. for all $0 \leq \vartheta' < \vartheta$ there exists an entangled state $\varrho(p, d)$ with $\text{Tr}(\varrho(p, d)W_0) = \vartheta'$.

Proof. The proof is essentially the same as the proof of Proposition 2. One just has to replace K by $B_{p,d}$ and therefore $KP(q)$ by $BP(q)$. □

3.2. Error estimates

Consider the case that one has measured $\text{Tr}(W_0\varrho(p, d)) = \alpha$ with $\alpha \in [0, \tau]$. For such expectation value one cannot make a sure decision whether ϱ is entangled or not. Nevertheless one can make a decision, if one accepts making some error. Here we want to estimate the probability of making an error. First we give some analytical bounds on the error, then we perform some numerical simulations to rate the error.

For an estimate of an error, we have to make one further assumption on $\varrho(p, d)$: we have to assume some probability distribution. We always assume that for all p and d , $\varrho(p, d)$ is uniformly distributed in $B_{p,d}$.

3.2.1. Analytical estimates

Our analytical estimation scheme relies on the fact that all states in the ball $B(1/4, 1/\sqrt{12})$ are separable. Thus we know at least some separable states yielding expectation values $\alpha \in [0, \tau]$. If we assume that $\varrho(p, d)$ is separable, this gives us an upper bound on the probability of the error.

For an illustration of this idea let us first assume that we know the fixed value of p and we have measured $Tr(W_0\rho(p, d)) = \alpha$. Via (16) we can compute the q such that $\rho(p, d) \in \mathcal{P}(q)$. We know then that $\rho(p, d) \in BP(q)$. If we now assume that $\rho(p, d)$ is separable, the probability of guessing right is given by the ratio of the volumes

$$e_r = \frac{\text{vol}(SP(q))}{\text{vol}(BP(q))}, \tag{29}$$

and the probability of being wrong is bounded by

$$e_w \leq E_w = \frac{\text{vol}(BP(q)) - \text{vol}(XP(q))}{\text{vol}(BP(q))}. \tag{30}$$

These terms can be calculated with the standard formulas for balls in higher dimensions: $\text{vol}(BP(q)) = (\pi^7/7!)((1-p)^2d^2 - 3/4(p-q)^2)^7$ and $\text{vol}(XP(q)) = (\pi^7/7!)(1/12 - 3/4q^2)^7$.

Since we do not know the value of p we have to maximize E_w over all p , then we arrive at

$$E_- = \sup_{p \in [0,1]} E_w = 1 - \frac{(\alpha(\alpha - \frac{1}{2})(d^2 - \frac{3}{4}))^7}{(d(\alpha + \frac{1}{2}))^{14}}. \tag{31}$$

This function is plotted together with numerical results in figure 1.

Let us mention that with the same method as above one can also estimate the error for other scenarios, for instance if one generally assumes that for $Tr(W_0\rho(p, d)) \geq 0$ ρ is separable without looking at the precise expectation value; we will not discuss further this direction here, however.

3.2.2. Numerical calculations

One can investigate the properties of the witness also with numerical calculations. For this purpose one can generate random matrices [16] and compare the

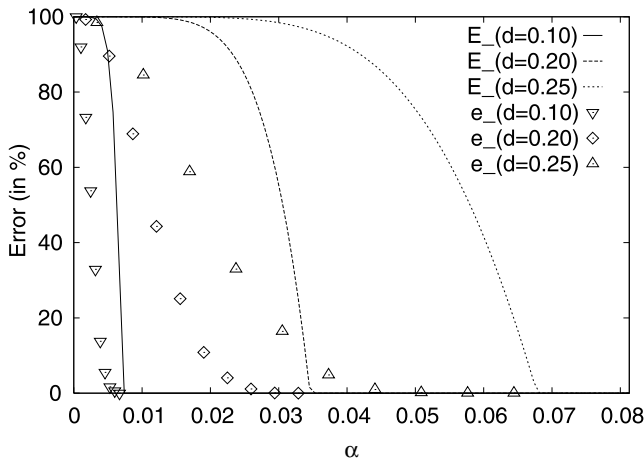


Figure 1. Comparison between the numerical estimate (e_-) and the analytical bound (E_-) as functions of the expectation value. The three curves are for three different values of d .

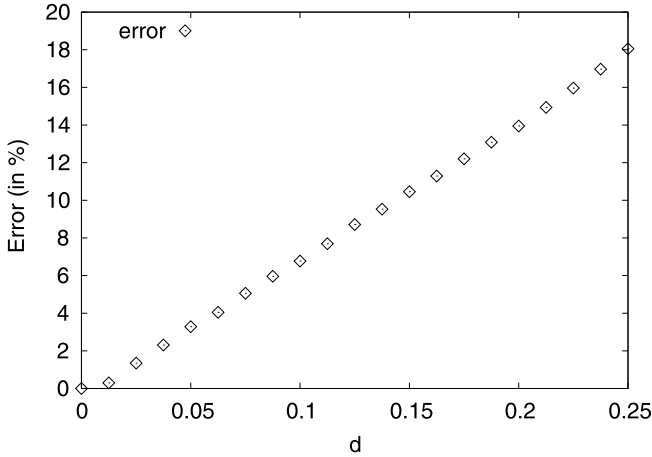


Figure 2. Probability of making an error (maximized over all p) when assuming $Tr(W_0\rho(p,d)) > 0 \Rightarrow \rho(p,d) \in S$ as a function of d .

result of the witness with the PPT criterion, which is a necessary and sufficient criterion of separability for 2×2 systems [5]. We have generated a sample of 50000 random matrices (in Hilbert–Schmidt norm) in the ball $B(1/4, 1/\sqrt{12})$. By scaling we have retrieved then a set of 50 000 random matrices $\rho(p, d)$ for all p and d . With these matrices we have obtained our numerical estimates.

First, we investigated the error estimate with E_- from equation (31). For this purpose we computed e_w and then e_- as the supremum of e_w over all p . This is shown in figure 1.

One may ask how big the error is, if one generally concludes from $Tr(W_0\rho(p,d)) \geq 0$ that $\rho(p,d) \in S$. We have estimated the error (maximized over all p). The result is given in figure 2.

4. Applications for three-qubit and $N \times N$ systems

First, we consider three-qubit systems. For these systems two types of tripartite entanglement are known: the GHZ-states and the W-states [17]. Also families of witnesses for these states are known [18] and we show how to decompose them locally. Then we say something about a decomposition of a projector in $N \times N$ systems.

4.1. Three qubits

Three qubits can be entangled in different ways: they might be separable, biseparable or fully tripartite entangled [18]. The genuine threepartite entanglement consists of two classes: the GHZ-class and the W-class [17].

Witnesses for detection of GHZ-type states and W-type states have also been constructed in [18]. Here we want to show that these witnesses can be decomposed within our scheme.

For the GHZ-class a witness is given by

$$W_{GHZ} = \frac{3}{4}\mathbb{1} - |GHZ\rangle\langle GHZ|, \tag{32}$$

where $|GHZ\rangle$ is a pure state of the GHZ-class: $|GHZ\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$. If ρ is a mixed state with $\text{Tr}(\rho W_{GHZ}) < 0$ the state ρ belongs to the GHZ-class. A decomposition of W_{GHZ} can be achieved with similar calculations as above. The result is:

$$W_{GHZ} = \frac{1}{8}(5 \cdot 1 \otimes 1 \otimes 1 - 1 \otimes \sigma_z \otimes \sigma_z - \sigma_z \otimes 1 \otimes \sigma_z - \sigma_z \otimes \sigma_z \otimes 1 - \sigma_x \otimes \sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_y \otimes \sigma_y + \sigma_y \otimes \sigma_y \otimes \sigma_x + \sigma_y \otimes \sigma_x \otimes \sigma_y). \quad (33)$$

This witness can be measured with five collective measurement settings: Alice, Bob and Charly have to perform correlated measurements in the z - z - z , x - x - x , x - y - y , y - x - y and the y - y - x direction.

For the W-states several witnesses are known. One example is the operator

$$W_{W1} = \frac{2}{3}\mathbb{1} - |W\rangle\langle W|, \quad (34)$$

where $|W\rangle$ is now a pure state of the W-class: $|W\rangle = 1/\sqrt{3}(|100\rangle + |010\rangle + |001\rangle)$. This witness detects states belonging to the W-class and the GHZ-class, i.e. its expectation value is positive on all biseparable and fully separable states. A decomposition is given by

$$W_{W1} = \frac{1}{8}(\frac{2}{3}\mathbb{1} \otimes 1 \otimes 1 - \sigma_z \otimes 1 \otimes 1 - 1 \otimes \sigma_z \otimes 1 - 1 \otimes 1 \otimes \sigma_z + \sigma_z \otimes \sigma_z \otimes 1 + \sigma_z \otimes 1 \otimes \sigma_z + 1 \otimes \sigma_z \otimes \sigma_z + 3 \cdot \sigma_z \otimes \sigma_z \otimes \sigma_z) - \frac{1}{4}(1 \otimes \sigma_x \otimes \sigma_x + 1 \otimes \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_y \otimes \sigma_y + \sigma_x \otimes 1 \otimes \sigma_x + \sigma_y \otimes 1 \otimes \sigma_y + \sigma_x \otimes \sigma_z \otimes \sigma_x + \sigma_y \otimes \sigma_z \otimes \sigma_y + \sigma_x \otimes \sigma_x \otimes 1 + \sigma_y \otimes \sigma_y \otimes 1 + \sigma_x \otimes \sigma_x \otimes \sigma_z + \sigma_y \otimes \sigma_y \otimes \sigma_z). \quad (35)$$

Although this decomposition is a little bit longer, only seven correlated measurements are necessary.

Another witness for W-class states is given by

$$W_{W2} = \frac{1}{2}\mathbb{1} - |GHZ\rangle\langle GHZ|. \quad (36)$$

This witness can be measured locally with the same decomposition as (33). It also can serve for a detection of states of the type $(1 - p)\mathbb{1}/8 + p|W\rangle\langle W|$, as explained in [18].

4.2. $N \times N$ systems

Now we want to generalize our results to higher dimensions. First we consider $N \times N$ systems, and at the end of this section we make some remarks about $N \times M$ systems.

A witness for an entangled state ρ with a non-positive partial transpose can be constructed just like in the two-qubit case. First, one computes one eigenvector corresponding to one negative eigenvalue of ρ^{T_B} . The partially transposed projector onto this vector is an entanglement witness. We want to decompose such witnesses for NPT states in this section. We only look at projectors, the partial transposition can be performed later.

Our discussion proceeds as follows: after explaining our notation we construct a decomposition of a projector onto a state with Schmidt rank l using about $2l$ measurements. This decomposition is a generalization of the decomposition for the two-qubit case. It is not clear whether this decomposition is optimal. Then we

derive a lower bound for the number of measurements needed if the Schmidt rank l is maximal. We show that if $l = N$ at least $l + 1$ measurements are necessary.

We first explain some notational and technical details (the reader should consult [19] for more explanations). We denote the real vector space of all Hermitean operators on \mathcal{H}_A by \mathcal{HS}_A . In this space one can use the orthogonal basis $\{1, G_i^A, i = 1 \dots N^2 - 1\}$ where the G_i^A are the traceless generators of the $SU(N)$, normalized to $Tr(G_i^2) = 1$. For $N = 2$ they are the Pauli matrices, for $N = 3$ the Gell-Mann matrices, etc. We can define $G_0^A := 1$ and can expand every projector (and any other element of \mathcal{HS}_A) in this basis:

$$|\phi\rangle\langle\phi| = \sum_{i=0}^{N^2-1} f_i G_i^A, \tag{37}$$

where the entries of the Bloch vector f_i are real, $f_0 = 1/N$, and from the fact that $|\phi\rangle\langle\phi|$ is a pure state it follows that

$$\sum_{i=1}^{N^2-1} f_i^2 = 1 - \frac{1}{N}. \tag{38}$$

We sometimes write $(f_0, \dots, f_{N^2-1}) = (f_0, \vec{f}) = \hat{f}$. It is easy to see that an operator described by \hat{g} is a projector onto a vector orthogonal to $|\phi\rangle$ if and only if \vec{g} fulfils (38) and

$$\langle \vec{f}, \vec{g} \rangle := \sum_{i=1}^{N^2-1} f_i g_i = -\frac{1}{N}. \tag{39}$$

One can also expand any projector (as every operator) on $\mathcal{H}_A \otimes \mathcal{H}_B$ as

$$|\psi\rangle\langle\psi| = \sum_{i,j=0}^{N^2-1} \lambda_{ij} G_i^A \otimes G_j^B \tag{40}$$

since the $G_i^A \otimes G_j^B$ form a product basis of the space $\mathcal{HS} = \mathcal{HS}_A \otimes \mathcal{HS}_B$.

Before we show our decomposition please note that it is easy to decompose any operator $A \in \mathcal{HS}$ into N^2 local measurements. One can always write

$$A = \sum_{i,j=0}^{N^2-1} \mu_{ij} G_i^A \otimes G_j^B = \sum_{i=0}^{N^2-1} G_i^A \otimes \left(\sum_{j=0}^{N^2-1} \mu_{ij} G_j^B \right) \tag{41}$$

to obtain such a decomposition. This is also a decomposition of the form (9).

Theorem 1: *Let $|\psi\rangle\langle\psi|$ be a projector onto a state with Schmidt rank l . If l is even, $|\psi\rangle\langle\psi|$ can be decomposed into $2l - 1$ local measurements. If l is odd, $|\psi\rangle\langle\psi|$ can be decomposed into $2l$ local measurements.*

Proof: If we have $|\psi\rangle = \sum_{i=1}^l s_i |ii\rangle$ we can write

$$|\psi\rangle\langle\psi| = \sum_{i=1}^l s_i^2 |ii\rangle\langle ii| + \sum_{i,j=1, i < j}^l s_i s_j K(i, j) \tag{42}$$

with $K(i, j) = |ii\rangle\langle jj| + |jj\rangle\langle ii|$. The first sum corresponds to one measurement and every of the $l(l - 1)/2$ terms of the second sum can be decomposed by defining for

every $K(i, j)$ the directions $|X_{i,j}^\pm\rangle = (1/\sqrt{2})(|i\rangle \pm |j\rangle)$, $|Y_{i,j}^\pm\rangle = (1/\sqrt{2})(|i\rangle \pm i|j\rangle)$ and writing:

$$K(i, j) = |X_{i,j}^+ X_{i,j}^+\rangle\langle X_{i,j}^+ X_{i,j}^+| + |X_{i,j}^- X_{i,j}^-\rangle\langle X_{i,j}^- X_{i,j}^-| - |Y_{i,j}^+ Y_{i,j}^+\rangle\langle Y_{i,j}^+ Y_{i,j}^+| - |Y_{i,j}^- Y_{i,j}^-\rangle\langle Y_{i,j}^- Y_{i,j}^-|, \tag{43}$$

as we have done before for 2×2 systems. This corresponds to two measurements for each $K(i, j)$.

The idea is now to sum up the terms from (43) for different $K(i, j)$ and $K(m, n)$ in a way that the terms from different $K(i, j)$ and $K(m, n)$ can be measured with one measurement.

Let us first consider the case that l is even. We have $l(l-1)/2$ index pairs (i, j) . These pairs can be grouped into $l-1$ sets of $l/2$ pairs in a way that in every set every index $1 \leq i \leq N$ appears exactly in one pair. For instance for $l = 4$ the 3 sets may be defined as $\{(1, 2), (3, 4)\}$, $\{(1, 3), (2, 4)\}$, $\{(1, 4), (2, 3)\}$. If we look at the $l/2$ $K(i, j)$ belonging to one set, the corresponding vectors $|X_{i,j}^\pm\rangle$ are mutually orthogonal, they form an orthogonal basis of \mathcal{H}_A and \mathcal{H}_B . So all these vectors can be viewed as eigenvectors of some Hermitean operator on \mathcal{H}_A and \mathcal{H}_B and can be measured with one measurement. The vectors $|Y_{i,j}^\pm\rangle$ can also be measured with one measurement. So we need two measurements for one set and $2(l-1)$ measurements for all $K(i, j)$. Finally we need one measurement for the first sum on the rhs of (58) and this completes the proof for even l .

If l is odd, we can similarly group the $l(l-1)/2$ index pairs into l sets of $(l-1)/2$ pairs. This time in every set every index appears at most one time, one index is missing in every set and every index is missing in exactly one set. As before we need two measurements for one set and therefore $2l$ measurements for all $K(i, j)$. For the first sum on the rhs of (42) we do not need another measurement since we can put vector $|i\rangle$ to the set of index pairs where i is missing. □

Now we want to give a lower bound for the number of required measurements for a projector. This bound is based on the same idea as the proof of Proposition 2 and needs three lemmata. In Lemma 1 we give a lower bound for the rank of some matrix of the form (14) in $N \times N$ -systems. In the Lemmata 2 and 3 we show that the matrix coming from one measurement has a low rank. Together this proves our bound.

Lemma 1: *If $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has the full Schmidt rank N then the matrix (λ_{ij}) in (40) has the full rank N^2 .*

Proof: First, notice that the rank of (λ_{ij}) is independent of the choice of the basis $G_i^A \otimes G_j^B$. If one has another basis $H_i^A \otimes H_j^B$ with $G_i^A = \sum_l a_{il} H_l^A$ and $G_j^B = \sum_r b_{jr} H_r^B$ the new matrix of coefficients is given by $(\lambda'_{ij}) = \sum_{i,j} a_{li}^T \lambda_{ij} b_{jr}$ and since the matrices (a_{il}) and (b_{jr}) have full rank the matrix (λ'_{ij}) has the same rank as (λ_{ij}) .

Now we simply construct an orthonormal product basis of \mathcal{HS} where (λ'_{ij}) is diagonal and the diagonal elements do not vanish. Starting from the Schmidt decomposition $|\psi\rangle = \sum_{i=1}^N s_i |ii\rangle$ we define on \mathcal{H}_A , as well as on \mathcal{H}_B :

$$P_k = |k\rangle\langle k|, \quad 1 \leq k \leq N \tag{44}$$

$$Q_{jk} = \frac{1}{\sqrt{2}}(|j\rangle\langle k| + |k\rangle\langle j|), \quad 1 \leq j < k \leq N \tag{45}$$

$$R_{jk} = \frac{i}{\sqrt{2}}(|j\rangle\langle k| - |k\rangle\langle j|), \quad 1 \leq j < k \leq N. \tag{46}$$

These N^2 operators form an orthonormal basis of \mathcal{HS}_A (resp. \mathcal{HS}_B), denoted by H_i^A (resp. H_i^B), and if one computes

$$\lambda'_{rs} = \sum_{\alpha, \beta=1}^N s_\alpha s_\beta \langle \alpha | H_r^A | \beta \rangle \langle \alpha | H_s^B | \beta \rangle, \tag{47}$$

one can directly verify that (λ'_{rs}) is in the basis $H_r^A \otimes H_s^B$ diagonal and has the full rank. □

Corollary 1: *If $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has the Schmidt rank l then the matrix (λ_{ij}) in (40) has the rank l^2 .*

Proof: The proof is essentially the same as the proof of Lemma 1. We can view $|\psi\rangle$ as a vector in a $l \times l$ -system. □

Corollary 2: *If $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has the Schmidt rank l than a decomposition in the sense of (9) requires l^2 Hermitean operators for every party.*

Proof: If one needed less, this would be a direct contradiction to Lemma 1 and Corollary 1. Please note that we have already computed this decomposition – see (41). □

Lemma 2: *Let $\vec{v}_1, \dots, \vec{v}_r \in \mathbb{R}^n$ be some vectors obeying the equations*

$$\langle \vec{v}_i, \vec{v}_j \rangle = C \neq 0 \quad \forall i \neq j. \tag{48}$$

\vec{v}_r should be uniquely defined by $\vec{v}_1, \dots, \vec{v}_{r-1}$ and the equations (48) while \vec{v}_{r-1} should not be uniquely defined by $\vec{v}_1, \dots, \vec{v}_{r-2}$ and the equations (48). Then we have

$$\vec{v}_r \in \text{Lin}(\vec{v}_1, \dots, \vec{v}_{r-1}) \tag{49}$$

and

$$\dim(\text{Lin}(\vec{v}_1, \dots, \vec{v}_r)) = r - 1 \tag{50}$$

where $\text{Lin}(\vec{v}_1, \dots, \vec{v}_r)$ denotes the linear subspace spanned by $\vec{v}_1, \dots, \vec{v}_r$.

Proof. We can split \vec{v}_r into two parts:

$$\vec{v}_r = \vec{v}_{r\parallel} + \vec{v}_{r\perp}, \tag{51}$$

where $\vec{v}_{r\parallel} \in \text{Lin}(\vec{v}_1, \dots, \vec{v}_{r-1})$ and $\vec{v}_{r\perp} \perp \text{Lin}(\vec{v}_1, \dots, \vec{v}_{r-1})$. Since \vec{v}_r is unique, it follows that $\vec{v}_{r\perp} = 0$ (otherwise $\vec{v}_r = \vec{v}_{r\parallel} - \vec{v}_{r\perp}$ would be a different solution) and the first part of the statement is proven. The equality in (50) comes from the fact that \vec{v}_{r-1} is not unique. □

Lemma 3: Let M be one LvNM in the sense of (7) expanded in the $G_i^A \otimes G_j^B$ basis:

$$M = \sum_{i,j=1}^N c_{ij} |A_i\rangle\langle A_i| \otimes |B_j\rangle\langle B_j| = \sum_{i,j=0}^{N^2-1} \mu_{ij} G_i^A \otimes G_j^B. \tag{52}$$

Then the $(N^2 - 1) \times (N^2 - 1)$ submatrix in the right bottom corner of the $N^2 \times N^2$ matrix (μ_{ij}) (called $(\mu_{ij})_{red} = (\mu_{ij})_{i,j=1,\dots,N^2-1}$) has the rank $N - 1$.

Proof: We can write any of the projectors $|A_i\rangle\langle A_i|$ and $|B_j\rangle\langle B_j|$ as Bloch vectors \hat{A}_i and \hat{B}_j (resp. \vec{A}_i and \vec{B}_j) with the help of (37). Then we have

$$(\mu_{ij})_{red} = \sum_{i,j=1}^N c_{ij} (\vec{A}_i)^T (\vec{B}_j). \tag{53}$$

This is a $(N^2 - 1) \times (N^2 - 1)$ matrix, since every $(\vec{A}_i)^T \vec{B}_j$ is a $(N^2 - 1) \times (N^2 - 1)$ matrix. The range of this matrix is spanned by the vectors $(\vec{A}_i)^T$. The vectors $(\vec{A}_i)^T$ correspond to the vectors $|A_i\rangle$, and they obey relations of the form (39). Furthermore, \vec{A}_N is uniquely determined by $\vec{A}_1, \dots, \vec{A}_{N-1}$, since $|A_N\rangle$ is uniquely determined by $|A_1\rangle, \dots, |A_{N-1}\rangle$. Thus, we can apply our Lemma 2, and the rank of $(\mu_{ij})_{red}$ is $N - 1$. □

Theorem 2: Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ have full Schmidt rank $N > 1$. Then a local measurement of the projector $|\psi\rangle\langle\psi|$ requires at least $N + 1$ measurements.

Proof: If we look at $|\psi\rangle\langle\psi|$ in the form (40) the matrix λ_{ij} has, according to Lemma 1, the full rank N^2 , the reduced matrix $(\lambda_{ij})_{red} = (\lambda_{ij})_{i,j=1,\dots,N^2-1}$ has a rank of at least $N^2 - 2$.

Since the matrix $(\mu_{ij})_{red}$ corresponding to a single LvNM has, according to Lemma 3, the rank $N - 1$ we need at least $(N^2 - 2)/(N - 1) = N + 1 - 1/(N - 1)$ measurements. This proves the statement for $N \geq 3$. For $N = 2$ please recall that we have already computed that the submatrix $(\lambda_{ij})_{red}$ is of rank $N^2 - 1 = 3$, not $N^2 - 2$. This proves the claim for the case $N = 2$. □

The question remains, which of these results remain valid for $N \times M$ -systems with $M > N$. The answer is simple: all results remain valid. Since the maximal Schmidt rank in a $N \times M$ -system is N , Theorem 1 can be proven in just the same way. Also the arguments which led to Theorem 2 can be applied.

5. Bound entangled states

In Hilbert spaces with dimensions higher than 2×3 , there exist entangled states with positive partial transpose, the *bound* entangled states [20, 21]. For this kind of state no general operational entanglement criterion is known and thus even complete knowledge of the density matrix may not suffice to decide whether a state is entangled or not. There exists, however, an important class of bound entangled states, the so-called ‘edge’ states [22], for which the optimal witness operators can be constructed explicitly. In situations where an experiment is aimed at the generation of an edge state our method of local decomposition of a witness provides, therefore, a genuine experimental test.

A state δ is called an edge state iff it cannot be represented as $\delta = q\delta' + (1 - q)\sigma_s$, where $0 \leq q < 1$, σ_s is a separable state and δ' is a state with a positive partial transpose. In other words, for all product vectors $|e, f\rangle$ and $\epsilon > 0$, $\delta - \epsilon|e, f\rangle\langle e, f|$ is not a bound entangled state anymore. This implies that the edge states lie on the boundary between the bound entangled states and the entangled states with non-positive partial transpose. They violate the range criterion [20] in an extremal sense, i.e. δ is an entangled edge state with a positive partial transpose iff for all product vectors $|e, f\rangle \in R(\delta)$, $|e, f^*\rangle \notin R(\delta^{T_B})$, where $R(\delta)$ denotes the range of δ .

The generic form of an entanglement witness for such a state δ is [23]

$$W = \bar{W} - \epsilon 1, \tag{54}$$

where

$$\bar{W} = (P + Q^{T_A}) \tag{55}$$

$$\epsilon = \inf_{|e, f\rangle} \langle e, f | \bar{W} | e, f \rangle, \tag{56}$$

and P and Q denote the projectors onto the kernel of δ and δ^{T_A} , respectively.

In the following we construct witnesses following this method for three kinds of bound entangled edge states from the literature, unextendable product basis (UPB) states introduced by Bennett *et al.* [24], ‘chessboard’ states from Bruß and Peres [25] and the bound entangled states in 2×4 dimensions introduced by P. Horodecki [20], and decompose them locally optimizing the number of projectors onto product states or the number of settings as explained above. Note that for this particular construction it is from an experimentalist’s point of view natural to decompose and optimize \bar{W} rather than W , because the term $\epsilon 1$ does not require any special setting.

5.1. UPB states in 3×3 dimensions

The states

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle(|0\rangle - |1\rangle)), & |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|2\rangle(|1\rangle - |2\rangle)), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle, & |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle, \\ |\psi_4\rangle &= \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle) \end{aligned} \tag{57}$$

form a UPB [24], i.e. they are orthogonal to each other and there is no other product vector orthogonal to all of them. Therefore the state

$$\varrho_{\text{UPB}} = \frac{1}{4} \left(1 - \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i| \right), \tag{58}$$

which is the projection on the space orthogonal to that spanned by the UPB, does not contain any product state in its range. Furthermore, it has a positive partial transpose due to the orthonormality of the states $|\psi_i\rangle$. Therefore, ϱ_{UPB} is an entangled edge state with a positive partial transpose.

The projectors P and Q are related to each other by

$$P_1 = Q_1^{T_A} = \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i|, \tag{59}$$

therefore we skip Q^{T_A} and write the witness as

$$W_{\text{UPB}} = \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i| - \epsilon 1. \tag{60}$$

Five measurements are necessary to measure this witness, one for each of the five projectors, since the UPB is constructed in such a way that no two projectors can be evaluated in the same basis. The main problem of this construction is to find ϵ . An analytical bound obtained by Terhal [26] gives

$$\epsilon \geq \frac{1}{9} \frac{(6 - \sqrt{30})(2 - \sqrt{3})}{6} \frac{(2 - \sqrt{3})}{2} \simeq 0.001297. \tag{61}$$

Numerical analysis leads however to the much bigger value $\epsilon \simeq 0.02842$.

In this case it is also interesting to optimize the number of local projection measurements needed of the total witness W_{UPB} . We extend the set $\{|\psi_i\rangle, i = 0, \dots, 3\}$ with the vectors

$$\begin{aligned} |\psi_5\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle), & |\psi_7\rangle &= \frac{1}{\sqrt{2}}|2\rangle(|1\rangle + |2\rangle), \\ |\psi_6\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|2\rangle, & |\psi_8\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)|0\rangle, \\ |\psi_9\rangle &= |11\rangle \end{aligned} \tag{62}$$

to an orthonormal basis which can be used to decompose the identity. Altogether we are then left with a pseudo-mixture containing 10 projectors. Denoting $B_1 = \{|0\rangle, |1\rangle, |2\rangle\}$, $B_2 = \{(|0\rangle - |1\rangle)/\sqrt{2}, |2\rangle, (|0\rangle + |1\rangle)/\sqrt{2}\}$, $B_3 = \{(|1\rangle - |2\rangle)/\sqrt{2}, |0\rangle, (|1\rangle + |2\rangle)/\sqrt{2}\}$, and $B_4 = \{(|0\rangle - |1\rangle)/\sqrt{2}, (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}, (|0\rangle + |1\rangle - 2|2\rangle)/2\}$, we easily see that measurement of W_{UPB} decomposed in this form requires six correlated settings for Alice and Bob: $B_1B_2, B_2B_1, B_1B_3, B_3B_1, B_4B_4$, and B_1B_1 , therefore it is unfeasible to decompose the whole witness from the number of settings point of view as noted above. By subtracting in equation (54) some positive operator I instead of 1, one can reduce the number of projectors in the decomposition of W_{UPB} to 9 – this gives an ONP, since the number of terms in any ONP must be larger than or equal to the rank of the witness, which is equal to 9. The idea is to form I as a convex sum of projectors onto $|\psi_i\rangle_{i=0,\dots,4}$ and onto four other product vectors that are obviously not orthogonal to the five UPB states, but can be chosen such that the set of the nine vectors forms a basis. If we choose as the additional vectors $|\bar{\psi}_i\rangle_{i=4,\dots,7}$ the decomposition contains nine projectors in five settings. The bound for ϵ has to be adapted as

$$\epsilon' = \inf_{|e,f\rangle} \frac{\langle e,f|\bar{W}|e,f\rangle}{\langle e,f|I|e,f\rangle}. \tag{63}$$

Numerical analysis leads to a value of $\epsilon' \simeq 0.0311$. Note that when the bound entangled state is affected by white noise, namely $\rho_p = p \cdot \rho_{\text{UPB}} + (1 - p)1/9$, the

witness given above is still suitable for the detection of entanglement. For the witness in equation (60), $Tr(W_{UPB}\rho_p) < 0$ when $p > (1 - 9\epsilon/5)$.

5.2. Chessboard states in 3×3 dimensions

The states introduced in [25] are constructed from four entangled vectors,

$$\rho_{cb} = N \sum_{i=1}^4 |V_i\rangle\langle V_i|, \tag{64}$$

$$\begin{aligned} |V_1\rangle &= (m, 0, s; 0, n, 0; 0, 0, 0) \\ |V_2\rangle &= (0, a, 0; b, 0, c; 0, 0, 0) \\ |V_3\rangle &= (n^*, 0, 0; 0, -m^*, 0; t, 0, 0) \\ |V_4\rangle &= (0, b^*, 0; 0, -a^*, 0; 0, d, 0), \end{aligned} \tag{65}$$

where $*$ denotes complex conjugation and $N = 1/\sum_j \langle V_j|V_j\rangle$. By choosing the phases of the $|V_i\rangle$ and the basis vectors, six of the parameters can be made real. Without loss of generality, t and s can be assumed to be complex. In matrix form, ρ_{cb} can then be written as

$$\rho_{cb} = N \begin{pmatrix} m^2 + n^2 & 0 & ms^* & 0 & 0 & 0 & nt^* & 0 & 0 \\ 0 & a^2 + b^2 & 0 & 0 & 0 & ac & 0 & bd & 0 \\ sm & 0 & |s|^2 & 0 & sn & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 + b^2 & 0 & bc & 0 & -ad & 0 \\ 0 & 0 & ns^* & 0 & m^2 + n^2 & 0 & -mt^* & 0 & 0 \\ 0 & ac & 0 & cb & 0 & c^2 & 0 & 0 & 0 \\ tn & 0 & 0 & 0 & -tm & 0 & |t|^2 & 0 & 0 \\ 0 & bd & 0 & -da & 0 & 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{66}$$

Bruß and Peres suggested two methods of ensuring that ρ_{cb} is bound entangled. We will employ the first one, i.e. we demand that $\rho_{cb} = \rho_{cb}^{T_A}$, which is fulfilled for $t = ad/m$ and $s = ac/n$ real and implies that $P = Q$. The kernel of ρ_{cb} (and of $\rho_{cb}^{T_A}$) is spanned by the (non-normalized) vectors

$$|k_1\rangle = |22\rangle \tag{67}$$

$$|k_2\rangle = \left(\frac{m}{n}, 0, -\frac{m^2 + n^2}{ac}; 0, 1, 0; 0, 0, 0\right) \tag{68}$$

$$|k_3\rangle = \left(0, -\frac{ac}{a^2 + b^2}, 0; -\frac{bc}{a^2 + b^2}, 0, 1; 0, 0, 0\right) \tag{69}$$

$$|k_4\rangle = \left(-\frac{ad}{mn}, 0, \frac{d}{c}; 0, 0, 0; 1, 0, 0\right) \tag{70}$$

$$|k_5\rangle = \left(0, -\frac{bd}{a^2 + b^2}, 0; \frac{ad}{a^2 + b^2}, 0, 0; 0, 1, 0\right). \tag{71}$$

In order to decompose the witness W in terms of projectors onto product states we first examine whether there are more product vectors in the kernel of ρ_{cb} , so we try to solve

$$(a_1|0\rangle + a_2|1\rangle + a_3|2\rangle) \otimes (b_1|0\rangle + b_2|1\rangle + b_3|2\rangle) = \sum_{i=1}^5 x_i |k_i\rangle. \tag{72}$$

When writing down the equations one can see that the x_i can be substituted by products of one a_j and one b_k . Then it is possible to solve the set of equations which is then linear in the parameters b_k . This in turn gives two equations for the a_j . The first solution is given by

$$|k'_4\rangle = |k_4\rangle - \frac{mn}{ac} |k_1\rangle = \left(-\frac{ad}{mn}, 0, 1\right) \otimes \left(1, 0, -\frac{mn}{ac}\right), \tag{73}$$

and with

$$\begin{aligned} \alpha_1 &\equiv (m^2 + n^2)bmn - (a^2 + b^2)am^2 \\ \alpha_3 &\equiv ad^2n^2 \\ \alpha_{13} &\equiv (m^2 + n^2)(mn + ab)d - 2abdm^2 \\ \gamma_1^{0,1} &\equiv \left(-\alpha_{13} \pm \sqrt{\alpha_{13}^2 - 4\alpha_1\alpha_3}\right) / 2\alpha_3 \\ \gamma_2^{0,1} &\equiv \left[\frac{1}{am^2} (bmn + d(mn + ab)\gamma_1^{0,1} + ad^2(\gamma_1^{0,1})^2)\right]^{1/2} \end{aligned} \tag{74}$$

the other solutions can be written as

$$|e, f\rangle = a_1 \left(1, \pm \gamma_2^{0,1}, \gamma_1^{0,1}\right) \otimes b_2 \left(\pm \frac{m^2\gamma_2^{0,1}}{mn + ad\gamma_1^{0,1}}, 1, \mp \frac{a^2 + b^2 + bd\gamma_1^{0,1}}{ac\gamma_2^{0,1}}\right). \tag{76}$$

The parameters a_1 and b_2 can be used to normalize the vectors. We found six product vectors in the kernels. Of those vectors, five will be linearly independent in general. Since they do not form an orthonormal set, we cannot construct P and Q from them. However, the witness can also be constructed by using instead of the projector onto the kernel of ρ_{cb} ($\rho_{cb}^{T_A}$) an operator \tilde{P} (\tilde{Q}) which is strictly positive on the range of the kernel of ρ_{cb} ($\rho_{cb}^{T_A}$). This will only affect the value of ϵ . Here $\tilde{P} = \tilde{Q}$ can be constructed by summing the projectors onto five linearly independent product vectors from the kernel of ρ_{cb} . Since the vectors are real, we have in addition that $\tilde{P} = \tilde{Q}^{T_A}$. Hence in general the ‘pre-witness’ \tilde{W} can be decomposed into five projectors onto product vectors requiring five settings to measure $Tr(W_{cb}\rho)$, one for each of the projectors.

5.3. Horodecki states in 2×4 dimensions

The positive operators introduced in [20] can be written in matrix form as

$$\rho_b = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+b) & 0 & 0 & \frac{1}{2}\sqrt{1-b^2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{1-b^2} & 0 & 0 & \frac{1}{2}(1+b) \end{pmatrix}, \tag{77}$$

where $b \in [0, 1]$. For $b = 0, 1$, the matrix ρ_b is separable, and bound entangled for all other values of b . In the following we assume $b \neq 0, 1$. The kernel of ρ_b is spanned by the entangled vectors

$$|k_1\rangle = \frac{1}{\sqrt{2}}(1, 0, 0, 0; 0, -1, 0, 0) \tag{78}$$

$$|k_2\rangle = \frac{1}{\sqrt{2}}(0, 1, 0, 0; 0, 0, -1, 0) \tag{79}$$

$$|k_3\rangle = \frac{1}{\sqrt{2+y^2}}(0, 0, 1, 0; y, 0, 0, -1), \tag{80}$$

where $y = \sqrt{(1-b)/(1+b)}$, so any vector in the kernel can be represented as

$$|k\rangle = (A, B, C, 0; yC, -A, -B, -C) \tag{81}$$

where A, B, C, D are complex parameters. For $|k\rangle$ to be a product vector, it must be of the form

$$|e, f\rangle = (r, s) \otimes (A', B', C', D') \equiv (r(A', B', C', D'); s(A', B', C', D')), \tag{82}$$

where r, s, A', B', C' are complex parameters. It can be readily checked that there is no possibility to write $|k\rangle$ in this form, therefore there is no product vector in the kernel of ρ_b . On the other hand, the kernel of ρ_b^{TB} is spanned by the entangled vectors

$$|k_1\rangle = \frac{1}{\sqrt{2}}(0, 0, 1, 0; 0, -1, 0, 0) \tag{83}$$

$$|k_2\rangle = \frac{1}{\sqrt{2}}(0, 0, 0, 1; 0, 0, -1, 0) \tag{84}$$

$$|k_3\rangle = \frac{1}{\sqrt{2+y^2}}(0, 1, 0, 0; 1, 0, 0, y), \tag{85}$$

and does not contain any product vector, either. Therefore, the decomposition of the witness from equation (54) in projectors onto product vectors is a rather tedious task. On the other hand, we can write down the witness W as in equation (54) and then decompose it as in equations (9) and (41):

$$W = \sum_{i=0}^3 \sum_{j=0}^{15} w_{ij} \sigma_i \otimes \tau_j \equiv \sum_{i=0}^3 \sigma_i \otimes \tilde{\tau}_i \tag{86}$$

in a straightforward manner. Here σ_i and τ_i , for $i > 0$, are the generators of the SU(2) and SU(4), respectively, while $\sigma_0 = 1_2/2$ and $\tau_0 = 1_4/4$.

The matrices $\tilde{\tau}$ turn out to be

$$\tilde{\tau}_0 = \begin{pmatrix} -c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{pmatrix} + (6 - 8\epsilon)\tau_0, \tag{87}$$

$$\tilde{\tau}_1 = \frac{1}{4} \begin{pmatrix} 0 & -cy^2 & 2cy & 0 \\ -cy^2 & 0 & -2 & 2cy \\ 2cy & -2 & 0 & -c(4 + y^2) \\ 0 & 2cy & -c(4 + y^2) & 0 \end{pmatrix}, \tag{88}$$

$$\tilde{\tau}_2 = \frac{i}{4} \begin{pmatrix} 0 & -cy^2 & -2cy & 0 \\ cy^2 & 0 & -2 & -2cy \\ 2cy & 2 & 0 & -c(4 + y^2) \\ 0 & 2cy & c(4 + y^2) & 0 \end{pmatrix}, \tag{89}$$

and

$$\tilde{\tau}_3 = -cy^2\tau_0, \tag{90}$$

where $c = 1/(2 + y^2)$. The number of correlated settings is four, but Alice and Bob need only three different settings each to measure the witness.

Summarizing, we have shown that for UPB states and ‘chessboard’ states in 3×3 systems, one needs five LvNMs to detect the witness, whereas for Horodecki states in 2×4 systems, four correlated measurements are necessary.

6. Conclusion

In this paper we have studied the problem of detection of entangled states using entanglement witnesses and few local measurements. Two optimization scenarios were discussed: the one corresponding to the decomposition of the witness operator into the optimal number of projectors on product vectors (ONP) and the one corresponding to the decomposition of the witness into the optimal number of settings of detecting devices (ONS). Several exact results and estimates have been obtained concerning optimal detection strategies for NPT states on 2×2 and $N \times M$ systems, entangled states in three-qubit systems, and bound entangled states in 3×3 and 2×4 systems. Despite numerous results and progress in understanding this problem, the general question of finding ONP and ONS for an arbitrary witness operator remains open.

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While finishing this paper we became aware of a recent preprint by A. Pittenger and M. Rubin [27], where it was shown how, if N is prime, a projector onto a pure maximally entangled state with full Schmidt rank can be measured with $N + 1$ LvNMs. Thus, our bound from Theorem 2 can be reached for this case.

References

- [1] BOUWMEESTER, D., EKERT, A., and ZEILINGER, A. (eds), 2000, *The Physics of Quantum Information* (Berlin: Springer); ALBER, G. *et al.*, 2001, *Quantum Information* (Berlin: Springer).
- [2] BENNETT, C. H. *et al.*, 1996, *Phys. Rev. Lett.*, **76**, 722; DEUTSCH, D. *et al.*, 1996, *Phys. Rev. Lett.*, **77**, 2818.
- [3] WHITE, A. G. *et al.*, 1998, *Phys. Rev. Lett.*, **83**, 3103; THEW, R. T. *et al.*, 2002, *Phys. Rev. A*, **66**, 012303.
- [4] LEWENSTEIN, M. *et al.*, 2000, *J. Mod. Opt.*, **47**, 2841.
- [5] PERES, A., 1996, *Phys. Rev. Lett.*, **77**, 1413; HORODECKI, M., HORODECKI, P., and HORODECKI, R., 1996, *Phys. Lett. A*, **223**, 1.
- [6] WERNER, R. F., and WOLF, M. M., 2000, *Phys. Rev. A*, **61**, 062102; see also WERNER, R. F., and WOLF, M. M., 2001, *Phys. Rev. A*, **64**, 032112; ZUKOWSKI, M., and BRUKNER, C., 2001, *Phys. Rev. Lett.*, **88**, 210401.
- [7] PERES, A., 1999, *Found. Phys.*, **29**, 589.
- [8] SANCHO, F., and HUELGA, S., 2000, *Phys. Rev. A*, **61**, 42303.
- [9] HORODECKI, P., quant-ph/0111082.
- [10] HORODECKI, P., and EKERT, A., 2002, *Phys. Rev. Lett.*, **89**, 127902.
- [11] GÜHNE, O. *et al.*, 2002, *Phys. Rev. A*, **66**, 062305.
- [12] TERHAL, B., 2001, *Lin. Alg. Appl.*, **323**, 61.
- [13] LEWENSTEIN, M. *et al.*, 2000, *Phys. Rev. A*, **62**, 052310; LEWENSTEIN, M. *et al.*, 2001, *Phys. Rev. A*, **63**, 044304; SANPERA, A. *et al.*, 2001, *Phys. Rev. A*, **63**, 050301.
- [14] SANPERA, A. *et al.*, 1998, *Phys. Rev. A*, **58**, 826.
- [15] For the discussion of volumes see: ŻYCZKOWSKI, K. *et al.*, 1998, *Phys. Rev. A*, **58**, 883; ŻYCZKOWSKI, K., 1999, *Phys. Rev. A*, **60**, 3496.
- [16] ŻYCZKOWSKI, K., and SOMMERS, H.-J., 2001, *J Phys. A*, **34**, 7111.
- [17] DÜR, W. *et al.*, 2000, *Phys. Rev. A*, **62**, 062314.
- [18] ACIN, A., BRUSS, D., LEWENSTEIN, M., and SANPERA, A., 2001, *Phys. Rev. Lett.*, **87**, 040401.
- [19] SCHLIENZ, J., and MAHLER, G., 1995, *Phys. Rev. A*, **52**, 4396.
- [20] HORODECKI, P., 1997, *Phys. Lett. A*, **232**, 333.
- [21] HORODECKI, M., HORODECKI, P., and HORODECKI, R., 1998, *Phys. Rev. Lett.*, **80**, 5239.
- [22] KRAUS, B., CIRAC, J. I., KARNAS, S., and LEWENSTEIN, M., 2000, *Phys. Rev. A*, **61**, 062303; see also LEWENSTEIN, M., CIRAC, J. I., and KARNAS, S., quant-ph/9903012; HORODECKI, P., LEWENSTEIN, M., VIDAL, G., and CIRAC, J. I., 2000, *Phys. Rev. A*, **62**, 032310.
- [23] LEWENSTEIN, M., KRAUS, B., CIRAC, J. I., and HORODECKI, P., 2000, *Phys. Rev. A*, **62**, 052310.
- [24] BENNETT, C. H., DIVINCENZO, D. P., MOR, T., SHOR, P. W., SMOLIN, J. A., and TERHAL, B.M., 1999, *Phys. Rev. Lett.*, **82**, 5385.
- [25] BRUSS, D., and PERES, A., 2000, *Phys. Rev. A*, **61**, 30301.
- [26] TERHAL, B. M., 2000, *Lin. Alg. Appl.*, **323**, 61.
- [27] PITTENGER, A., and RUBIN, H., to appear in *Phys. Rev. A*.