Optimal estimation of multiple phases

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We study the issue of simultaneous estimation of several phase shifts induced by commuting operators on a quantum state. We derive the optimal positive-operator-valued measure corresponding to the multiple-phase estimation. In particular, we discuss the explicit case of the optimal detection of double phase for a system of identical qutrits and generalize these results to optimal multiple-phase detection for *d*-dimensional quantum states.

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I. INTRODUCTION

The issue of phase estimation has important applications in quantum computation and quantum information theory. For example, it was shown that the existing quantum algorithms can be described in a unified way as quantum interference processes among different computational paths where the result of the computation is encoded in a phase shift [1]. The design of optimal phase measurement procedures is also crucial in various tasks of atomic physics, such as methods for precision spectroscopy [2] and quantum interferometric experiments in quantum optics.

The problem of the optimal estimation of the value of a phase shift experienced by a quantum state has been extensively studied [3], and, in particular, a method to derive the optimal measurement procedure in the phase-covariant case was reported in Ref. [4]. More recently, a general formulation of the phase estimation problem was considered in Ref. [5], where a method to derive the optimal positive-operator-valued measurement (POVM) [6] for a generally degenerate phase-shift operator was developed. In this work, we introduce the problem of simultaneous estimation of several phase shifts undergone by a quantum physical system. More specifically, we address the problem of estimating the values of M-independent phase shifts ϕ_j $(j=1,M)$, pertaining to the unitary transformation

$$\rho_{\{\phi_j\}}=\exp\left(-i\sum_{j=1}^{M}\phi_j\hat{H}_j\right)\rho_0\exp\left(i\sum_{j=1}^{M}\phi_j\hat{H}_j\right), \qquad (1)$$

where $\hat{H}_j$ represent M commuting self-adjoint operators, which are, in general, degenerate on the Hilbert space $\mathcal{H}$ of the considered quantum system and each of them has a discrete spectrum S_j (S_j can be, for example, Z, N, or Z_q, $q>0$) [7]. In Eq. (1) ρ_0 is a generic initial pure state $|\psi_0\rangle\langle\psi_0|$ describing a quantum system with arbitrary dimension. We want to point out that the scenario of simultaneous estimation of several phases may be useful to improve the efficiency of quantum information processing tasks where several variables are encoded into phases in the same quantum states.

The paper is organized as follows. In Sec. II we derive a general treatment of the multiple phase estimation problem, extending to the multiple-phase case the approach presented in Ref. [5] for the case of single-phase estimation. In Sec. III we derive the optimal POVM and the corresponding estimation fidelity for a system of N identically prepared "equatorial" three-dimensional systems. We want to point out that the possibility of encoding information in the states of three-dimensional systems has been the object of several recent studies, for example, in the context of quantum cloning [8] and quantum cryptography [9]. In Sec. IV we extend these results derived for qutrits to the case of quantum systems with arbitrary finite dimension d. Finally, we summarize and comment the results presented in Sec. V.

II. OPTIMAL POVM FOR MULTIPLE-PHASE ESTIMATION

In this section we derive the optimal POVM corresponding to the simultaneous estimation of several phase shifts experienced by a pure state $|\psi_0\rangle$, belonging to the Hilbert space $\mathcal{H}$ that undergoes the unitary transformation (1). Following the approach of Ref. [5], we treat the estimation problem in the general framework of quantum estimation theory [6]. According to this framework, we first define a cost function $C(\{\overline{\phi_j}\},\{\phi_j\})$ that depends on the set of the M estimated values $\{\overline{\phi_j}\}$, which are the results of the estimation procedure, and on the set of the M true values $\{\phi_j\}$. This function weights the errors for the estimates $\{\overline{\phi_j}\}$ given the true values $\{\phi_j\}$. The estimation problem then consists in minimizing the average cost $\overline{C}$ of the procedure defined as

$$\overline{C}=\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2\cdots\int_0^{2\pi}d\phi_M p_0(\{\phi_j\})\int_0^{2\pi}d\overline{\phi_1}\times\int_0^{2\pi}d\overline{\phi_2}\cdots\int_0^{2\pi}d\overline{\phi_M}C(\{\overline{\phi_j}\},\{\phi_j\})p(\{\overline{\phi_j}\}|\{\phi_j\}), \qquad (2)$$

where $p_0(\{\phi_j\})$ is the *a priori* probability density for the true values $\{\phi_j\}$ and $p(\{\overline{\phi_j}\}|\{\phi_j\})$ is the conditional probability of estimating the set of values $\{\overline{\phi_j}\}$ given the true values $\{\phi_j\}$. The average cost is minimized by optimizing the POVM $d\mu(\{\overline{\phi_j}\})$ which appears in the definition of the conditional probability as follows:

$$p(\{\overline{\phi_j}\}|\{\phi_j\})d\overline{\phi}_1 d\overline{\phi}_2 \cdots d\overline{\phi}_M = \mathrm{Tr}\Bigg[d\mu(\{\overline{\phi_j}\}) \times \exp\left(-i\sum_{j=1}^{M} \phi_j \hat{H}_j\right)\rho_0 \exp\left(i\sum_{j=1}^{M} \phi_j \hat{H}_j\right)\Bigg]. \quad (3)$$

In this work we consider the general scenario where all the values $\{\phi_j\}$ are *a priori* uniformly distributed, i.e., the probability density is simply given by $p_0(\{\phi_j\})=(1/2\pi)^M$. Moreover, we consider the case where the errors in the estimates are weighted independently of the values ϕ_j of the phases, but they depend only on the values of $\overline{\phi_j}-\phi_j$, so that the cost function becomes an even function of M variables, i.e. $C(\{\overline{\phi_j}\},\{\phi_j\})\equiv C(\{\overline{\phi_j}-\phi_j\})$. From these requirements, it follows that also the conditional probability corresponding to the optimal estimation procedure will depend only on the variables $\overline{\phi_j}-\phi_j$, and therefore the optimal POVM will be phase covariant, i.e., of the form

$$d\mu(\{\overline{\phi_j}\}) = \exp\left(-i\sum_{j=1}^{M} \overline{\phi_j}\hat{H}_j\right)\chi \times \exp\left(i\sum_{j=1}^{M} \overline{\phi_j}\hat{H}_j\right)\frac{d\overline{\phi_1}}{2\pi}\frac{d\overline{\phi_2}}{2\pi}\cdots\frac{d\overline{\phi_M}}{2\pi}. \quad (4)$$

In the above equation χ is a positive operator satisfying the completeness constraints needed for the normalization of the POVM $\int d\mu(\{\phi_j\})=\mathbb{1}$, where $\mathbb{1}$ denotes the identity operator. Actually, using Eq. (3) and the invariance of the trace under cyclic permutations it can be shown that $p(\{\overline{\phi_j}\}|\{\phi_j\}) \equiv p(\{\overline{\phi_j}-\phi_j\})$ if and only if $d\mu(\{\overline{\phi_j}\})$ is covariant. Therefore the optimization of the phase estimation procedure can be performed by finding the positive operator χ that minimizes the average cost for a given cost function $C(\{\phi_j\})$ and a generic initial state ρ_0.

We will now show explicitly how to derive the optimal POVM for a broad class of cost functions and initial states ρ_0. First of all we will choose the representation where all the operators $\hat{H}_j$ are diagonal. We have assumed that operators $\hat{H}_j$ commute, so we can identify a common basis of eigenvectors. Operators $\hat{H}_j$ are generally degenerate, and we will denote by $|\{n_j\}\rangle_\nu$ a choice of (normalized) eigenvectors corresponding to eigenvalue n_j for operator $\hat{H}_j$, by $\Pi_{\{n_j\}}$ the projector onto the corresponding degenerate eigenspace and by ν a degeneracy index, whose maximum value corresponds to the dimension of the degenerate eigenspace.

We will now generalize the projection method developed in Ref. [5] and define $\mathcal{H}_\parallel$ as the Hilbert space spanned by the (normalized) vectors $|\{n_j\}\rangle \propto \Pi_{\{n_j\}}|\psi_0\rangle \neq 0$ with the choice of the arbitrary phases such that $\langle\{n_j\}|\psi_0\rangle > 0$. We can then write the Hilbert space of the system as $\mathcal{H}=\mathcal{H}_\parallel\otimes\mathcal{H}_\perp$, where component $\mathcal{H}_\perp$ is spanned by states that are orthogonal to $|\psi_0\rangle$. Hence the POVM can be chosen of the block diagonal form on $\mathcal{H}_\parallel\otimes\mathcal{H}_\perp$, i.e., $d\mu(\{\phi_j\})=d\mu_\parallel(\{\phi_j\})\oplus d\mu_\perp(\{\phi_j\})$. In this way the component $d\mu_\perp(\{\phi_j\})$ of the POVM acting on $\mathcal{H}_\perp$ can be chosen arbitrarily because it does not contribute to the average cost. Therefore, the optimization of the estimation procedure can be performed by optimizing only the component $d\mu_\parallel(\{\phi_j\})$ of the POVM.

In order to optimize the POVM we can assume that $\Pi_{\{n_j\}}|\psi_0\rangle \neq 0$ for all values of $\{n_j\}$, since the resulting POVM will be optimal also for states having zero projection for some of these values. Due to the covariance property (4) and to the argument followed above, we can also write $\chi = \chi_\parallel \oplus \chi_\perp$. Thus, the problem reduces to finding the positive operator $\chi_\parallel$ that minimizes the cost $\overline{C}$ in Eq. (2). In order to accomplish this task, we first rewrite Eq. (2) more conveniently as

$$\overline{C} = \int_0^{2\pi}\frac{d\phi_1}{2\pi}\int_0^{2\pi}\frac{d\phi_2}{2\pi}\cdots\int_0^{2\pi}\frac{d\phi_M}{2\pi}C(\{\phi_j\}) \times \mathrm{Tr}\Bigg[\chi\exp\left(-i\sum_{j=1}^{M}\phi_j\hat{H}_j\right)\rho_0\exp\left(i\sum_{j=1}^{M}\phi_j\hat{H}_j\right)\Bigg]. \quad (5)$$

We will now express the operator $\chi_\parallel$ on the $|\{n_j\}\rangle$ basis as follows:

$$\chi_\parallel = \sum_{\{n_j\},\{m_j\}} |\{n_j\}\rangle\langle\{m_j\}|\chi_{\{n_j\}\{m_j\}}. \quad (6)$$

The positivity condition for operator χ implies the inequalities

$$|\chi_{\{n_j\}\{m_j\}}| \leqslant \sqrt{\chi_{\{n_j\}\{n_j\}}\chi_{\{m_j\}\{m_j\}}} = 1, \quad (7)$$

where the last equality comes from the POVM completeness relation $\int d\mu_\parallel(\phi)=\mathbb{1}_\parallel$.

The cost functions we will consider are 2π-periodic functions in variables $\{\phi_j\}$ and, therefore, they can be written as

$$C(\{\phi_j\}) = -\sum_{l_1,l_2,\ldots,l_M=-\infty}^{\infty} c_{\{l_j\}}\exp\left(i\sum_j l_j\phi_j\right) \quad (8)$$

with condition $c_{\{l_j\}}=c_{\{-l_j\}}$ due to the fact that the cost is a real and even function. By performing the integrals in Eq. (5) and exploiting the relation $\int_0^{2\pi}d\phi e^{i(n-m)\phi}=\delta_{n,m}/2\pi$, we arrive at the following form of the average cost:

$$\overline{C} = -c_0 - \sum_{\{l_j\}\neq 0} c_{\{l_j\}} \sum_{\{m_j-n_j\}=\{l_j\}} \langle\psi_0|\{n_j\}\rangle \times \langle\{m_j\}|\psi_0\rangle\chi_{\{n_j\}\{m_j\}}, \quad (9)$$

where the expression $\{l_j\}\neq 0$ under the first summation symbol means that the sum contains all the values of indexes l_j apart from the case where they are all zero [this contribution corresponds to the term c_0 in Eq. (9)] and the expression $\{m_j-n_j\}=\{l_j\}$ under the second summation means that the equality $m_j-n_j=l_j$ must hold for all values of index j.

Let us now consider the following inequality:

$$\mathrm{sgn}(c_{\{l_j\}})\sum_{\{m_j-n_j\}=\{l_j\}}\langle\psi_0|\{n_j\}\rangle\langle\{m_j\}|\psi_0\rangle\chi_{\{n_j\}\{m_j\}}$$
$$\leqslant\sum_{\{m_j-n_j\}=\{l_j\}}\langle\psi_0|\{n_j\}\rangle||\langle\{m_j\}|\psi_0\rangle|, \tag{10}$$

where we remind that we chose $\langle\{n_j\}|\psi_0\rangle>0$. The above relation becomes an equality if the conditions

$$\chi_{\{n_j\}\{m_j\}}=\mathrm{sgn}(c_{\{m_j-n_j\}}) \tag{11}$$

can be fulfilled. In this case the minimum cost takes the simple form

$$\overline{C}=-c_0-\sum_{\{l_j\}\neq 0}c_{\{l_j\}}\sum_{\{m_j-n_j\}=\{l_j\}}|\langle\psi_0|\{n_j\}\rangle||\langle\{m_j\}|\psi_0\rangle|. \tag{12}$$

Notice, however, that the positivity of $\chi_\|$ is not generally guaranteed for any set of values of $\mathrm{sgn}(c_{\{l_j\}})$.

Let us now define a general class of cost functions, which extends that considered by Holevo [4], with

$$c_{\{l_j\}}\geqslant 0,\quad \forall\{l_j\}\neq 0. \tag{13}$$

For this class, conditions (11) are trivially satisfied for all the sets of values $\{n_j\}$ and the optimal POVM takes the explicit form

$$d\mu_\|(\{\phi_j\})=\frac{d\phi_1}{2\pi}\ldots\frac{d\phi_M}{2\pi}|e(\{\phi_j\})\rangle\langle e(\{\phi_j\})|, \tag{14}$$

where vectors $|e(\{\phi_j\})\rangle$ are defined as

$$|e(\{\phi_j\})\rangle=\sum_{\{n_j\}}\exp\left(i\sum_j n_j\phi_j\right)|\{n_j\}\rangle. \tag{15}$$

In the following sections we will illustrate more explicitly the results presented here by considering specific examples.

III. DOUBLE-PHASE ESTIMATION FOR QUTRITS

As a simple application of the concepts presented above, let us consider the optimal double-phase estimation for N identical three-dimensional quantum systems (qutrits) all in the state

$$|\psi(\phi,\theta)\rangle=\frac{1}{\sqrt{3}}(|0\rangle+e^{i\phi}|1\rangle+e^{i\theta}|2\rangle), \tag{16}$$

where $\{|0\rangle,|1\rangle,|2\rangle\}$ represents a basis for the qutrit. In the language of the preceding section we identify $\phi_1=\phi$, $\phi_2=\theta$, $\hat{H}_1=|1\rangle\langle 1|$, $\hat{H}_2=|2\rangle\langle 2|$, and $|\psi_0\rangle=(|0\rangle+|1\rangle+|2\rangle)/\sqrt{3}$ for each qutrit. For the composite system of N qutrits we have $\hat{H}_1=\Sigma_{k=1}^N|1\rangle\langle 1|_k$ and $\hat{H}_2=\Sigma_{k=1}^N|2\rangle\langle 2|_k$, where $|j\rangle\langle j|_k$ denotes the projection operator onto the state $|j\rangle$ of the kth qutrit. Operators $\hat{H}_1$ and $\hat{H}_2$ commute, and they are diagonalized in the basis $|N-n_1-n_2,n_1,n_2\rangle_\nu$ of states where $N-n_1-n_2$ qutrits are in state $|0\rangle$, n_1 in state $|1\rangle$ and n_2 in the state $|2\rangle$. The symbol ν represents the degeneracy index of the corresponding subspace, and, in particular, it ranges from 1 to $N!/(N-n_1-n_2)!n_1!n_2!$ Since the state of the N qutrits is symmetric under any permutation performed on the qutrits, the states $|\{n_j\}\rangle$ defined in the preceding section by the projection method correspond in this case to the symmetric normalized states of the N qutrits, which we will simply denote as $|N-n_1-n_2,n_1,n_2\rangle_s$ (such a state is an equally weighted superposition of $N!/(N-n_1-n_2)!n_1!n_2!$ components corresponding to all the possible permutations of states with $N-n_1-n_2$ qutrits in state $|0\rangle$, n_1 in state $|1\rangle$, and n_2 in state $|2\rangle$).

In this case the optimal POVM (14) for the cost functions of the generalized Holevo form (13) takes the form

$$d\mu(\phi,\theta)\equiv\frac{d\phi}{2\pi}\frac{d\theta}{2\pi}|e(\phi,\theta)\rangle\langle e(\phi,\theta)|, \tag{17}$$

where

$$|e(\phi,\theta)\rangle=\sum_{n_1=0}^{N}\sum_{n_2=0}^{N}{}_{-n_1}e^{i(n_1\phi+n_2\theta)}|N-n_1-n_2,n_1,n_2\rangle_s.$$

Let us now compute the fidelity of the optimal double phase estimation procedure. As a cost function we can choose, for example, $1-F$, where F is the fidelity of the estimated state $|\psi(\overline{\phi},\overline{\theta})\rangle$ with respect to the true state $|\psi(\phi,\theta)\rangle$. This cost belongs to class (13), and therefore the corresponding optimal POVM is the one written above. This choice of the cost function is particularly interesting because the fidelity is the figure of merit usually adopted to describe other processes in the quantum information theory, such as cloning transformations, and therefore it allows a direct comparison of the efficiency of optimal phase estimation with other procedures.

By the covariance of the procedure we can write the fidelity as

$$F(\phi,\psi)=|\langle\psi_0|\psi(\phi,\theta)\rangle|^2=\tfrac{1}{9}[3+2\cos\phi+2\cos\psi$$
$$+2\cos(\phi-\psi)], \tag{18}$$

where

$$|\psi_0\rangle=\frac{1}{\sqrt{3^N}}\sum_{j=0}^{N}\sum_{k=0}^{N-j}\sqrt{M(N,j,k)}|N-j-k,j,k\rangle_s. \tag{19}$$

In the above equation we have defined $M(N,j,k)\equiv N!/(N-j-k)!j!k!$.

The average fidelity $\overline{F}$ of the procedure is then given by

$$\overline{F}\equiv\int F(\phi,\psi)\mathrm{Tr}[\rho_0 d\mu(\phi,\psi)]=\frac{1}{9}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\psi}{2\pi}[3$$
$$+2\cos\phi+2\cos\psi+2\cos(\phi-\psi)]|\langle\psi_0|\psi(\phi,\theta)\rangle|^2. \tag{20}$$

By performing the integration in Eq. (20), we have

$$\overline{F}=\frac{1}{3}+\frac{1}{3^{N+2}}\sum_{j,p=0}^{N}\sum_{k=0}^{N-j}\sum_{q=0}^{N-p}\sqrt{M(N,j,k)M(N,p,q)}\times[\delta_{k,q}(\delta_{j,p+1}+\delta_{j+1,p})+\delta_{j,p}(\delta_{k,q+1}+\delta_{k+1,q})+\delta_{j+1,p}\delta_{k,q+1}+\delta_{j,p+1}\delta_{k+1,q}]=\frac{1}{3}+\frac{2}{3^{N+1}}\sum_{j=0}^{N-1}\sum_{k=0}^{N-j-1}M(N,j,k)\sqrt{\frac{N-j-k}{j+1}}. \quad (21)$$

We want to point out that fidelity (21) corresponding to the optimal double-phase estimation for qutrits is always smaller than that for equatorial qubits, given in Ref. [10], where a single phase is estimated.

We want also to stress that, as in the case of qubits, there is a relation between optimal double-phase estimation and optimal cloning for states of form (16). Actually, the fidelity of the optimal double-phase estimation (21) for a single qutrit ($N=1$) coincides with the cloning fidelity for the optimal $1\rightarrow M$ cloning transformations, which take a single input equatorial qutrit and produce M output copies [11], in the limit of an infinite number of output copies, i.e., $M\rightarrow\infty$. Moreover, this result is consistent also with the relation between optimal state estimation [12] and optimal cloning [13] for input qutrits whose state is completely unknown [not restricted to the form (16)].

We want to point out that other figure of merits could be considered in order to evaluate the efficiency of the phase estimation procedure. For example, a mean periodic "variance" $V(\phi,\theta)=2(\sin^2\phi/2+\sin^2\theta/2)$ could be considered as a cost function. In this case the cost function still belongs to class (13) and, therefore, by explicitly calculating the average variance in a similar way as for the average fidelity, we arrive at the form

$$\overline{V}\equiv\int V(\phi,\psi)\mathrm{Tr}[\rho_0 d\mu(\phi,\psi)]=2-\frac{2}{3^N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-j-1}M(N,j,k)\sqrt{\frac{N-j-k}{j+1}}. \quad (22)$$

IV. MULTIPLE-PHASE ESTIMATION FOR SYSTEMS WITH ARBITRARY DIMENSION

In this section we generalize the results derived above for qutrits to the case of multiple phase estimation for systems with arbitrary finite dimension d (qudits). We will consider the optimal multiple-phase estimation for N identical d-dimensional quantum systems all in the state

$$|\psi(\{\phi_j\})\rangle=\frac{1}{\sqrt{d}}(|0\rangle+e^{i\phi_1}|1\rangle+e^{i\phi_2}|2\rangle+\cdots+e^{i\phi_{d-1}}|d-1\rangle), \quad (23)$$

where $\{|0\rangle,|1\rangle,|2\rangle,\ldots,|d-1\rangle\}$ represents a basis for each system.

In the language of Sec. II, we are considering the estimation problem for $d-1$ phases corresponding to operators $\hat{H}_j=|j\rangle\langle j|$, $j=1,\ldots,d-1$, and $|\psi_0\rangle=(|0\rangle+|1\rangle+|2\rangle+\cdots+|d-1\rangle)/\sqrt{d}$ for each system. For the composite system of N qudits, we have $\hat{H}_j=\Sigma_{k=1}^{N}|j\rangle\langle j|_k$, where as in the preceding section $|j\rangle\langle j|_k$ denotes the projection operator onto state $|j\rangle$ of the kth qudit. Operators $\hat{H}_j$ commute, and they are diagonalized in the basis $|n_0,n_1,n_2,\ldots,n_{d-1}\rangle_\nu$ of the where n_0 qudits are in the state $|0\rangle$, n_1 qudits are in the state $|1\rangle$, and so on, with $\Sigma_{j=0}^{d-1}n_j=N$. As in the case of qutrits, ν represents the degeneracy index of the corresponding subspace and, in particular, it ranges from 1 to the multinomial $N!/(N-n_1-n_2-\cdots n_{d-1})!n_1!n_2!\cdots n_{d-1}!$ Analogously to the case of qutrits, the POVM is optimized by choosing the symmetric normalized states of the N qudits, which we will simply denote as $|n_0,n_1,n_2,\ldots,n_{d-1}\rangle_s$.

The optimal POVM for the cost functions of the generalized Holevo form (13) is given by Eq. (14), with $M=d-1$ and

$$|e(\{\phi_j\})\rangle=\sum_{\{n_j\}}\exp\left(i\sum_{j=1}^{d-1}n_j\phi_j\right)|n_0,n_1,n_2,\ldots,n_{d-1}\rangle_s. \quad (24)$$

In the above equation the sum over $\{n_j\}$ means that variables n_j take all the possible non-negative values compatible with constraint $\Sigma_{j=0}^{d-1}n_j=N$.

Let us now compute the fidelity of the optimal multiple-phase estimation procedure derived above. As in the case of qutrits, we choose a cost function of the form $1-F$, where F is the fidelity of the estimated state $|e(\{\overline{\phi_j}\})\rangle$ with respect to the true state $|e(\{\phi_j\})\rangle$. This cost belongs to class (13), and therefore the corresponding optimal POVM is the one mentioned above. By the covariance of the procedure we can write the fidelity as

$$F(\{\phi_j\})=|\langle\psi_0|\psi(\{\phi_j\})\rangle|^2=\frac{1}{d^2}\left[d+2\sum_{j=1}^{d-1}\cos\phi_j+2\sum_{j>k}\cos(\phi_j-\phi_k)\right], \quad (25)$$

where

$$|\psi_0\rangle=\frac{1}{\sqrt{d^N}}\sum_{\{n_j\}}\sqrt{\frac{N!}{n_0!n_1!n_2!\ldots n_{d-1}!}}\times|n_0,n_1,n_2,\ldots,n_{d-1}\rangle_s. \quad (26)$$

The average fidelity $\overline{F}$ of the procedure is now given by

$$\overline{F}=\frac{1}{d^2}\int_0^{2\pi}\frac{d\phi_0}{2\pi}\cdots\int_0^{2\pi}\frac{d\phi_{d-1}}{2\pi}F(\{\phi_j\})|\langle\psi_0|\psi(\{\phi_j\})\rangle|^2. \quad (27)$$

By performing the integrations in Eq. (27) we have

$$\overline{F}=\frac{1}{d}+\frac{d-1}{d^{N+1}}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-n_1-1}\cdots\sum_{n_{d-1}=0}^{N-n_1-n_2-\cdots-1}\times\frac{N!}{(N-n_1-n_2-\cdots n_{d-1})!n_1!n_2!\cdots n_{d-1}!}\times\sqrt{\frac{N-n_1-n_2-\cdots n_{d-1}}{n_1+1}}. \quad (28)$$

Notice that the fidelity decreases as a function of dimension *d*. For example, in the case of multiple-phase estimation on the state of a single qudit it takes the simple form

$$\overline{F_1}=\frac{2d-1}{d^2}. \quad (29)$$

We also want to point out that the above fidelity is larger than the fidelity of estimation of a single qudit in a completely unknown pure state [not restricted to be of form (23)]. Actually, the fidelity of such a universal procedure, which we will call $\overline{F}_{univ,1}$, is given by [12]

$$\overline{F}_{univ,1}=\frac{2}{d+2}. \quad (30)$$

V. CONCLUSIONS

In this paper we have addressed the problem of simultaneous estimation of several phase shifts induced by a unitary transformation acting on a quantum system. We have derived in a general way the optimal estimation procedure for an arbitrary number of phase shifts and for a wide class of cost functions. We have then specialized the results obtained to the case of ''equatorial'' qutrits and then generalized them to the case of a quantum system with arbitrary finite dimension. An interesting result that was emphasized in this work is the connection between optimal double-phase estimation and optimal double-phase-covariant cloning for qutrits. We expect also that such a connection is valid in arbitrary finite dimension.

Before closing the paper, we want to point out that the scenario of simultaneous estimation of several phases considered in this work may be exploited to design schemes where several variables are encoded into phases in the same quantum states and in this way the efficiency of quantum information processing tasks may be improved.

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