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On the Entanglement Structure in Quantum Cloning

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We study the entanglement properties of the output state of a universal cloning machine. We analyse in particular bipartite and tripartite entanglement of the clones, and discuss the "classical limit" of infinitely many output copies.

KEY WORDS: entanglement structure; quantum cloning.

1. INTRODUCTION

A fundamental law in quantum physics is the no-cloning theorem,[1,2] which tells us that it is impossible to copy an unknown quantum state perfectly. This feature is a direct consequence of the linearity of the Schrödinger equation.

From a physical point of view, it should better not be allowed to copy an unknown quantum state perfectly, because otherwise a conflict with other well-established laws of physics would arise: first, perfect quantum cloning would allow to use an entangled EPR-state that is shared between two remote parties to send information across the distance of the parties, where the speed of the signalling would only depend on the speed of the cloning machine and a subsequent measurement, thus allowing superluminal signalling. Second, perfect cloning would allow to produce infinitely many identical copies of a given spin state, half of which could be measured, e.g., in the x-basis, half of them in the y-basis, therefore

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allowing to measure the expectation value of non-commuting observables simultaneously.

Since the beginning of the development of the field of quantum information the no-cloning theorem has also been viewed as one of the most fundamental differences between classical and quantum information theory. The impossibility of perfectly cloning an unknown quantum state has far-reaching consequences: for instance, it disables a spy from copying a quantum signal and resending the original, and is thus the reason for the security of quantum cryptography. At the same time, however, it also disables us from making a "quantum backup," i.e., to keep a copy of an unknown quantum state for the purpose of error correction.

The no-cloning theorem only tells us that it is impossible make a *perfect* copy of an unknown quantum state. In the recent years, much work has been done in order to explore the limits for *approximate* cloning transformations.[(3–8)] This is often loosely referred to as quantum cloning. Experimental realisations of quantum cloning have been suggested in quantum optics[(9)] and cavity QED,[(10)] and experiments have been performed in quantum optics[(11)] and nuclear magnetic resonance experiments.[(12)]

Here, we want to investigate an aspect of quantum cloning that has not received any attention so far, namely we want to study the entanglement properties of the output state of a cloning device. The motivation for this study is two-fold: first, we wish to understand some fundamental properties of quantum cloning, by shedding light on a certain limit of quantum cloning—the case of producing infinitely many clones. It was shown that the optimal fidelity of this case corresponds to the optimal state estimation process,[(6)] i.e., a cloner with infinitely many copies could be implemented by optimally estimating the input and then producing a product state of infinitely many copies by using this knowledge. In this sense this case is referred to as "classical limit," see also Ref. 4. Here we want to ask the question: is it justified to call the case of infinitely many copies the classical limit, when judging by the entanglement structure of the output state?

Second, we want to investigate the multipartite entanglement properties of the cloning output. One arrives at the shape of this specific entangled state by maximising a scalar quantity, namely the fidelity, i.e., the overlap $F = \langle\psi|\,\varrho_{\rm out}\,|\psi\rangle$, where $|\psi\rangle$ is the input state and $\varrho_{\rm out}$ the one-particle reduced density operator of the output. Does the cloning output have some significance in nature, i.e., does it resemble a state that appears naturally, e.g., in the ground state of some quantum statistical system? This question is motivated by recent studies of multiparticle entanglement in this direction.[(13–16)]

2. ENTANGLEMENT PROPERTIES OF THE 1 → 2 CLONING STATE

Let us commence by looking into the most simple case, namely the output of a universal cloner that takes any pure state of one qubit as input and produces two identical copies with optimal fidelity 5/6. In the next section we will then look at general universal cloning transformations. Throughout this paper we will only consider two-dimensional states and symmetric universal cloning transformations, i.e., transformations for which the fidelity of all output copies is equal and their quality does not depend on the form of the input state. For these universal transformations we can study without loss of generality the entanglement structure of the cloning output for an input that is the basis state $|0\rangle$. This is given by

$$\mathcal{U}\,|0\rangle\,|0\rangle\,|a\rangle=\sqrt{\frac{2}{3}}\,|0\rangle\,|0\rangle\,|0\rangle+\sqrt{\frac{1}{6}}\,(|0\rangle\,|1\rangle+|1\rangle\,|0\rangle)\,|1\rangle, \tag{1}$$

where the first qubit is the original, the second is the clone and the third is an auxiliary system, usually called ancilla. Clearly, the total state is entangled. How is the entanglement distributed over the state, i.e., how much are the two clones or a clone and an ancilla entangled with each other? The reduced density matrices $\varrho_{\rm cc}$, for two clones and $\varrho_{\rm ca}$ for one clone and one ancilla are given by

$$\varrho_{\rm cc}=\begin{pmatrix}\frac{2}{3}&0&0&0\\0&\frac{1}{6}&\frac{1}{6}&0\\0&\frac{1}{6}&\frac{1}{6}&0\\0&0&0&0\end{pmatrix},\qquad \varrho_{\rm ca}=\begin{pmatrix}\frac{2}{3}&0&0&\frac{1}{3}\\0&\frac{1}{6}&0&0\\0&0&0&0\\\frac{1}{3}&0&0&\frac{1}{6}\end{pmatrix}. \tag{2}$$

Thanks to the analytical formula for the concurrence by Wootters[17] it is easy to calculate the entanglement of formation for these bipartite density matrices. As shown in Ref. 13, a density matrix of the shape

$$\sigma=\begin{pmatrix}a&0&0&0\\0&b&c&0\\0&c^*&d&0\\0&0&0&e\end{pmatrix} \tag{3}$$

in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ has the concurrence

$$C(\sigma)=2\max(|c|-\sqrt{ae},0). \tag{4}$$

The entanglement of formation E_F can then be expressed as a function of C, namely

$$E_F = -\tfrac{1}{2}(1+\sqrt{1-C^2})\log_2[\tfrac{1}{2}(1+\sqrt{1-C^2})]$$
$$-\tfrac{1}{2}(1-\sqrt{1-C^2})\log_2[\tfrac{1}{2}(1-\sqrt{1-C^2})]. \quad (5)$$

In the case of the cloning output from Eq. (2) this leads to $C_{12} = 1/3$ for the two clones and $C_{23} = 2/3$ for clone and ancilla. Using the relation (5) between the concurrence and the entanglement of formation we arrive at $E_{F,12} \simeq 0.19$ for the state of the two clones and $E_{F,23} \simeq 0.55$ for the state of one clone and the ancilla. Note that the entanglement between clone and ancilla is higher than between the two clones.

Let us also study the entanglement properties of the total pure state (1). It is obviously genuinely three-party entangled, but there exist two different inequivalent classes of three-party entangled states,[18] namely the W- and the GHZ-class—to which one does it belong? Let us remind the reader that states of the form

$$|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2} \quad (6)$$

are called GHZ-states (these states can be detected by the Mermin inequality[19]), and states of the form

$$|\mathrm{W}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3} \quad (7)$$

are called W-states.[18] For pure states of three qubits there is a simple criterion to establish whether an entangled state belongs to the GHZ-class or to the W-class, namely the 3-tangle.[20] It is straightforward to verify that the 3-tangle is zero for the state (1), and therefore we can conclude that the total cloning output is in the W-class. We want to point out that this is very reasonable: the 1-particle reduced density matrices for GHZ-states are maximally mixed, and therefore could not represent high fidelity copies.

Finally, we point out that the state (1) happens to have the property that the three tangles add up to 1, i.e., $C_{12}^2 + C_{13}^2 + C_{23}^2 = 1$. This value lies the interval for this sum that is allowed by the sum rules of Ref. 20. Remember that for a GHZ-state the tangles add up to zero, because the bipartite reduced density matrices are separable, and they add up to $4/3$ for the W-state.

3. BIPARTITE ENTANGLEMENT IN THE OUTPUT OF A UNIVERSAL CLONER

Let us now generalise the above ideas to an $N \to M$-cloner for qubits, i.e., a cloner that takes N inputs and produces M outputs. What is our expectation for this case? How will the bipartite entanglement scale with the number M of outputs? The case $M \to \infty$ is often referred to as classical limit, and it has indeed been shown in Ref. 6 that the fidelity for producing infinitely many copies can be also reached by making an optimal state estimation on the input and then producing M identical uncorrelated states that correspond to the guessed state.

The optimal cloning transformation is given by Ref. 4

$$U_{N,M}\,|N\psi\rangle\,|(N-M)s\rangle\,|R\rangle = \sum_{j=0}^{M-N} \alpha_j(N,M)\,|(M-j)\,\psi, j\psi^{\perp}\rangle\,|R_j(\psi)\rangle, \tag{8}$$

where $|N\psi\rangle$ is the state of N qubits all in state $|\psi\rangle$, $|s\rangle$, and $|R\rangle$ are arbitrary input states of the $M-N$ copies and the state of the ancilla respectively, $|k\psi, j\psi^{\perp}\rangle$ is the normalised symmetric state of $k+j$ qubits (with k qubits in state $|\psi\rangle$ and j qubits in the orthogonal state $|\psi^{\perp}\rangle$), and

$$\alpha_j(N,M) = \sqrt{\frac{N+1}{M+1}}\,\sqrt{\frac{(M-N)!\,(M-j)!}{(M-N-j)!\,M!}}\,. \tag{9}$$

The output states of the ancilla are composed of $M-1$ qubits and are given by

$$|R_j(\psi)\rangle = |(M-1-j)\,\psi^*, j(\psi^*)^{\perp}\rangle, \tag{10}$$

where ψ^* denotes the complex conjugation of the wavefunction ψ.

This transformation is symmetric in the space of the clones, and therefore leads to the same fidelity for all clones. The symmetry of this transformation also means that the reduced density matrix for two clones will be of the form (3), with the further simplification of $b=c=d$. Note that *any* two clones will have the same reduced density matrix.

From (8) one calculates the elements of the reduced bipartite density matrix, with the notation as in (3):

$$\begin{aligned} a(N,M) &= \sum_{j=0}^{M-N} \alpha_j^2(N,M)\cdot\frac{(M-j)(M-j-1)}{M(M-1)}, \\ c(N,M) &= \sum_{j=0}^{M-N} \alpha_j^2(N,M)\cdot\frac{j(M-j)}{M(M-1)}, \\ e(N,M) &= \sum_{j=0}^{M-N} \alpha_j^2(N,M)\cdot\frac{j(j-1)}{M(M-1)}. \end{aligned} \tag{11}$$

What do these coefficients tell us? Let us first study in detail the simple case of $N=1$, extending the results of the previous section to an arbitrary number of output copies. For $N=1$ the above coefficients take the form

$$a(1,M)=\frac{3M+2}{6M},\qquad c(1,M)=\frac{1}{6},\qquad e(1,M)=\frac{M-2}{6M}. \tag{12}$$

As we can see, the concurrence surprisingly vanishes for $M\geqslant 3$. In the case of universal cloning with $N=2$ we have a similar behaviour: the coefficients take the form

$$a(2,M)=\frac{3M^2-2}{5M(M-1)},\qquad c(2,M)=\frac{3M^2-5M-2}{20M(M-1)},$$
$$e(2,M)=\frac{M^2-5M+6}{10M(M-1)}. \tag{13}$$

As we can easily verify, the concurrence vanishes for $M\geqslant 4$.

For a generic value of N it is not straightforward to discuss the form of the coefficients (11). However, we can analyse the limit $M\to\infty$. Each of the coefficients consists of a finite sum, let us call the terms in the sum a_j, c_j, and e_j. How does $\sqrt{ae}$ compare with c? For large M we find that $\sqrt{a_je_j}\approx c_j$, and therefore we find for the concurrence

$$C=2\max\left(0-\sqrt{\sum_{i\neq j}a_ie_j},\,0\right)=0. \tag{14}$$

This means that our intuition about any two clones being in a separable state for $M\to\infty$ is confirmed. However, we also realise that in this limiting case $\sqrt{ae}$ is much larger than c. Maybe the concurrence already vanishes for a finite M, and if so, for which? We have seen that for $N=1$ the concurrence vanishes already for $M=3$, and for $N=2$ it vanishes for $M=4$. So, does this hold in general, i.e., is there no entanglement for the $N\to N+2$-cloner? Well, first of all we can easily see that in the $N\to N+1$ case there is always entanglement, because $e(N,N+1)=0$ and $c(N,N+1)\neq 0$. Moreover, the explicit calculation of the $N\to N+2$ case gives the following results

$$a(M-2,M)=\frac{M^4-5M^2+8}{M^2(M^2-1)},\qquad c(M-2,M)=\frac{2(M^2-3)}{M^2(M^2-1)},$$
$$e(M-2,M)=\frac{4}{M^2(M^2-1)}. \tag{15}$$

As we can see, the corresponding concurrence is always zero for $M \geqslant 3$, and it is therefore reasonable to expect that entanglement really vanishes when the number of output copies exceeds the number of inputs by at least two.

One can also study the entanglement between a clone and the ancilla. Here we use the form of the ancilla as given in Eq. (10). The density matrix of an output copy and an ancilla qubit can be also written in the form (3), where now the basis is $\{|01\rangle, |00\rangle, |11\rangle, |10\rangle\}$:

$$\begin{aligned} a(N,M) &= f(N,M)\sum_{j=0}^{M-N} j(M-j)\,\frac{(M-j)!}{(M-N-j)!}\,, \\ b(N,M) &= f(N,M)\sum_{j=0}^{M-N} (M-j)(M-j-1)\,\frac{(M-j)!}{(M-N-j)!}\,, \\ c(N,M) &= f(N,M)\sum_{j=0}^{M-N-1} (j+1)\,\sqrt{\frac{M-j-1}{M-N-j}}\,\frac{(M-j)!}{(M-N-j-1)!}\,, \\ d(N,M) &= f(N,M)\sum_{j=0}^{M-N} j^2\,\frac{(M-j)!}{(M-N-j)!}\,, \\ e(N,M) &= f(N,M)\sum_{j=0}^{M-N} j(M-j-1)\,\frac{(M-j)!}{(M-N-j)!}\,, \end{aligned} \tag{16}$$

where $f(N,M)=\frac{N+1}{M(M^2-1)}\frac{(M-N)!}{M!}$. Again, for generic N we can only study the limit $M\to\infty$, using a similar argument as for the entanglement between the clones: we label the jth term in the sum for the coefficient a in Eq. (16) a_j, and accordingly for coefficients c and e. Then for infinitely large M we find that $c_j^2 \leqslant a_j e_j$, and therefore the concurrence vanishes.

The form of the coefficients in this case is more complicated than before. As an illustration, we report the analytic results for the case $N=1$, where the above coefficients take the simpler form

$$\begin{aligned} a(1,M) &= d(1,M) = \frac{1}{6}\,, & b(1,M) &= \frac{3M+2}{6M}\,, \\ c(1,M) &= \frac{M+2}{6M}\,, & e(1,M) &= \frac{M-2}{6M}\,. \end{aligned} \tag{17}$$

The concurrence in this case is given by

$$C(1,M) = \frac{1}{3}\left(\frac{M+2}{M} - \sqrt{\frac{M-2}{M}}\right). \tag{18}$$

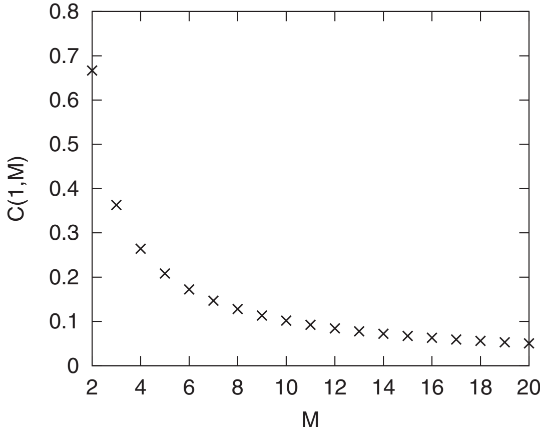

Fig. 1. Concurrence $C(1, M)$ for the state of a clone and an ancilla qubit as a function of the number of output copies M. The corresponding analytic form is given in Eq. (18).

As we can see, $C(1, M)$ is always different from zero for any value of M and vanishes in the limit $M \to \infty$. The behaviour of $C(1, M)$ is plotted in Fig. 1. This result is a bit surprising: while correlations of quantum nature vanish very quickly with increasing M for two output copies, they survive for all finite values of M for states of one output clone and one ancilla.

Let us interpret and summarise our results. The global output state of a universal symmetric quantum cloner is an entangled state. However, the clones in general do not carry any bipartite entanglement, unless we consider a very low number of copies: we have found that the entanglement does *not* gradually decrease with the number of copies, but vanishes already for $M = 3$, if one input is given, and for $M = 4$, if two inputs are given. Therefore, judging only by the bipartite entanglement properties, one *cannot* conclude that the limit $M \to \infty$ corresponds to the "classical" limit, as the quantum part of the bipartite correlations already vanishes for a very low number of outputs. We also found that the cloning states do not resemble the ground states of quantum statistical systems, which are typically either separable states or bear bipartite entanglement, whereas here a generic state is entangled, but without bipartite entanglement between the clones.

In order to characterise the entanglement structure of the global cloning state fully, one would need to study the entanglement properties of subsystems of any size. In the following section we consider the case of a tripartite subsystem.

4. TRIPARTITE ENTANGLEMENT IN THE OUTPUT OF A UNIVERSAL CLONER

In Sec. 3 we have shown that the total output of a $1 \to 2$ cloner, that is a pure state of two clones and the ancilla, corresponds to a W-state. If there is no bi-partite entanglement for the clones in a $1 \to 3$ cloner, maybe there is tripartite entanglement? Unfortunately, the problem of characterising the entanglement properties is not solved for general mixed states of three qubits. However, we will show some partial results for the cloning states.

Let us study the reduced density matrix $\varrho_{123}(N, M)$ for three clones in general, i.e., as a function of the number N of inputs and number M of outputs. From Eq. (8) we find

$$\varrho_{123}(N, M)$$
$$= \sum_{j=0}^{\min[M-N,\, M-3]} \alpha_j^2 \cdot \frac{\binom{M-3}{j}}{\binom{M}{j}} \cdot |000\rangle\langle 000| + \sum_{j=1}^{\min[M-N,\, M-2]} \alpha_j^2 \cdot \frac{\binom{M-3}{j-1}}{\binom{M}{j}} \cdot 3P_{W_{100}}$$
$$+ \sum_{j=2}^{\min[M-N,\, M-1]} \alpha_j^2 \cdot \frac{\binom{M-3}{j-2}}{\binom{M}{j}} \cdot 3P_{W_{110}} + \sum_{j=3}^{M-N} \alpha_j^2 \cdot \frac{\binom{M-3}{j-3}}{\binom{M}{j}} \cdot |111\rangle\langle 111|, \qquad (19)$$

where α_j depends on N and M and was defined in Eq. (9), and $P_{W_{100}}$ denotes the projector onto the pure W-state where one of the three qubits is in the state 1. In general, it is not known for which coefficients a mixture of this type is entangled. Therefore, let us restrict to two special cases.

The reduced density matrix for the three clones in the $1 \to 3$ cloner is found to be

$$\varrho_{123}(1, 3) = \tfrac{1}{2}\, |000\rangle\langle 000| + \tfrac{1}{3}\, P_{W_{100}} + \tfrac{1}{6}\, P_{W_{110}}\,. \qquad (20)$$

This is a mixture of two different W-states and a product state. Therefore, according to the classification of Ref. 21 it is at most in the W-class, i.e., not in the GHZ-class.

When one wants to detect the entanglement properties of multipartite states, the Mermin inequalities immediately[19] come to ones mind. However, in their original form they are violated by the GHZ-states, but satisfied by the W-states: these two inequivalent classes of states show different forms of the violation of local realism. A generalisation of the Mermin inequality which is also violated for W-states was introduced in Ref. 22. However, testing this generalised Mermin inequality for the state from Eq. (20) does not show a violation.

Let us also study the case of the $2 \to 3$ cloner, where the reduced density matrix for the three clones is given by

$$\varrho_{123}(2, 3) = \tfrac{3}{4}\,|000\rangle\langle 000| + \tfrac{1}{4}\,P_{W_{100}}\,. \tag{21}$$

which is a mixture of only one W-state and a product state. Again, it is at most in the W-class, i.e., not in the GHZ-class. And again, the generalised Mermin inequality from Ref. 22 is not violated by this state.

As we did not find a violation of a Mermin inequality, the three-qubit states given above could be in principle separable, bi-separable or in the W-class. However, we can also study the partial transpose with respect to Alice of a state of the general form

$$\varrho_{123}(N, M) = p_0\,|000\rangle\langle 000| + p_1 P_{W_{100}} + p_2 P_{W_{110}} + p_3\,|111\rangle\langle 111|, \tag{22}$$

with $p_i > 0$ and $\sum_i p_i = 1$. Notice that due to symmetry all three partial transposes with respect to one subsystem have the same structure. In general, the partial transpose turns out to be non-positive iff either $p_1^2 > 3p_0p_2$ or $p_2^2 > 3p_1p_3$. This means that for either $p_0 = 0$ or $p_3 = 0$ the partial transpose is non-positive, independently of the other probabilities. Therefore the cloning states given in Eqs. (20) and (21) cannot be separable.[23] Indeed, although there is no bipartite entanglement in the $1 \to 3$-cloner, we have found that there is tripartite entanglement. We also see immediatley from Eq. (19) that for the $N \to N+1$ cloner the partial transpose is always non-positive, as the last two terms in the equation vanish. Furthermore, for the $N \to N+2$ cloner the last term in Eq. (19) vanishes, and thus there is always tripartite entanglement for this case, contrary to the bipartite case.

Let us finally study tripartite entanglement for the general case of one input, i.e., the $1 \to M$-cloner, especially in the limit $M \to \infty$. The corresponding weights p_i, which were defined in Eq. (22), are given by

$$\begin{aligned} p_0(1, M) &= \frac{4M+3}{10M}, \qquad & p_1(1, M) &= \frac{3M+1}{10M}, \\ p_2(1, M) &= \frac{2M-1}{10M}, \qquad & p_3(1, M) &= \frac{M-3}{10M}. \end{aligned} \tag{23}$$

For $M \to \infty$ one finds immediately that the partial transposes are positive, and thus there is no free tripartite entanglement for infinitely many copies. In fact, for $M > 4$ the partial tranposes are positive, and thus tripartite entanglement vanishes already for small numbers of output copies, analogously to the bipartite case.

5. CONCLUSIONS

In summary, we have studied the entanglement structure at the output of an approximate cloning device. Regarding bipartite entanglement between two cloning outputs, we have found the surprising result that it vanishes for the $N \rightarrow N+2$ cloner. Thus, bipartite entanglement does not only disappear for the "classical" limit of infinitely many copies, but already when the number of outputs exceeds the number of inputs by two. Using one input, we showed that the entanglement between clone and ancilla, however, only vanishes in the limit of infinitely many copies. We also proved on the other hand that tripartite entanglement is always present for the $N \rightarrow N+2$ cloner. This entanglement generically seems to be of the W-type, rather than of the GHZ-type. We hope that these results will be useful for the further understanding of multipartite entanglement.

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