

Entanglement in quantum-optical bilinear devices

Matteo G A Paris

Theoretical Quantum Optics Group, Istituto Nazionale di Fisica della Materia—Unità di Pavia, Dipartimento di Fisica ‘Alessandro Volta’, Università degli Studi di Pavia, via Bassi 6, I-27100 Pavia, Italy

Received 2 September 1998, in final form 6 April 1999

Abstract. We examine the entanglement arising from the bilinear coupling of two nearly single-mode optical beams. We consider both the case of active and passive linear devices. The two modes are initially prepared in a pair of uncorrelated pure Gaussian states, and the degree of entanglement is analytically evaluated in terms of the excess information entropy at the output. A general formula is obtained and relevant cases are analysed in some detail.

Keywords: Entanglement, excess entropy, information, Gaussian states

1. Introduction

Entanglement is a peculiar feature of quantum mechanics. Within a quantum mechanical description, if two initially uncorrelated systems start to interact, they can no longer be described as separate entities. Rather, we should admit that the outcomes of any measurement performed on the system reflect some property of the two parts as a whole. This behaviour does not depend on the positions of the two subsystems after the interaction, which may be so far apart that no influence can propagate from one subsystem to the other during the time of measurement. In fact, quantum correlations play a role on a different level to classical ones, as they are related to the nonlocal properties of the theory.

In the domain of quantum optics, entangled states of a two-mode field are produced in a number of linear and nonlinear processes. Entangled photon pairs are utilized in fundamental tests of quantum mechanics by Bell-type correlation experiments [1–4], whereas practical applications of entanglement are currently being developed in the fields of quantum communication [5, 6], information [7, 8] and computation [9, 10].

Quantitative measurement of the entanglement of a composite quantum system is one of the fundamental problems of quantum information theory [11–14]. For a two-mode pure state the degree of entanglement can be quantified by means of the corresponding excess information entropy [15–18]. This quantity has been first utilized to describe the entanglement between two squeezed modes of the radiation field [15]. Subsequently, it has been successfully applied to study the entanglement in a number of systems as, for example, a two-level atom coupled to a cavity mode [19], a multi-mode field in a passive network [20], and the two optical beams exiting a Mach–Zehnder interferometer [21]. It should be mentioned that for a pure state the excess information entropy represents the unique measure of entanglement [22], whereas for general mixed states some

problems arise, and different definitions have been suggested [23–25].

In this paper we focus our attention on the entanglement arising from the bilinear coupling of two harmonic oscillators, modelling two nearly single-mode optical beams. In fact, among the different quantum-optical processes, devices that are governed by bilinear Hamiltonians play a special role, both from the theoretical and the experimental point of view. On the one hand, bilinear Hamiltonians correspond to simple algebraic structures, and hence many useful mathematical tools can be exploited for calculations [26–28]. On the other hand, they correspond to realistic devices currently forming the basis of experiments performed in quantum optics laboratories [29]. The present study is motivated by the interest in how to create strong correlations between two modes starting from available sources. Specific examples involving linear devices have been previously considered [15, 30], and the appearance of entangled states has been demonstrated. However, no systematic approach has been presented so far. Here, we consider the general case of bilinear coupling, and derive a formula that allows us to evaluate the degree of entanglement for both passive and active devices, and for a wide class of input signals. We also point out that the knowledge of the degree of entanglement at the output is useful in order to distinguish classical and nonclassical working regimes of linear devices, depending on whether or not quantum correlations have been established.

If a_1 and a_2 with $[a_1, a_1^\dagger] = 1$ and $[a_2, a_2^\dagger] = 1$ denote the field operators of two nearly single-mode radiation fields, a Hamiltonian of the form

$$\hat{H}_L \propto a_1^\dagger a_2 + a_1 a_2^\dagger, \quad (1)$$

describes a number of passive devices, including linear attenuators, beam splitters, linear couplers and frequency converters [30–36]. Conversely, a Hamiltonian of the form

$$\hat{H}_A \propto a_1^\dagger a_2^\dagger + a_1 a_2, \quad (2)$$

corresponds to processes such as parametric down-conversion and phase-insensitive linear amplification [37–41]. The Hamiltonian \hat{H}_A models active devices, and describes the effective interaction taking place in a classically pumped nonlinear crystal, the pumping mode assuring energy conservation, though not participating to the quantum dynamics [42–45].

We examine the situation in which the two modes are initially uncorrelated. The input signal is thus described by a factorized density matrix $\hat{\rho}_{\text{IN}} = \hat{\rho}_1 \otimes \hat{\rho}_2$, and we consider the case of both modes excited in a Gaussian state, namely a state described by a Gaussian Wigner function. This is a wide class of single-mode states, whose density matrix can be written in the general form [46, 47]

$$\hat{\rho}_G = \hat{D}(\alpha)\hat{S}(\zeta)\hat{v}\hat{S}^\dagger(\zeta)\hat{D}^\dagger(\alpha), \quad (3)$$

where $\hat{D}(\alpha) = \exp[\alpha a^\dagger - \bar{\alpha}a]$ is the displacement operator, $S(\zeta) = \exp[\frac{1}{2}(\zeta^2 a^{\dagger 2} - \bar{\zeta}^2 a^2)]$ the squeezing operator, and \hat{v}_{N} denotes the density matrix of a chaotic (thermal) state with $N = \text{Tr}[\hat{v}_{\text{N}} a^\dagger a]$ average number of photons, that is

$$\hat{v}_{\text{N}} = \frac{1}{1+N} \left(\frac{N}{1+N} \right)^{a^\dagger a}. \quad (4)$$

The class of states described by $\hat{\rho}_G$ includes thermal, coherent, and squeezed states, namely the relevant examples of classical, quasiclassical, and nonclassical states available by common quantum optical sources. In the following, we actually restrict our attention to the subclass of pure states described by (3).

In the next section we briefly review the use of excess entropy as a quantitative measure of two-mode entanglement and derive a formula for the entropy of a generic Gaussian state. In section 3 we consider both passive and active linear devices, and illustrate the evolution of the two-mode Wigner function. In section 4 a general formula for the entanglement arising from bilinear coupling is obtained, and some relevant examples are discussed in detail. Section 5 closes the paper by summarizing the results.

2. Excess entropy as a measure of entanglement

The measurement of a classical random variable A is always associated with the gain of some amount of information. Indeed, each realization of the experiment removes the uncertainty about which one of the possible outcomes $\{a_k\}$ may turn out. The relevant information depends only on the probabilities $\{P_k\}$ associated with the events $\{a_k\}$ and is given by the Shannon information entropy [48]

$$I(A) = - \sum_k P_k \log P_k. \quad (5)$$

The simultaneous measurement of two classical random variables A and B gives the outcome $\{a_k, b_j\}$ with probability $\{P_{kj}\}$. The joint information is thus given by

$$I(A, B) = - \sum_{kj} P_{kj} \log P_{kj}. \quad (6)$$

The information of the A -measurement regardless of the outcome of B is again given by (5), where $P_k = \sum_j P_{kj}$

is now the marginal probability. Similarly, the information of the B -measurement regardless of the outcome of A is governed by the marginal probability $P_j = \sum_k P_{kj}$. The index of correlation between the two random variables is given by [49]

$$I_{AB} = I(A) + I(B) - I(A, B). \quad (7)$$

In order to describe quantum correlations a suitable generalization of formula (7) is needed. This can be obtained by means of von Neumann entropy of a quantum state. The entropy of a two-mode state $\hat{\rho}$ is defined as

$$S[\hat{\rho}] = - \text{Tr}_{12}\{\hat{\rho} \log \hat{\rho}\}. \quad (8)$$

Conversely,

$$S[\hat{\rho}_1] = - \text{Tr}_1\{\hat{\rho}_1 \log \hat{\rho}_1\} \quad S[\hat{\rho}_2] = - \text{Tr}_2\{\hat{\rho}_2 \log \hat{\rho}_2\}, \quad (9)$$

are the single-mode entropies of $\hat{\rho}_1 = \text{Tr}_2\{\hat{\rho}\}$ and $\hat{\rho}_2 = \text{Tr}_1\{\hat{\rho}\}$, namely the state of modes 1 and 2, respectively, as obtained by tracing out the other mode from the total density matrix. A measure of quantum correlation is given by the excess entropy [15, 16]

$$I_e = S[\hat{\rho}_1] + S[\hat{\rho}_2] - S[\hat{\rho}], \quad (10)$$

which formalizes the idea that the stronger the correlations in the two-mode state, the more disordered should be the two modes taken separately. The quantity I_e is always non-negative [50] and, due to additivity of entropy [51], is zero for factorized (uncorrelated) states $\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2$. If $\hat{\rho}$ describes a pure state, we have $S[\hat{\rho}] = 0$ and thus $S[\hat{\rho}_1] = S[\hat{\rho}_2]$, so that the excess entropy reduces to $I_e = 2S[\hat{\rho}_1]$, which is maximized when $\hat{\rho}_1$ is a chaotic state of the form (4). Using these considerations, we introduce the degree of entanglement ϵ of the two-mode state $\hat{\rho}$ as the normalized excess entropy

$$\epsilon = \frac{1}{T[N_1] + T[N_2]} (S[\hat{\rho}_1] + S[\hat{\rho}_2] - S[\hat{\rho}]), \quad (11)$$

where

$$T[N_j] \equiv S[\hat{v}_{N_j}] = (1 + N_j) \log(1 + N_j) - N_j \log N_j, \quad (12)$$

is the entropy of a thermal state with an average number of photons $N_j = \text{Tr}_j[\hat{\rho}_j a_j^\dagger a_j]$ equal to that of each partial trace, respectively. The degree of entanglement ϵ ranges from zero to unity, with $\epsilon = 0$ for a factorized state and $\epsilon = 1$ for a maximally entangled state [15, 16], namely a pure state whose partial traces coincide with a couple of thermal states. For pure states the degree ϵ represents the unique choice for a quantitative measure of entanglement [22], whereas for general mixed states the notion of inseparability has been introduced [52–54], and different measures have been suggested [23–25]. The point is that for mixed states the two subsystems' entropies may differ and the excess entropy represents a combination of entanglement and classical correlations [55].

2.1. Entropy of a Gaussian state

The evaluation of the degree of entanglement in (11) implies the evaluation of the von Neumann entropy. In this section we study the entropy of a single-mode Gaussian state. The problem has been already considered in [56, 57], where the entropy has been expressed in terms of the first two moments of the field operators a and a^\dagger . Here, we derive a more compact formula containing only parameters of the Wigner function.

In principle, to evaluate the entropy we need to diagonalize the density matrix $\hat{\rho}$ in some orthogonal basis $\{|\psi_k\rangle\}$ of the Hilbert space (δ_{kj} denotes the Kronecker delta)

$$\begin{aligned}\hat{\rho} &= \sum_k p_k |\psi_k\rangle\langle\psi_k| & \langle\psi_k|\psi_j\rangle &= \delta_{kj} \\ S[\hat{\rho}] &= -\text{Tr}[\hat{\rho} \log \hat{\rho}] = -\sum_k p_k \log p_k.\end{aligned}\quad (13)$$

However, it should be noticed that any unitary transformation \hat{U} applied to $\hat{\rho}$ does not change the value of its entropy, as it transforms an orthogonal basis to an orthogonal basis. As a consequence, the entropy of a Gaussian state $\hat{\rho}_G$ of the form (3) can always be evaluated as the entropy of a thermal state. We now proceed to find the correct average number of photons of this ‘entropy-equivalent’ thermal state in terms of the parameters entering the Wigner function.

Starting from the definition of the Wigner function describing the single-mode state $\hat{\rho}$

$$\begin{aligned}W(x, y) &= \int_{\mathbb{R}} d\mu \int_{\mathbb{R}} d\nu \exp\{2i(\nu x - \mu y)\} \\ &\quad \times \text{Tr}[\hat{\rho} \hat{D}(\mu_1 + i\nu_1)],\end{aligned}\quad (14)$$

3.1. Evolution of the Wigner function

For the purpose of evaluating entropies and entanglement it is convenient to analyse the dynamics of linear devices in terms of the two-mode Wigner function, which is defined as follows:

$$W(x_1, y_1; x_2, y_2) = \int_{\mathbb{R}} d\mu_1 \int_{\mathbb{R}} d\nu_1 \int_{\mathbb{R}} d\mu_2 \int_{\mathbb{R}} d\nu_2 \times \exp\{2i(\nu_1 x_1 - \mu_1 y_1 + \nu_2 x_2 - \mu_2 y_2)\} \times \text{Tr}[\hat{\rho} \hat{D}(\mu_1 + i\nu_1) \hat{D}(\mu_2 + i\nu_2)]. \quad (30)$$

By using (28) and (29), together with equation (30) it is possible to obtain the evolved Wigner function in terms of its initial expression. In the case of a beam splitter one has

$$W_{\text{OUT}}(x_1, y_1; x_2, y_2) = W_{\text{IN}}(x_1 \cos \delta - x_2 \sin \delta, y_1 \cos \delta - y_2 \sin \delta, x_1 \sin \delta + x_2 \cos \delta, y_1 \sin \delta + y_2 \cos \delta), \quad (31)$$

whereas for the case of linear amplification one obtains

$$W_{\text{OUT}}(x_1, y_1; x_2, y_2) = W_{\text{IN}}(x_1 \cosh \gamma + x_2 \sinh \gamma, y_1 \cosh \gamma + y_2 \sinh \gamma, x_1 \sinh \gamma + x_2 \cosh \gamma, y_1 \sinh \gamma + y_2 \cosh \gamma). \quad (32)$$

Our initial state $\hat{\rho}_{\text{IN}} = \hat{\rho}_1 \otimes \hat{\rho}_2$ corresponds to the factorized Wigner function

$$W_{\text{IN}}(x_1, y_1; x_2, y_2) = W_{\text{IN}}^1(x_1, y_1) W_{\text{IN}}^2(x_2, y_2), \quad (33)$$

where both $W_{\text{IN}}^j(x_j, y_j)$, $j = 1, 2$ are Gaussian single-mode Wigner functions of the form (18). By inserting the explicit expression of $W_{\text{IN}}(x_1, y_1; x_2, y_2)$ in (31) or (32) one obtains the output two-mode Wigner function for the two devices, respectively. In both cases, $W_{\text{OUT}}(x_1, y_1; x_2, y_2)$ is expressed by a rather long formula, which we do not report here. We only notice that $W_{\text{OUT}}(x_1, y_1; x_2, y_2)$ is no longer factorized, thus indicating the appearance of correlations between the two interacting modes. The output states obtained by partial traces $\hat{\rho}_{\text{OUT}}^1 = \text{Tr}_2[\hat{\rho}_{\text{OUT}}]$ and $\hat{\rho}_{\text{OUT}}^2 = \text{Tr}_1[\hat{\rho}_{\text{OUT}}]$ are described by the two single-mode Wigner functions

$$W_{\text{OUT}}^1(x_1, y_1) = \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dy_2 W_{\text{OUT}}(x_1, y_1; x_2, y_2) \quad (34)$$

$$W_{\text{OUT}}^2(x_2, y_2) = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 W_{\text{OUT}}(x_1, y_1; x_2, y_2),$$

obtained by integration over the degrees of freedom of the other mode. These two single-mode Wigner functions are still of Gaussian form. By denoting output mean values and variances by capital letters we have

$$\begin{aligned} A_1 &= a_1 \cos \delta + a_2 \sin \delta & \Sigma_{1x}^2 &= \sigma_{1x}^2 \cos^2 \delta + \sigma_{2x}^2 \sin^2 \delta \\ B_1 &= b_1 \cos \delta + b_2 \sin \delta & \Sigma_{1y}^2 &= \sigma_{1y}^2 \cos^2 \delta + \sigma_{2y}^2 \sin^2 \delta \\ A_2 &= a_1 \sin \delta - a_2 \cos \delta & \Sigma_{2x}^2 &= \sigma_{1x}^2 \sin^2 \delta + \sigma_{2x}^2 \cos^2 \delta \\ B_2 &= b_1 \sin \delta - b_2 \cos \delta & \Sigma_{2y}^2 &= \sigma_{1y}^2 \sin^2 \delta + \sigma_{2y}^2 \cos^2 \delta, \end{aligned} \quad (35)$$

in the case of the beam splitter and

$$\begin{aligned} A_1 &= a_1 \cosh \gamma - a_2 \sinh \gamma & \Sigma_{1x}^2 &= \sigma_{1x}^2 \cosh^2 \gamma + \sigma_{2x}^2 \sinh^2 \gamma \\ B_1 &= b_1 \cosh \gamma - b_2 \sinh \gamma & \Sigma_{1y}^2 &= \sigma_{1y}^2 \cosh^2 \gamma + \sigma_{2y}^2 \sinh^2 \gamma \\ A_2 &= -a_1 \sinh \gamma + a_2 \cosh \gamma & \Sigma_{2x}^2 &= \sigma_{1x}^2 \sinh^2 \gamma + \sigma_{2x}^2 \cosh^2 \gamma \\ B_2 &= -b_1 \sinh \gamma + b_2 \cosh \gamma & \Sigma_{2y}^2 &= \sigma_{1y}^2 \sinh^2 \gamma + \sigma_{2y}^2 \cosh^2 \gamma, \end{aligned} \quad (36)$$

for the linear amplifier.

4. Entanglement at the output

Using the results obtained in previous sections we are now ready to evaluate the entanglement due to passive and active linear devices. As already mentioned, the input signals are uncorrelated, and thus the input entanglement ϵ_{IN} is zero, namely

$$S[\hat{\rho}_{\text{IN}}] = S[\hat{\rho}_{\text{IN}}^1] + S[\hat{\rho}_{\text{IN}}^2]. \quad (37)$$

On the other hand, the global density matrices $\hat{\rho}_{\text{IN}}$ and $\hat{\rho}_{\text{OUT}}$ are connected to each other by the unitary transformations \hat{U}_τ or \hat{U}_G . Therefore, $S[\hat{\rho}_{\text{IN}}] = S[\hat{\rho}_{\text{OUT}}]$ and one can also write

$$S[\hat{\rho}_{\text{OUT}}] = S[\hat{\rho}_{\text{IN}}^1] + S[\hat{\rho}_{\text{IN}}^2]. \quad (38)$$

Finally, we note that the partial traces are Gaussian states both at the input and at the output, and thus their entropies can be expressed as in section 2. Inserting the proper quantities in equation (11) we arrive at the final expression for the output entanglement

$$\epsilon_{\text{OUT}} = \frac{T[N_1^*] + T[N_2^*] - T[n_1^*] - T[n_2^*]}{T[N_1] + T[N_2]}. \quad (39)$$

Remarkably, equation (39) contains only entropies of thermal states, which can be easily evaluated by means of (12) and (23). In equation (39)

$$\begin{aligned} n_1^* &= 2\sigma_{1x}\sigma_{1y} - \frac{1}{2} \\ n_2^* &= 2\sigma_{2x}\sigma_{2y} - \frac{1}{2}, \end{aligned} \quad (40)$$

denote the equivalent average thermal photons of input signals, whereas

$$\begin{aligned} N_1^* &= 2\Sigma_{1x}\Sigma_{1y} - \frac{1}{2} \\ N_2^* &= 2\Sigma_{2x}\Sigma_{2y} - \frac{1}{2}, \end{aligned} \quad (41)$$

are the corresponding thermal photons for the output partial traces. The quantities

$$\begin{aligned} N_1 &= \text{Tr}_{12}[\hat{\rho}_{\text{OUT}} a_1^\dagger a_1] = \text{Tr}_1[\hat{\rho}_{\text{OUT}}^1 a_1^\dagger a_1] \\ &= A_1^2 + B_1^2 + \Sigma_{1x}^2 + \Sigma_{1y}^2 - \frac{1}{2} \end{aligned} \quad (42)$$

$$\begin{aligned} N_2 &= \text{Tr}_{12}[\hat{\rho}_{\text{OUT}} a_2^\dagger a_2] = \text{Tr}_2[\hat{\rho}_{\text{OUT}}^2 a_2^\dagger a_2] \\ &= A_2^2 + B_2^2 + \Sigma_{2x}^2 + \Sigma_{2y}^2 - \frac{1}{2}, \end{aligned}$$

are the total average number of photons for the output partial traces. The last equality in equations (42) follows from the fact that averaging over the Wigner function gives the

symmetrically ordered moments of field operators [59]; in particular, we have that

$$\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy W(x, y)(x^2 + y^2) = \text{Tr} \left[\hat{\rho} \frac{a^\dagger a + a a^\dagger}{2} \right] = \text{Tr}[\hat{\rho}(a^\dagger a + \frac{1}{2})]. \quad (43)$$

Equation (39), together with (35) and (36), is a quite general result that allows us to evaluate the entanglement that arises from the bilinear coupling of a pair of Gaussian states, both in the case of passive and active devices. Here, we utilize (39) to analyse some relevant examples in detail. As we are going to consider only pure states at the input, we will always have $T[n_1^*] = T[n_2^*] = 0$.

4.1. Amplification and mixing of coherent signals

The mixing of a pair of coherent states in a linear coupler, or in a beam splitter, is the only linear process that does not lead to entanglement at the output. This can be seen from equation (39), or more directly from the Schrödinger evolution of the input signal. If $|\psi_{\text{IN}}\rangle = |\alpha\rangle \otimes |\beta\rangle$ is the input state, we have

$$|\psi_{\text{OUT}}\rangle = \hat{U}_\tau |\psi_{\text{IN}}\rangle = |\sqrt{\tau}\alpha + \sqrt{1-\tau}\beta\rangle \otimes |-\sqrt{1-\tau}\alpha + \sqrt{\tau}\beta\rangle, \quad (44)$$

which shows that the output state consists of a pair of independent coherent states.

On the other hand, the linear amplification of coherent signals does lead to entanglement at the output. The input state is pure and thus we have zero input entropy. The output state is also pure, such that the two partial traces are equal, $N_1^* = N_2^* = N^*$ and the entanglement at the output can be rewritten as

$$\epsilon_{\text{OUT}} = \frac{2T[N^*]}{T[N_1] + T[N_2]}. \quad (45)$$

In the simple case of an initially unexcited idler mode $|\psi_{\text{IN}}\rangle = |\alpha\rangle \otimes |0\rangle$, the equivalent thermal photons are given by $N^* = G - 1$, whereas the total number of photons in the two output modes are given by $N_1 = G|\alpha|^2 + G - 1$ and $N_2 = (G - 1)(|\alpha|^2 + 1)$, respectively. In figure 1(a) we report the degree of entanglement as a function of the gain of the amplifier for different values of the initial coherent amplitude of the signal. As ϵ_{OUT} is an increasing function of the gain, any input signal would, in principle, lead to a maximum entangled state at the output for strong enough amplification. In practice, the increasing rate of ϵ_{OUT} versus the gain rapidly decreases with input intensity, so that highly entangled states are obtained only for weak input signals. For any fixed value of the gain the degree of entanglement is a decreasing function of the input intensity (see figure 1(b)). This means that, in principle, for the amplification of a coherent signal the classical limit of uncorrelated output is recovered in the limit of very large intensity.

We now consider an input signal consisting of a couple of coherent states with the same number of photons but with different phases, in formula $|\psi_{\text{IN}}\rangle = |\alpha\rangle \otimes |\beta\rangle$, with $\beta = \alpha \exp(i\phi)$. In this case it is straightforward to show that

$$N_1^* = N_2^* = G \quad (46)$$

$$N_1 = N_2 = [|\alpha| + \sqrt{G}]^2 - 2|\alpha|\sqrt{G(G+1)} \cos \phi,$$

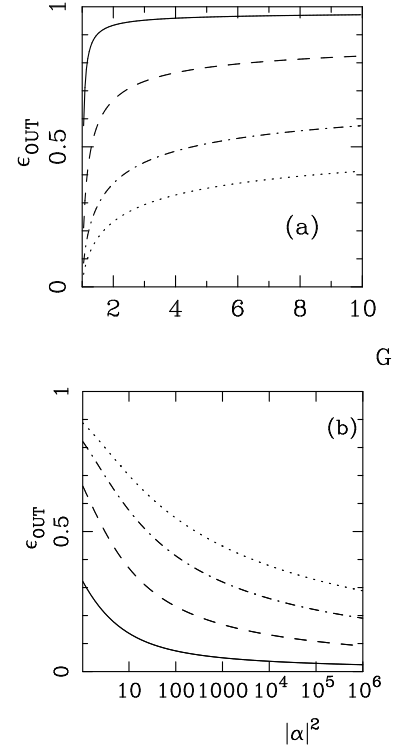


Figure 1. (a) The degree of entanglement at the output of a linear amplifier fed by a coherent signal as a function of the gain of the amplifier. Curves for different values of the input intensity are reported: $|\alpha|^2 = 0.1$ (full curve), $|\alpha|^2 = 1$ (dashed), $|\alpha|^2 = 10$ (dot-dashed), and $|\alpha|^2 = 100$ (dotted). (b) The degree of entanglement as a function of the input intensity $|\alpha|^2$ for different values of the gain of the amplifier. $G = 1.1$ (full curve), $G = 2$ (dashed), $G = 10$ (dot-dashed) and $G = 100$ (dotted), respectively.

such that the relative phase plays a major role in determining the output entanglement. In figure 2 we report the degree of entanglement as a function of the relative phase for different values of the input intensity $|\alpha|^2 = |\beta|^2$ and the amplification gain G . As is apparent from the plots, the entanglement strongly depends on the relative phase: the variation is more pronounced for more excited states, whereas the amplification gain sets the oscillation range.

4.2. Mixing of a pair of squeezed vacuums

It has been suggested that feeding a Mach–Zehnder interferometer with single-mode or two-mode squeezed light may lead to the formation of entangled states in a wide range of degree of entanglement [21, 60]. We argue that such an effect can also be observed at the output of a beam splitter fed by a couple of single-mode squeezed vacuums. Indeed, apart from a single-mode rotation, a Mach–Zehnder interferometer is equivalent to a single beam splitter.

As an input state we consider a couple of squeezed vacuums with opposite squeezing parameter

$$|\psi_{\text{IN}}\rangle = \hat{S}(r)|0\rangle \otimes \hat{S}(-r)|0\rangle, \quad (47)$$

which corresponds to a Gaussian two-mode Wigner function with zero mean and variances $\sigma_{1x}^2 = \sigma_{2y}^2 = e^{2r}/4$ and

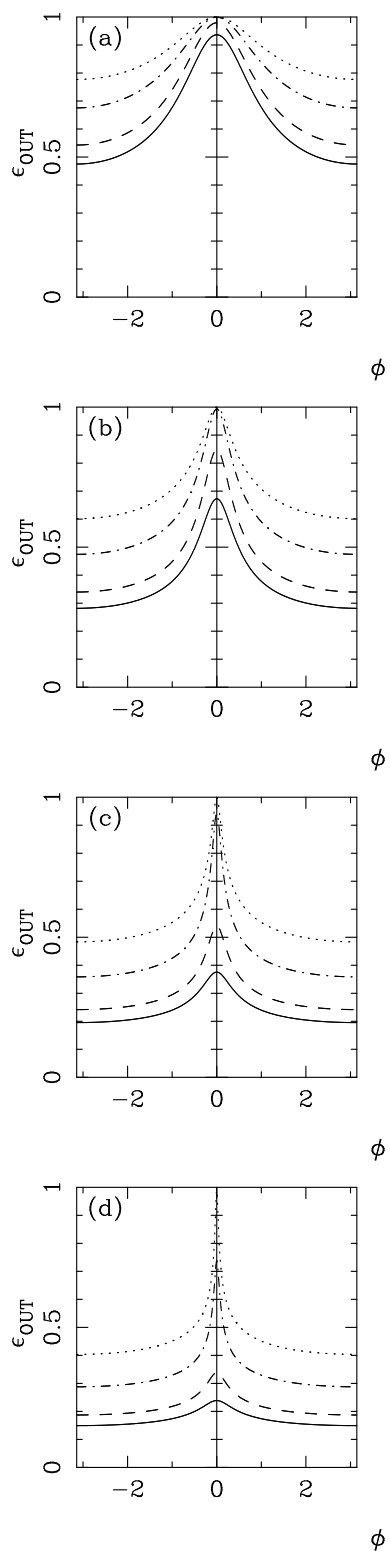


Figure 2. (a) The degree of entanglement at the output of a linear amplifier fed by a couple of coherent signals as a function of the

both for passive and active devices, as well as for different kinds of input signals.

The mixing of coherent states in a beam splitter is the only bilinear process not leading to entanglement at the output. On the other hand, the linear amplification of a coherent signal does produce entanglement. In this case the degree of entanglement is a decreasing function of the input intensity, and therefore the classical limit of uncorrelated output signals can be recovered in the very large intensity regime. In the case of an amplification process with excited idler mode the output entanglement strongly depends on the relative phase between the two modes.

Finally, the mixing of a pair of squeezed vacuums at a beam splitter has also been analysed. It has been shown that the output state ranges from a totally disentangled state to a maximally entangled state depending on the transmissivity of the beam splitter.

As a concluding remark, we point out that for the Gaussian input states considered here the degree of entanglement at the output of a linear device is always a measurable quantity. The parameters involved in the expression of the output entanglement, in fact, are just the mean values and the variances of the single-mode Wigner functions $W_{\text{OUT}}^1(x_1, y_1)$ and $W_{\text{OUT}}^2(x_2, y_2)$ which, in turn, correspond to the real and the imaginary part of the complex field amplitudes and to the fluctuations of a couple of field quadratures. More precisely, as regards the mean values we have

$$\begin{aligned} \langle a_j \rangle &= \text{Tr}_j[\rho_{\text{OUT}}^j a_j] & j &= 1, 2 \\ A_j &= \text{Re} \langle a_j \rangle & B_j &= \text{Im} \langle a_j \rangle, \end{aligned} \quad (51)$$

whereas the variances Σ_{jx}^2 and Σ_{jy}^2 are obtained as the extreme values $\max_{\phi} \langle \widehat{\Delta x}_{\phi}^2 \rangle$ and $\min_{\phi} \langle \widehat{\Delta x}_{\phi}^2 \rangle$, with the maximization/minimization procedure needed to individuate the principal axis of squeezing. Remarkably, all these quantities can be jointly reconstructed by the measurements of the two single-mode Q -functions $\langle \alpha | \rho_{\text{OUT}}^j | \alpha \rangle$, $j = 1, 2$, which can be accomplished by means of heterodyne [61, 62], eight-port homodyne [63–65] or six-port homodyne [66, 67] detection schemes.

Acknowledgments

The author would like to thank Valentina De Renzi for encouragement and valuable discussions. This work is part of the INFM project PRA-97 CAT. The author has been partially supported by *Accademia Nazionale dei Lincei* through the *Giuseppe Borgia* award.

References

- [1] Ou Z Y and Mandel L 1988 *Phys. Rev. Lett.* **61** 50
- [2] Rarity J G and Tapster P R 1990 *Phys. Rev. Lett.* **64** 2495
- [3] Rubin M H and Shih Y H 1992 *Phys. Rev. A* **45** 8138
- [4] Kwiat P G, Mattle K, Weinfurter H and Zeilinger A 1995 *Phys. Rev. Lett.* **75** 4337
- [5] Bennet C H 1995 *Phys. Today* **48** 24
- [6] Bennet C H and Di Vincenzo D P 1995 *Nature* **377** 389
- [7] Boschi D, Branca S, De Martini F, Hardy L and Popescu S 1998 *Phys. Rev. Lett.* **80** 1121
- [8] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
- [9] Feynman R 1982 *Int. J. Theor. Phys.* **21** 467
- [10] Deutsch D and Josza R 1992 *Proc. R. Soc. A* **439** 553
- [11] Werner R F 1989 *Phys. Rev. A* **40** 4277
- [12] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
- [13] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett. A* **223** 1
- [14] Horodecki M, Horodecki P and Horodecki R 1998 *Acta Phys. Slov.* **48** 141
- [15] Barnett S M and Phoenix S J D 1989 *Phys. Rev. A* **40** 2404
- [16] Barnett S M and Phoenix S J D 1991 *Phys. Rev. A* **44** 535
- [17] Lindblad G 1973 *Commun. Math. Phys.* **33** 305
- [17] Lindblad G 1974 *Commun. Math. Phys.* **39** 111
- [17] Lindblad G 1975 *Commun. Math. Phys.* **40** 147
- [18] Vedral V, Plenio M B, Jacobs K and Knight P L 1997 *Phys. Rev. A* **56** 4452
- [19] Phoenix S J D and Knight P L 1991 *Phys. Rev. A* **44** 6023
- [20] Törma P and Jex I 1997 *J. Mod. Opt.* **44** 875
- [21] Paris M G A 1999 *Phys. Rev. A* **59** 1615
- [22] Popescu S and Rohrlich R 1997 *Phys. Rev. A* **56** R3319
- [23] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 *Phys. Rev. Lett.* **78** 2275
- [24] Bennet C H, Di Vincenzo D P, Smolin J and Wootters W K 1997 *Phys. Rev. A* **54** 3814
- [25] Vedral V and Plenio M B 1998 *Phys. Rev. A* **57** 1619
- [26] Truax D R 1985 *Phys. Rev. D* **31** 1988
- [27] Ban M 1993 *J. Opt. Soc. Am. B* **10** 1347
- [28] Paris M G A 1996 *Phys. Lett. A* **217** 78
- [29] Bachor H A 1998 *A Guide to Experiments in Quantum Optics* (Weinheim: VCH)
- [30] Campos R A, Saleh B E A, Teich M C 1989 *Phys. Rev. A* **40** 1371
- [31] Ou Z Y, Hong C K and Mandel L 1987 *Opt. Commun.* **63** 118
- [32] Fearn H and Loudon R 1987 *Opt. Commun.* **64** 485
- [33] Fearn H 1990 *Quantum Opt.* **2** 103
- [34] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev. A* **33** 4033
- [35] Janszky J, Sibilia C, Bertolotti M and Yushin Y 1988 *J. Mod. Opt.* **35** 1757
- [36] Janszky J, Adam P, Bertolotti M and Sibilia C 1992 *Quantum Opt.* **4** 163
- [37] Mollow B R 1967 *Phys. Rev.* **162** 1256; 1973 *Phys. Rev. A* **8** 2684
- [38] Mollow B R and Glauber R J 1967 *Phys. Rev.* **160** 1076
- [39] Klyshko D N 1968 *Sov. Phys. JETP* **28** 522
- [40] Caves C M 1982 *Phys. Rev. D* **26** 1817
- [41] Hong C K, Friberg S and Mandel L 1985 *J. Opt. Soc. Am. B* **2** 494
- [42] Carusotto S 1975 *Phys. Rev. A* **11** 1629
- [43] Hillery H and Zubairy M S 1984 *Phys. Rev. A* **29** 1275
- [44] Crouch D D and Braunstein S L 1988 *Phys. Rev. A* **38** 4696
- [45] Scharf G and Walls D F 1984 *Opt. Commun.* **50** 245
- [46] Adam G 1995 *J. Mod. Opt.* **42** 1311
- [47] Bužek V, Adam G and Drobný G 1996 *Ann. Phys., NY* **245** 37
- [48] Jumarie G 1990 *Relative Information* (Berlin: Springer)
- [49] Kullback S 1959 *Information Theory and Statistics* (New York: Wiley)
- [50] Araki H and Lieb E H 1970 *Commun. Math. Phys.* **18** 160
- [51] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
- [52] Popescu S 1994 *Phys. Rev. Lett.* **72** 797
- [53] Popescu S 1995 *Phys. Rev. Lett.* **74** 2619
- [54] Gisin N 1996 *Phys. Lett. A* **210** 151
- [55] Bennett C H 1998 *Phys. Scr.* **T76** 210
- [56] Peřinová V, Křepelka J, Peřina J, Lukš A and Szlachetka P 1986 *Opt. Acta* **33** 15
- [57] Lukš A and Peřinová V 1989 *Czech. J. Phys. B* **39** 392
- [58] Yuen H P and Chan V W S 1983 *Opt. Lett.* **8** 177
- [59] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
- [59] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
- [60] Böhmer B and Leonhardt U 1995 *Opt. Commun.* **118** 181
- [61] Yuen H P and Shapiro J H 1980 *IEEE Trans. Inform. Theory* **26** 78

M G A Paris

- [62] Shapiro J H and Wagner S S 1984 *IEEE J. Quantum Electron.* **20** 803
- [63] Walker N G and Carrol J E 1986 *Opt. Quantum Electron.* **18** 355
- [64] Walker N G 1987 *J. Mod. Opt.* **34** 15
- [65] Lay Y and Haus H A 1989 *Quantum Opt.* **1** 99
- [66] Zucchetti A, Vogel W and Welsch D G 1996 *Phys. Rev. A* **54** 856
- [67] Paris M G A, Chizhov A and Steuernagel O 1997 *Opt. Commun.* **134** 117