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# CLASSICAL CAPACITY OF FREE-SPACE OPTICAL COMMUNICATION

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The classical-information capacity of lossy bosonic channels is studied, with emphasis on the far-field free space channel.

## 1 Classical capacity

A prominent landmark in the extension of Shannon information theory to the quantum domain is the realization that any particular physical system can store only a finite amount of information. As a consequence, the quantity of information that can be exchanged in any communication process is limited by the available physical resources. Two of the main results in this regard are the Holevo bound[1] and the Holevo-Schumacher-Westmoreland (HSW) theorem.[2] The former limits the amount of information that can be retrieved after storage in a quantum system. For an encoding alphabet whose $j$th "letter" is the state $\sigma_j$, the mutual information between the encoded and the decoded message—independent of the decoding strategy—cannot exceed the Holevo information $\chi$, defined by

$$\chi(p_j, \sigma_j) \equiv S(\sum_j p_j \sigma_j) - \sum_j p_j S(\sigma_j) \;, \tag{1}$$

where $p_j$ are the prior probabilities of the message states and $S(\sigma) \equiv -\mathrm{Tr}[\sigma \log_2 \sigma]$ is the von Neumann entropy. The HSW result, on the other hand, extends Shannon's noisy-channel capacity theorem to the quantum regime by showing that the Holevo bound (1) is asymptotically achievable.

The classical capacity $C$ of a channel is, by definition, the maximum number of bits per channel use that can be communicated reliably. Shannon's theorem obtains $C$ by maximizing the mutual information between the input and the output over all possible encoding and decoding strategies. The HSW theorem, however, finds $C$ by maximizing the Holevo information of the output state over all possible quantum codings, independent of the decoding, while regularizing $\chi$ over successive uses of the channel. In particular, consider a channel described by the completely-positive (CP) map $\mathcal{E}$. The capacity $C_n$ for $n$ uses of this channel is given by

$$C_n \equiv \max_{p_j,\sigma_j} \chi(p_j, \mathcal{E}^{\otimes n}[\sigma_j]) \ , \tag{2}$$

where the maximization is over all possible coding strategies, i.e., over all possible prior probabilities $\{p_j\}$ and code states $\{\sigma_j \in \mathcal{H}^{\otimes n}\}$ for $n$ uses of the channel. ($\mathcal{H}$ is the Hilbert space of input states for a single channel use.) The aforementioned regularization over successive channel uses then yields the capacity $C$ via

$$C = \sup_n C_n/n. \tag{3}$$

This step is necessary because it is not yet known whether the Holevo information is superadditive, i.e., if employing codewords that are entangled over successive channel uses enhances communication performance.[2, 3]

## 2 Thermal Bosonic channel

In this paper we study the capacity of bosonic communication channels with Gaussian noise, viz., the information is carried by photons and $\mathcal{E}$ is a linear bosonic map[4] that evolves Gaussian input states into Gaussian output states.[5] Of particular interest is the lossy bosonic channel with thermal noise, in which the received electromagnetic field modes couple to both the transmitter and to a thermal reservoir. This channel coupling can be described by a beam-splitter relation whose transmitter-to-receiver transmissivity, for a given field mode, equals that mode's quantum efficiency. Let $a_k$ and $b_k$ be the annihilation operators of the $k$th input and reservoir modes. The beam splitter evolution for these modes is governed by the unitary operator

$$U = \bigotimes_k \exp\left[-\arctan\sqrt{\eta_k^{-1} - 1}(a_k b_k^\dagger - a_k^\dagger b_k)\right] \ , \tag{4}$$

where $\eta_k \in [0,1]$ is the transmissivity, so that

$$\begin{array}{lclcl} a_k & \longrightarrow & U^\dagger a_k U & = & \sqrt{\eta_k}\, a_k + \sqrt{1-\eta_k}\, b_k \\ b_k & \longrightarrow & U^\dagger b_k U & = & -\sqrt{1-\eta_k}\, a_k + \sqrt{\eta_k}\, b_k \ . \end{array} \tag{5}$$

The associated CP-map is therefore

$$\mathcal{E}[\rho] = \mathrm{Tr}_E\left[U\, \rho \otimes \rho_E\, U^\dagger\right] \ , \tag{6}$$

where $\rho$ is the channel's input state (multi-mode output from the transmitter) and $\rho_E$ is the thermal state of the environment, i.e., the isotropic Gaussian state corresponding to independent Bose-Einstein distributions over photon number states for each mode:

$$\rho_E \equiv \bigotimes_k \frac{1}{M_k+1}\left(\frac{M_k}{M_k+1}\right)^{b_k^\dagger b_k} \ , \tag{7}$$

with $M_k$ being the average photon number in the $k$th environment mode. Because $\mathcal{H}$ for the electromagnetic field is infinite dimensional, we must restrict the energy available to preclude transmission of infinite information via a single mode. For each use of the channel, we thus constrain the average number of input photons $\mathrm{Tr}[a_k^\dagger a_k \rho]$ to be at most $N_k$. This energy-constrained channel has been extensively studied,[5, 6, 7, 8] and the Holevo bound has been proven for the constrained infinite-dimensional case.[6] As yet, however, no capacity derivation exists for this channel.

It has been conjectured that $C$ obeys[5, 9]

$$C = \sum_k \left[ g(\eta_k N_k + (1-\eta_k) M_k) - g((1-\eta_k) M_k) \right] \ , \tag{8}$$

where $g(x) \equiv (x+1)\log_2(x+1) - x\log_2 x$. This conjecture implies that the Holevo information for the channel (6) is *not* superadditive, rendering the regularization in Eq. (3) unnecessary because $C = C_1$. Equation (8) was recently demonstrated[10] for the pure-loss special case in which the thermal reservoir is at zero temperature, so that $M_k = 0$, $\forall k$. For the general case, the capacity conjecture Eq. (8) arises from the coincidence between a lower bound for the capacity and a conjectured upper bound. That $C$ cannot be less than the right-hand side of Eq. (8) is easily shown. We choose $n = 1$ and employ a code in which each mode $k$ is an independent mixture of coherent states $|\mu\rangle_k$ weighted with the Gaussian probability distribution [9, 11]

$$p_k(\mu) = \exp[-|\mu|^2/N_k]/(\pi N_k) \ . \tag{9}$$

This corresponds to feeding the channel the input state

$$\rho = \bigotimes_k \int \mathrm{d}\mu \ p_k(\mu) \ |\mu\rangle_k\langle\mu| \ , \tag{10}$$

which is a thermal state that contains no entanglement or squeezing. The Holevo information for this coding is then found to be given by Eq. (8).

Our incomplete demonstration that the right-hand side of (8) is an upper bound for $C$ proceeds as follows. The subadditivity of the von Neumann entropy implies

$$C_n \leqslant n \max_{\rho\in\mathcal{H}} S(\mathcal{E}[\rho]) - \min_{R\in\mathcal{H}^{\otimes n}} S(\mathcal{E}^{\otimes n}[R]) \ , \tag{11}$$

where the maximum is taken over density matrices $\rho$ of a single use of the channel, while the minimum is taken over density matrices $R$ of $n$ uses of the channel, with both optimizations required to satisfy the input photon-number constraint. An argument presented in the pure-loss special case[10] can be used to show that the state (10) achieves the maximum needed in (11). Our capacity proof would be complete were the minimum needed in (11) achieved by this same coherent-state encoding, because straightforward evaluation of (11) then gives the right-hand side of Eq. (8). Indeed, because convexity allows us to restrict the minimization in (11) to pure states, and because displacing a state—whether pure or mixed—does not change its von Neumann entropy, demonstrating that a vacuum-state input minimizes the output entropy for a single channel use[9, 12] is sufficient to prove the upper bound, and hence the capacity conjecture.

Unfortunately, the preceding minimum output entropy property of the vacuum-state input is still unproven, but it is quite reasonable from a physical point of view. Consider a beam splitter with one input port illuminated by a pure state $|\psi\rangle$ with zero mean-field ($\langle\psi|a_k|\psi\rangle = 0, \forall k$), and the other by thermal light. We expect that the $|\psi\rangle$ which minimizes the entropy at one of the output ports is the vacuum, because this seems like it should be the state that will become least entangled with the reservoir, and hence acquire the least von Neumann entropy in passing through the beam splitter. This then is our fundamental conjecture: that an input vacuum state achieves the minimum output entropy. We now present a collection of preliminary results that lend credence to this claim.[12]

### *Minimum output entropy*

For brevity, we will limit our treatment here to the single-mode case. Our conjecture states that the output entropy of the channel reaches its minimum for a vacuum-state input, i.e.,

$$\min_{\rho} S(\mathcal{E}[\rho]) = g((1-\eta)M), \tag{12}$$

where $\eta$ is the mode transmissivity and $M$ is the average photon number of the thermal environment. By studying the directional derivatives of $S(\mathcal{E}[\rho])$ along linear trajectories, we have proven that coherent states are *local* minima for $S(\rho')$, where $\rho' \equiv \mathcal{E}[\rho]$ is the channel's output state.[a] We have also proven that such a minimum is a global minimum for some special classes of input states, i.e., when the sender is allowed to use only Gaussian states or Fock states. The general input case is more demanding to analyze. Here, however, we have derived several lower bounds for the global output entropy, which we plot in Fig. 1. When applied to the negative contribution in the Holevo quantity (1), each of these bounds provides an upper bound on the channel capacity $C$. Although these minimum-entropy lower bounds do not attain $g((1-\eta)M)$, the mismatch is very small (especially in the high-noise regime), so that the channel capacity conjecture is a good approximation for many practical purposes. Indeed, superadditivity of $\chi$ would yield at most a modest increase in the capacity over that predicted by (8).

Additional support for our conjecture comes from our proof that coherent-state inputs minimize integer-order output Rényi entropies, i.e., $S_k(\rho') \equiv -[\ln \mathrm{Tr}(\rho'^k)]/(k-1)$, for $k = 2, 3, \ldots$ We have also shown that coherent-state inputs minimize the output Wehrl entropy, a measure of the output state's phase-space localization, given by

$$W(\rho') \equiv -\int \frac{\mathrm{d}^2\alpha}{\pi}\ \langle\alpha|\rho'|\alpha\rangle\ \ln\langle\alpha|\rho'|\alpha\rangle, \tag{13}$$

where $|\alpha\rangle$ is the coherent state.

In what follows we will ignore the question of whether (8) is indeed the the thermal bosonic channel's capacity by focusing our attention on the zero-temperature (pure-loss) limit for which we have already proven[10] that (8) is the channel capacity.

### *Input average energy constraint*

In keeping with common practice in communication theory, we shall relax our mode-by-mode average photon number constraint to allow the $\{N_k\}$ to take on any values consistent with

[a] An alleged proof of (12) has appeared;[13] careful analysis of this proof reveals that it too only demonstrates that the coherent states provide local minima for the output entropy.

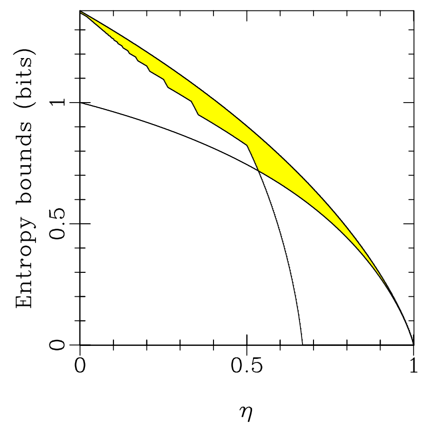


Fig. 1. Bounds on the minimal output entropy for the thermal channel $\mathcal{E}$ versus transmissivity $\eta$ for $M = 1/2$. The top curve is the upper bound $g((1-\eta)M)$; the other curves are the lower bounds discussed in [12]. The minimal output entropy is constrained to lie in the gray region between these bounds.

the average energy restriction

$$\sum_k \hbar\omega_k N_k = E \;, \tag{14}$$

where $\omega_k$ is the frequency of the $k$th mode. The channel capacity under this constraint is found from (8) by maximizing it over all $\{N_k\}$ satisfying (14),

$$C = \max_{\{N_j\}} \sum_k K(N_k) \;, \tag{15}$$

where $K(N_k) \equiv g(\eta_k N_k + (1-\eta_k)M_k) - g((1-\eta_k)M_k)$. This constrained maximization can be solved through the Lagrange multiplier technique,[10, 11] which reduces the problem to solving

$$\frac{\partial}{\partial N_j}\left[\sum_k K(N_k) - \frac{1}{\Omega \ln 2}\sum_k \omega_k N_k\right] = 0 \;, \tag{16}$$

where $1/(\Omega \ln 2)$ is the Lagrange multiplier. The solution to (16) is

$$N_k = \frac{1/\eta_k}{e^{\omega_k/(\eta_k\Omega)} - 1} - \frac{1-\eta_k}{\eta_k} M_k \;. \tag{17}$$

However, because $N_k$ is an average photon number, we must discard negative results in Eq. (17), viz., modes whose $N_k$ values from (17) are negative are not used in the communication and must be prepared in the vacuum ($N_k = 0$). Then, choosing $\Omega$ to satisfy the constraint (14), we obtain the channel capacity by substituting $N_k$ into Eq. (8).

## 3 Free-space optical communication

As an explicit example of the energy constrained, pure-loss channel we now treat the case of free-space optical communication. The propagation geometry we shall consider is shown in Fig. 2: the transmitter's output is a quantized optical field radiated through a circular aperture of area $A_t$, and the receiver's input is the quantized optical field collected through another circular aperture $A_r$. These apertures are separated by an $L$-m-long vacuum propagation path, so that the CP map for this channel derives from the quantum version of the Huygens-Fresnel principle.[14] For our purposes, it is more convenient to use the associated normal-mode description that is provided by classical electromagnetic-field analysis,[15, 16] which decomposes the field into generalized prolate spheroidal mode functions indexed by integer parameters $N, n$. Because of diffraction, each $A_t$-to-$A_r$ mode encounters a sub-unity, frequency-dependent transmissivity, $\eta_{N,n}$, that is a monotonically increasing function of frequency. Closed-form results for the $\{\eta_{N,n}\}$ are not known, but they have the following asymptotic behavior in terms of the normalized frequency $x = \omega/\omega_0$, where $\omega_0 \equiv 2\pi c\, L/\sqrt{A_r A_t}$ is the Fresnel frequency:[16]

$$\eta_{N,n}(x) \simeq \begin{cases} \left[\dfrac{n!(n+N)!\, x^{2n+N+1}}{(2n+N)!(2n+N+1)!}\right]^2 & x \ll 1 \\ 1 - \pi \dfrac{2^{3N+6n+4}\, x^{2n+N+1} e^{-4x}}{n!(n+N)!} & x \gg 1 \ . \end{cases} \tag{18}$$

Our calculations employ an interpolation of these asymptotic results, see Fig. 3. Moreover, we initially discretize frequency in terms of a $T$-s-long transmission interval, i.e., $\omega_j = 2\pi j/T$ for integer $j$.

The capacity of this pure-loss channel is then given by Eq. (8) with $k$ being the collective index $\{N, n, j\}$, and $M_k = 0, \forall k$. To keep this capacity finite, we impose the average energy constraint

$$E = \sum_{N,n} D_{N,n} \sum_j \hbar\omega_j N_j(N,n) \ , \tag{19}$$

where $D_{N,n}$ is a degeneracy factor[16] which equals 1 for $N = n = 0$ and 2 otherwise. The Lagrange multiplier technique now yields the optimal photon-number distribution

$$N_j(N,n) = \frac{1/\eta_{N,n}(\frac{\omega_j}{\omega_0})}{\exp[\omega_j/(\Omega\ \eta_{N,n}(\frac{\omega_j}{\omega_0}))] - 1} \ , \tag{20}$$

where $\Omega$ must be determined from Eq. (19). Its value cannot be recovered analytically, but

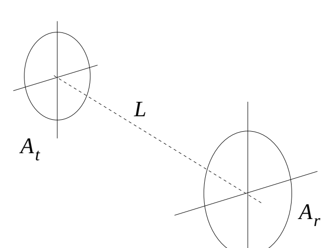


Fig. 2. Propagation geometry for free-space optical communication

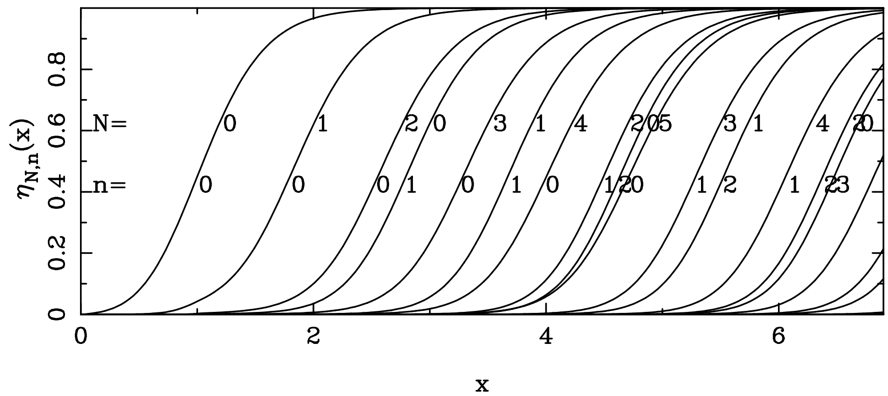

Fig. 3. Modal transmissivities $\eta_{N,n}(x)$ versus normalized frequency.

it can be found numerically from

$$\frac{2\pi\, P}{\hbar\omega_0^2} = \sum_{N,n} D_{N,n} \int_0^\infty \frac{\mathrm{d}x}{\eta_{N,n}(x)} \, \frac{x}{\exp[x\, \omega_0/(\Omega\, \eta_{N,n}(x))] - 1} \,, \tag{21}$$

where $P = E/T$ is the transmission power, see, e.g., Fig. 4. (Equation (21) replaces a sum by

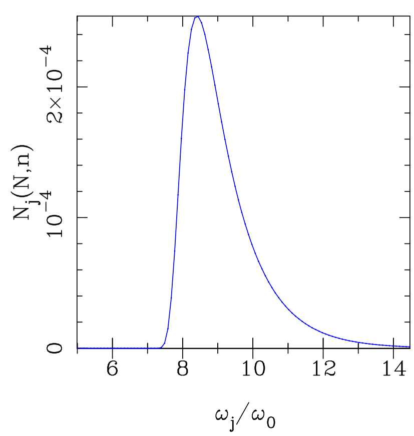

Fig. 4. Plot of $N_j(N,n)$ versus normalized frequency, for $N = 1$ and $n = 3$, obtained by substituting the numerical solution of (21) when $2\pi\, P/(\hbar\omega_0^2) = 2$ into (20). The average photon number approaches zero at both low and high frequencies. The former behavior is due to the poor transmissivity. The latter is a consequence of the high energy cost of the mode: given the energy constraint (15) it is not optimal to use high frequencies to transfer information.

an integral; this is a good approximation[8, 17] when $P \gg \hbar\omega_0^2$.) Substituting (20) into (8) yields,

$$C = \frac{\omega_0\, T}{2\pi} \sum_{N,n} D_{N,n} \int_0^\infty \mathrm{d}x \; g\left( \frac{1}{\exp[x\, \omega_0/(\Omega\, \eta_{N,n}(x))] - 1} \right) \,, \tag{22}$$

for the capacity per channel use. Dividing by $T$ then gives us the channel's information rate.[6, 8, 11] Figure 5 plots this information rate for several cases in which different numbers of modes are employed.

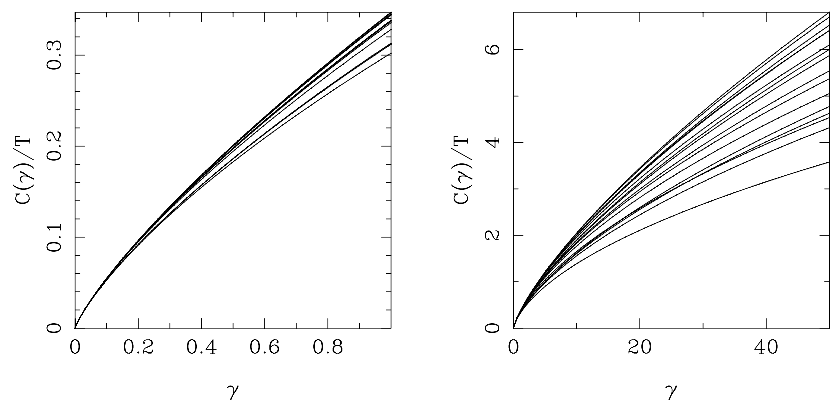


Fig. 5. Information rate (capacity in bits/sec) versus normalized power $\gamma \equiv 2\pi\, P/(\hbar\omega_0^2)$ as the number of modes employed is increased.. The bottom curve includes only the $N = n = 0$ mode. Capacity increases as more modal contributions are included; the remaining curves contain contributions from modes with $n = 0, 1, 2, 3$ and $N = 0, 1, 2, 3$. For $\gamma \ll 1$, the channel capacity is power limited, and essentially all the capacity is provided by the maximum-transmissivity ($N = n = 0$) mode.

### *Far field propagation*

The channel model given above simplifies significantly if we assume that only a single spatial mode in the transmitter couples a significant amount of power to the receiver. Such is the case at low Fresnel numbers, $D(\omega) \equiv (\omega/\omega_0)^2 \ll 1$.[14, 16] To stay within this far-field regime, with fixed aperture areas, we must impose a maximum transmitter frequency, $\omega_c$, obeying $D(\omega_c) \ll 1$, Equation (18) then implies that $\eta_{0,0}(\omega/\omega_0) \simeq D(\omega)$ for $0 \le \omega \le \omega_c$, so that the capacity becomes

$$C = \frac{\omega_c\, T}{2\pi y_0} \int_0^{y_0} \mathrm{d}x\; g\!\left(\frac{1}{e^{1/x} - 1}\right) , \tag{23}$$

where $y_0$ is a dimensionless parameter (inversely proportional to the Lagrange multiplier) which is determined from the power constraint

$$\frac{2\pi\, P}{\hbar\omega_0^2} = \int_0^{y_0} \frac{\mathrm{d}x}{x} \frac{1}{e^{1/x} - 1} . \tag{24}$$

Although $C$ is proportional to $\omega_c$, this factor cannot be increased without bound because of the far-field assumption. Figure 6 plots $C$ versus $P$ obtained by numerical evaluation of Eqs. (23) and (24).

### *Far-field capacity with coherent state encoding*

The far-field, free-space capacity result from the preceding section provides an ultimate benchmark against which the capacities achieved on this channel with the coherent state encoding

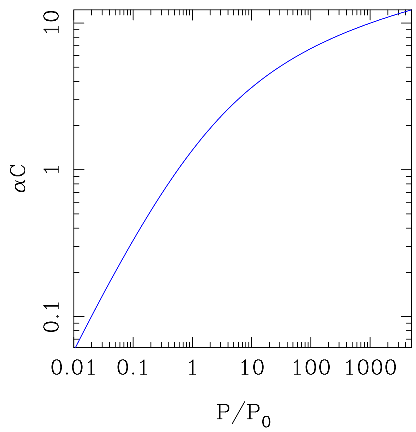


Fig. 6. Normalized far-field, free-space channel capacity versus normalized power, where $\alpha \equiv 2\pi/(\omega_c T)$ and $P_0 \equiv \hbar\omega_0^2/(2\pi)$ ).

(10) and coherent (homodyne or heterodyne) detection may be compared. Summing the Shannon capacities of these coherent detection schemes we find that[8]

$$C_{\text{coherent}} = \max_{N_k} \sum_k \xi \log_2(1 + \eta_k N_k/\xi^2), \tag{25}$$

where $\xi = 1/2$ for homodyne detection and $\xi = 1$ for heterodyne detection, and the maximization must be performed under the energy constraint (14). The Lagrange multiplier maximization now yields the optimal intensity

$$I_k(I_0) = \hbar\omega_k N_k(I_0) = \max\left\{ I_0 - \frac{\hbar\omega^2\xi^2}{\omega_k}, 0 \right\}, \tag{26}$$

where $I_0$ is the Lagrange multiplier.

It is convenient to recast the preceding intensity and capacity results in terms of the following normalized quantities: $I'(\omega') \equiv I(\omega)\omega_c/\hbar{\omega_0}^2$, where $\omega' \equiv \omega/\omega_c$ is a dimensionless frequency, and $C'_{\text{coherent}} \equiv C_{\text{coherent}}(2\pi)/\omega_c T$. These normalized quantities are, in turn, functions of the normalized power $P' \equiv P/P_0$, where $P_0 = \hbar\omega^2/2\pi$, as given by the following relations:

$$I'(\omega') = \max\left\{ I'_0 - \frac{\xi^2}{\omega'}, 0 \right\}, \tag{27}$$

$$C'_{\text{coherent}} = \frac{\xi}{\ln 2}\left( \frac{1}{I'_0} - 1 + \ln I'_0 \right), \quad \text{and} \tag{28}$$

$$P' = \xi^2 \left( I'_0 - 1 - \ln I'_0 \right), \tag{29}$$

with $I'_0 \equiv I_0\omega_c/\hbar\omega^2$. Equation (27) is a power-allocation formula that is similar to the capacity achieving 'water-filling' power allocation for parallel additive white Gaussian noise channels in classical information theory. For our far-field free-space channel, the 'water-filling' starts from the highest allowed frequency $\omega_c$, and then progresses to lower frequencies. If there is a non-zero *minimum* frequency $\omega_{\text{min}}$ that may be used in the transmission, then $P'$ has a

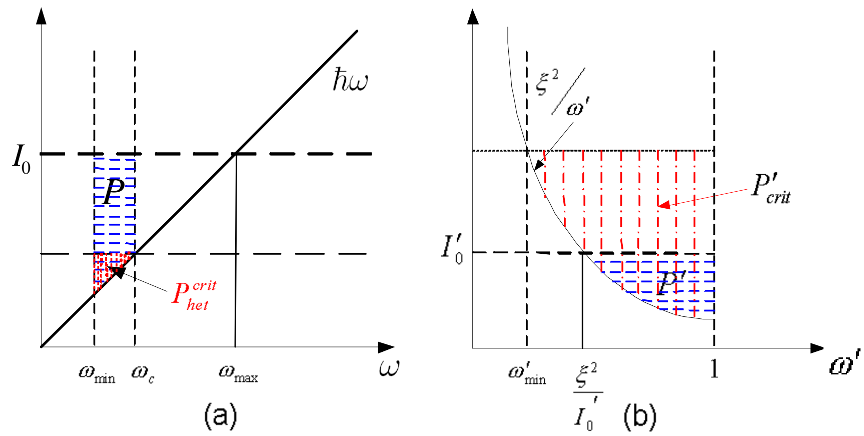


Fig. 7. Normalized capacity-achieving power allocation versus normalized frequency for coherent detection: (a) lossless channel; (b) far-field, free-space channel.

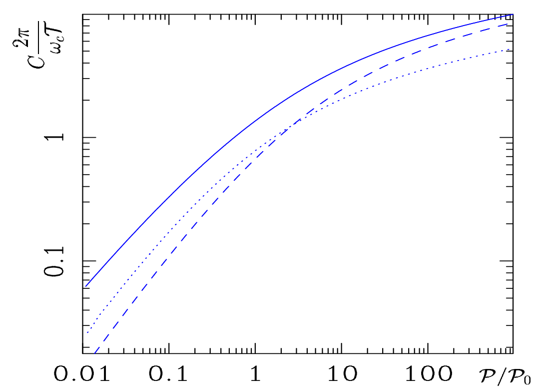


Fig. 8. Normalized capacities (in bits) for the far-field, free-space channel versus normalized power. The solid line is the ultimate (maximum Holevo information) capacity, the dashed line is the heterodyne detection capacity, and the dotted line is the homodyne detection capacity.

critical value $P'_{\text{crit}}$, above which the power 'fills' up vertically in the region $\omega' \in [\omega'_{\text{min}}, 1]$, where $\omega'_{\text{min}} \equiv \omega_{\text{min}}/\omega_0$. This capacity-achieving power allocation for the far-field channel, shown in Fig. 7(b), stands in marked contrast to that for the lossless power-constrained wideband channel, shown in Fig. 7(a), wherein power is preferentially allocated to low frequencies.[8]

Figure 8 compares the normalized coherent-detection capacities of the far-field, free-space channel with $C' \equiv C/\omega_c T$, the normalized version of Eq. (23), the ultimate (maximum Holevo information) capacity. At high powers, this figure suggests that the heterodyne detection capacity approaches the ultimate capacity. It is straightforward to show, analytically, that this is indeed the case, viz., $C'/C'_{\text{heterodyne}} \to 1$ as $P \to \infty$.

## 4 Conclusions

In this chapter we summarized the evidence we have developed in support of the conjectured capacity formula for the bosonic channel with thermal noise. Because we have already proven this conjecture in the zero-temperature special case, we discussed in detail the application of that result to the free space optical channel, paying particular attention to the far-field propagation regime. We also presented far-field, free-space propagation capacities for transmission systems that use coherent-state encoders and coherent (homodyne or heterodyne) detection. It turned out that that the heterodyne capacity for this channel approached the ultimate (maximum Holevo information) capacity in the limit of high input power.

It is an honor for us to contribute a chapter so intimately concerned with Holevo information on the occasion of Professor Alexander Holevo's 60th birthday. This work was funded by ARDA, NRO, NSF, and by ARO under a MURI program.

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