QUANTUM BINARY DECISION AND THE ULTIMATE BOUND ON INTERFEROMETRIC PRECISION

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Abstract

Quantum hypothesis testing has been applied to the analysis of high-sensitive interferometric measurements and a general bound on interferometric precision has been obtained in terms of the photon number fluctuations of the mode carrying the phase information.

1 Introduction

The precise determination of phase-shifts is an important issue in quantum measurement theory. On one hand there are fundamental applications of interferometry, such as in gravitational-wave antennae, whereas, on the other hand, the phase-number complementarity is a delicate matter, due to the absence of a proper phase observable in quantum mechanics. Here, we are going to employ quantum decision theory as a suitable framework to discuss high-sensitive interferometric measurements. We do not concern to any specific measurement device and we do not discuss the feasibility of optimized measurement. Rather, our aim is that of deriving an ultimate bound on interferometric precision, depending only on the quantum state carrying the phase information.

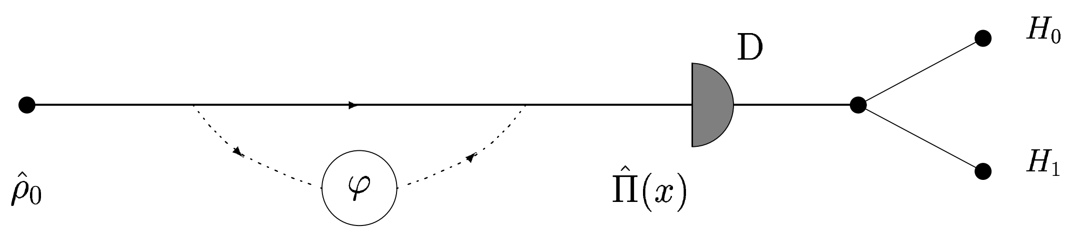

Figure 1: Abstract outline of an interferometric detection scheme.

A schematic diagram of a generic interferometric setup is shown in Fig. 1. A fixed state of radiation, say $\hat{\rho}_0$, is initially prepared, travels along the interferometer and it is eventually measured by some detectors, denoted by D. The outcome of the latter is described by an operator-valued probability measure $\hat{\Pi}(x)$, being $x \in X$ the set of the possible outgoing values for the measured quantity. If some environmental parameter changes then also the optical path is subjected to variation, thus leading to a phase-shift φ on the signal mode. The aim of the detection scheme $\hat{\Pi}(x)$

is that of discriminating in between $\hat{\rho}_0$ and its phase-shifted version $\hat{\rho}_1 = \exp\{i\hat{n}\varphi\}\hat{\rho}_0 \exp\{-i\hat{n}\varphi\}$, which results if some perturbations have been occurred. An optimized interferometer is able to tell the $\hat{\rho}$'s apart for φ smaller as possible. Here, the phase-shift φ plays the role of a parameter labeling the perturbed state $\hat{\rho}_1$. The interest of this approach lies in the fact that it does not refer to any specific detection scheme for the final stage of the interferometer. Thus, ultimate quantum limit on the interferometric precision can be obtained, depending only on the quantum state of radiation carrying the phase information along the interferometer. One should mention that usually, in optimizing interferometry, just the opposite route has been followed. After fixing some interferometric setup precision has been optimized over the states of radiation [1, 2, 3, 4, 5].

2 Quantum binary decision

First, consider a decision problem requiring to discriminate among M hypotheses $H_1, \cdots, H_M$ regarding a quantum system. Hypothesis H_k asserts that the density operator of the system is $\hat{\rho}_k$, $(k = 0, \cdots, M)$. To establish a *quantum decision strategy* a measurement scheme should be chosen. The latter is described by an operator-valued probability measure $\hat{\Pi}(x)$, defined on the Hilbert space of the quantum system under examination. If this measurement is performed on the system when hypothesis H_k is true, then the conditional probability of choosing hypothesis H_j is given by

$$\mathrm{P}(x \equiv j|\varphi_k) \;=\; \mathrm{Tr}\left\{\rho_k \hat{\Pi}(j)\right\} . \tag{1}$$

Here, x denotes the random variable that is to be observed, and φ a parameter labeling the unknown state of the system.

Let us now suppose to be able to specify the *costs* C_{ij} of choosing hypothesis H_i when H_j is true. In this case the average (expected) cost of the whole observational strategy is given by

$$\bar{C} \;=\; \sum_{i=1}^{M}\sum_{j=1}^{M} \xi_j C_{ij} P(i|j) , \tag{2}$$

ξ_k being the *a priori* probability of the hypothesis H_k. An operator-valued probability measure $\hat{\Pi}(x)$, which minimizes the average cost $\overline{C}$ individuates an optimal *Bayesian* strategy for quantum hypothesis testing.

In the case of binary decision one can easily verify [6] that the optimal probability measure is actually projection valued, namely it corresponds to the measurement of a self-adjoint operator. The average cost becomes

$$\bar{C} = \xi_1 C_{11} + \xi_2 C_{12} - \xi_2 (C_{12} - C_{22}) \sum_{\eta_i > 0} \eta_i \;, \tag{3}$$

where η_i are the eigenvalues of the operator $\hat{\rho}_2 - \gamma\hat{\rho}_1$, with

$$\gamma \;=\; \frac{\xi_1 (C_{21} - C_{11})}{\xi_2 (C_{12} - C_{22})} \,. \tag{4}$$

Unfortunately, *a priori* probabilities ξ_j and *costs* C_{ij} cannot be easily individuated in an interferometric setup, so that some different approach should be analyzed. Actually, for the case of binary decision, the task of finding an optimal measurement can be nicely pursued by adopting a different strategy, corresponding to the Neyman-Pearson criterion. The latter will be illustrated in the following. We denote by P_{01} the probability of wrong inference for the hypothesis H_1, namely that one of inferring H_1 when H_0 is true. In hypothesis testing formulation this is usually referred to as *false alarm probability*. Conversely, we denote by P_{11} the *detection probability*, that is the probability of inferring H_1 when it is actually true. Now, which is the best measurement to discriminate between $\hat{\rho}_0$ and $\hat{\rho}_1$? If these two states are mutually orthogonal the problem has a trivial solution. It is a matter of measuring the observable for which $\hat{\rho}_0$ and $\hat{\rho}_1$ are eigenstates. However, this is not our case, as it is well known that no orthogonal set of phase-eigenstates is available in quantum optics. In the following we consider nonorthogonal $\hat{\rho}_0$ and $\hat{\rho}_1$ and we focus our attention on pure states $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$ as input for the interferometer. The Neyman-Pearson criterion for binary decision [7]. reads as follows. First, we have to fix a value for the false alarm probability P_{01}. Then, we have to find the measurement strategy $\hat{\Pi}(x)$ which maximizes the detection probability P_{11}. As a general definition, each measurement strategy which maximizes the detection probability P_{11} for a fixed value of false alarm probability P_{01} is considered as a Neyman-Pearson optimized detection for binary hypothesis testing. We mention, for comparison, that what is minimized in a Bayesian approach is instead the average probability of error.

It was shown by Helstrom [6] and Holevo [8] that the Neyman-Pearson quantum binary decision problem could be reduced to solving the eigenvalue problem for the operator

$$\hat{\Pi}(x|\lambda) = \hat{\rho}_1 - \lambda\hat{\rho}_0 \;, \tag{5}$$

which represents the optimized measurement scheme. The parameter λ is a Lagrange multiplier. Different values of λ correspond to different values of the false alarm probability, namely to a different Neyman-Pearson strategies. Once the eigenvalues problem for $\hat{\Pi}(x|\lambda)$ has been solved it results that only positive eigenvectors contribute to the detection probability P_{11} [6, 9]. Thus the decision strategy is transparent: after a measurement of the quantity $\hat{\Pi}(x|\lambda)$ if the outcome is positive we infer that perturbation hypothesis H_1 is true. Conversely, we infer null hypothesis H_0 when obtaining negative outcome. By expanding the eigenstates of $\hat{\Pi}(x|\lambda)$ in terms of $|\psi_0\rangle$ and $|\psi_1\rangle$ the Lagrange multiplier λ can be eliminated from the expression of detection probability which results

$$P_{11} = \begin{cases} \left[\sqrt{P_{01}\kappa} + \sqrt{(1-P_{01})(1-\kappa)}\right]^2 & 0 \le P_{01} \le \kappa \\ 1 & \kappa \le P_{01} \le 1 \end{cases} \;. \tag{6}$$

In Eq. (6) κ denotes the square modulus of the overlap between perturbed and unperturbed state, in formula

$$\kappa = |\langle\psi_0|\psi_1\rangle|^2 = |\langle\psi_0|\exp\{i\hat{n}\varphi\}|\psi_0\rangle|^2 \;. \tag{7}$$

The overlap depends both on the initial state and on the occurred phase-shift φ. It is obvious that if the overlap is small, it is easy to discriminate between the two states. Thus, it is possible to obtain strategies with large detection probability without paying the price of an also large false alarm probability. On the contrary, if the overlap becomes appreciable it is difficult to discriminate the states. In the limit of complete overlap the perturbed and the unperturbed states become indistinguishable. Detection probability is now equal to false alarm probability and the decision strategy is just a matter of guessing after each random measurement outcome. In Fig. 2 we show the detection probability P_{11} of Eq. (6) as a function of the false alarm probability P_{01} and the overlap strength κ.

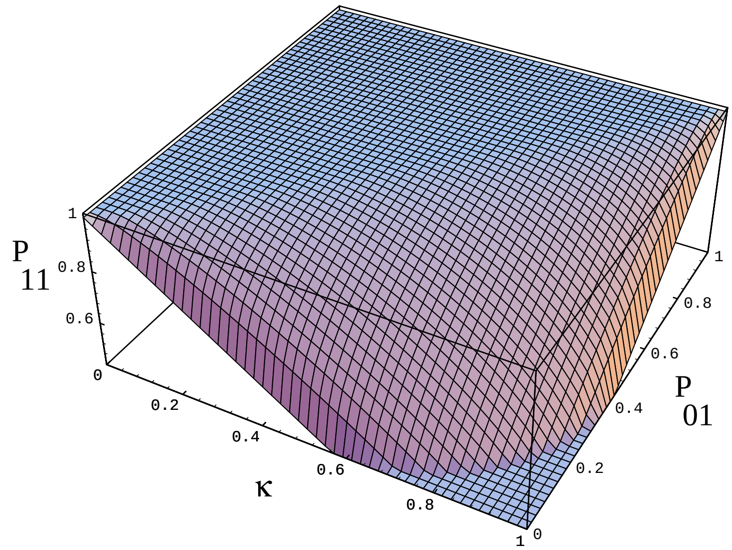

Figure 2: Detection probability P_{11} versus the false alarm probability P_{01} and the overlap κ.

Choosing a value for the false alarm probability is a matter of convenience, depending on the specific problem this approach would be applied. The maximum tolerable value for P_{01} increases with the expected number of measurement outcomes, and conversely a very low rate detection scheme needs a very small false alarm probability. The latter is the case of interferometry, in the following we always will consider small value for P_{01}.

3 Ultimate bound on interferometric precision

Once an input state for the interferometer has been specified the probability measure in Eq. (5) defines the detection scheme to be performed in order to implement an optimized interferometer. Optimality is in the Neyman-Pearson sense, namely that detection (5) maximizes the detection probability P_{11} at a fixed tolerable value of the false alarm probability P_{01}.

The interferometric strategy is thus the following: the initial state $\hat{\rho}_0$ is prepared and left free to travel along the interferometer. A set of measurements for the quantity (5) is then performed and from the data record we have to infer the state of radiation at the output of the interferometer. From this inference we can discriminate in between the two hypothesis, namely we are able to monitor the optical path of the light beam.

The input state is fixed in advance, therefore the detection probability depends only on the accepted value for the false alarm probability and on actual value of the phase-shift φ. The minimum detectable value of the phase-shift, denoted by φ_M, is defined by the relation

$$P_{11}(\varphi_M; P_{01}) = \frac{1}{2}\,. \tag{8}$$

A lower value for P_{11}, in fact, would make the measurements record useless, as no readable informations can be extracted in that case.

Let us consider customary coherent states as an illustrative example. Without loss of generality we can set the phase of initial state to be zero, so that we have

$$|\psi_0\rangle = \exp\{-\frac{1}{2}\alpha^2\}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}\,|k\rangle \qquad \alpha\in\mathbf{R}\,. \tag{9}$$

The photon distribution of a coherent state is Poissonian with mean given by $N\equiv\langle\alpha|\hat{n}|\alpha\rangle=\alpha^2$ and the overlap can be easily evaluated to be

$$\kappa = \left|\langle|\psi_0|e^{i\hat{n}\varphi}|\psi_0\rangle\right|^2 = \exp\{-2\alpha^2(1-\cos\varphi)\}\,. \tag{10}$$

Interferometric detection is frequently involved with low rate processes. We mention that among applications of high sensitive interferometry one of the most interesting regards the detection of gravitational waves. The reader may agree that this is a prototype for *very* low rate process [10]. Therefore, we have, as a general requirement, to ask for a small false alarm probability: a convenient setting reads $P_{01}\leq\kappa$. Inserting Eq. (6) in Eq. (8) the relation for the minimum detectable phase shift becomes

$$\frac{1}{2} = \left[\sqrt{P_{01}\kappa}+\sqrt{(1-P_{01})(1-\kappa)}\right]^2\,, \tag{11}$$

that is,

$$\kappa = \sqrt{\frac{1+\sqrt{P_{01}(1-P_{01})}}{2}}\,. \tag{12}$$

Finally, upon substituting (10) in Eq. Eq. (12) and expanding for small φ we have

$$\varphi_M = \sqrt{\log\left(\frac{2}{1+\sqrt{P_{01}(1-P_{01})}}\right)}\frac{1}{\sqrt{N}}\,, \tag{13}$$

which represents the lower bound on minimum detectable phase-shift for *any* interferometer based on coherent states. The bound in Eq. (13) is well known and represent the lower bound on precision also for interferometer based on classical state of radiation. It is usually termed shot-noise limit.

Let us now consider a generic pure state at the input of the interferometer

$$|\psi_0\rangle = \sum_{k=0}^{\infty} c_k|k\rangle\,. \tag{14}$$

Still we consider zero initial phase, thus the coefficients $\{c_k\}_{k\in\mathbf{N}}$ are real numbers. For the overlap we have

$$\kappa = \left|\sum_{k=0}^{\infty} c_k^2 e^{ik\varphi}\right|^2 = \left(\sum_{k=0}^{\infty} c_k^2 \cos k\varphi\right)^2 + \left(\sum_{k=0}^{\infty} c_k^2 \sin k\varphi\right)^2 , \tag{15}$$

and, up to second order in the phase-shift,

$$\kappa = 1 - \varphi^2 \Delta N^2 , \tag{16}$$

where $\Delta N^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$ denotes the photon number fluctuations of the considered state. By substitution in Eq. (12) we obtain the lower bound on the minimum detectable perturbation

$$\varphi_M = \sqrt{\frac{1-\sqrt{P_{01}(1-P_{01})}}{2}}\frac{1}{\Delta N} . \tag{17}$$

Eq. (17) represents a quite general result. It indicates that the minimum detectable phase-shift φ_M shows an inverse scaling relative to the photon number fluctuations rather than the photon number intensity. Eq. (17) is not surprisingly, however it is worth noticing that we derived it in a direct way from the binary problem approach, i.e. we did not make use of any uncertainty phase-number *pseudo* relation $\Delta N \Delta\varphi \sim 1$. The latter, in fact, can be derived only in a heuristic way [11] and thus possesses only a limited validity.

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References

[1] B. Yurke, S. L. McCall, J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[2] M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).

[3] M. J. Holland and K Burnett, Phys. Rev. Lett. **71**, 1355 (1993).

[4] G. M. D'Ariano, M. G. A. Paris, Phys. Rev. A**49**, 3022 (1994).

[5] M. G. A. Paris, Phys. Lett. A**201**, 132 (1995).

[6] C.W.Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, 1976).

[7] J. Neyman, E. Pearson, Proc. Camb. Phil. Soc. **29**, 492 (1933); Phil. Trans. Roy. Soc. London A**231**, 289 (1933).

[8] A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory*, (North-Holland, 1982).

[9] A. S. Holevo, J. Multivar. Anal. **3**, 337 (1973); H. P. Yuen, R. S. Kennedy, M. Lax, IEEE Trans. Inf. Theory, IT**21**, 125 (1975).

[10] K. Thorne, Rev. Mod. Phys. **52**, 285 (1980); W. W. Chou, Rev. Mod. Phys. **57**, 61 (1985).

[11] A. S. Lane, S. Braunstein, C. M. Caves, Phys. Rev. A**47**, 1667 (1993).