DYNAMICS OF PURITY IN NOISY CHANNELS

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We study the evolution of purity for Gaussian states of single-mode continuous variable systems, in general Gaussian environments. We derive initial and asymptotic conditions which, at any given time, maximize purity. These prove to be initial coherent states evolving in thermal baths or initial squeezed states evolving in 'orthogonally squeezed' baths. The extension of this method to Schroedinger cat states is discussed.

1 Introduction

As well known, due to the unavoidable interaction with its environment, every pure state of a quantum system involved in some physical process encounters decoherence. In particular, an initial pure quantum state is corrupted and gets mixed while, more generally, an initially mixed state becomes more and more mixed. Unfortunately, in most relevant applications of fundamental quantum mechanics, one would like to dispose of states as close as possible to pure ones. For instance, in the emerging field of quantum information and computation, controlling decoherence constitutes a basic challenge. [1] Moreover, the evolution of decoherence retains a *per se* importance towards the full understanding of the behaviour of open quantum systems. Therefore, it should be clear that the evolution of decoherence of relevant quantum systems is an issue of great interest, on both the fundamental and the applied point of view.

In particular, quantum optical systems seem to be more and more promising, either for applications in quantum information or for testing fundamental quantum mechanics. [2] In this paper we address the evolution of the degree of mixedness of some quantum optical systems in configurations of major phenomenological relevance. Namely, we will deal with a single-mode dissipative Gaussian channel (both thermal and squeezed), considering various initial conditions (single-mode Gaussian states and generalized cat states). Some preliminary study about two-mode states will be briefly shown as well.

The degree of mixedness of the quantum state will be characterized by its purity $\mu \equiv \mathrm{Tr}(\varrho^2)$. The quantity μ is conjugate to the so called linear entropy, namely $S_l = 1 - \mu$ (holding for continuous variable systems). The purity of a quantum state is obviously strictly positive (going to 0 for the ideal completely mixed state) and takes the value 1 for pure states.

2 Gaussian states

Let us consider a n-mode continuous variable system, described by the Hilbert space $\mathcal{H} = \otimes_i \mathcal{H}_i$ resulting from the tensor product of Fock spaces $\mathcal{H}_i$, with annihilation operators a_i and quadrature operators $\hat{x}_i = (a_i + a_i^\dagger)/\sqrt{2}$ and $\hat{p}_i = -i(a_i - a_i^\dagger)/\sqrt{2}$. We recall that the set of Gaussian states is defined as the set of states with Gaussian characteristic functions and quasi-probability distributions. Therefore, Gaussian states are fully determined by their first and second statistical moments that is, respectively, by the vector of mean values $\bar{X}$ and by the covariance matrix $\boldsymbol{\sigma}$

$$\bar{X}_i \equiv \langle \hat{X}_i \rangle \, , \quad \sigma_{ij} \equiv \frac{1}{2}\langle \{\hat{X}_i, \hat{X}_j\} \rangle - \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle \, , \tag{1}$$

where $\hat{X}_1 = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$ and $\langle \hat{o} \rangle = \mathrm{Tr}\,(\varrho \hat{o})$ for the operator $\hat{o}$. The Wigner function $W(X)$ of a Gaussian state, defined as the Fourier transform of the symmetrically ordered characteristic function [3], reads, in terms of the phase space variable vector $X \equiv (x_1, p_1, \ldots, x_n, p_n)$

$$W(X) = \frac{\mathrm{e}^{-\frac{1}{2}(X-\bar{X})\boldsymbol{\sigma}^{-1}(X-\bar{X})^T}}{\pi\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}} \, . \tag{2}$$

Heisenberg uncertainty principle is equivalent to the condition

$$\boldsymbol{\sigma} + \frac{i}{2}\boldsymbol{\Omega} \geq 0 \, , \tag{3}$$

where $\boldsymbol{\Omega}$ is the symplectic form for n couples of canonical variables: $\boldsymbol{\Omega} = \oplus_i \boldsymbol{\omega}$, resulting from the direct sum of n 2×2 matrices $\omega_{ij} = \delta_{i+1,j} - \delta_{i,j+1}$.

Because of the well known properties of the Wigner phase space representation, one can write

$$\mu = \mathrm{Tr}\,(\varrho^2) = \frac{\pi}{2^n}\int_{\mathbb{R}^{2n}} W^2 \,\mathrm{d}^n x \,\mathrm{d}^n p = \frac{1}{2^n\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}} \, . \tag{4}$$

Thus a Gaussian state with covariance matrix $\boldsymbol{\sigma}$ is pure iff $\mathrm{Det}\,\boldsymbol{\sigma} = 1/2^{2n}$.

We recall that a single-mode Gaussian state can always be written as

$$\varrho = D(\bar{\alpha}) S(r,\varphi) \nu_{\bar{n}} S^\dagger(r,\varphi) D^\dagger(\bar{\alpha}) \, , \tag{5}$$

where $\bar{\alpha} = \bar{x} + i\bar{p}$, $\nu_{\bar{n}}$ is a thermal state with average photon number $\bar{n}$, $D(\bar{\alpha}) = \exp(\bar{\alpha}a^\dagger - \bar{\alpha}^* a)$ denotes the displacement operator and $S(r,\varphi) = \exp(\frac{1}{2} r\,\mathrm{e}^{-i2\varphi} a^2 - \frac{1}{2} r\,\mathrm{e}^{i2\varphi} a^{\dagger 2})$ is the squeezing operator. A convenient parametrization of single-mode Gaussian states can then be achieved replacing the σ_{ij}'s by n, r, φ, which have a more direct phenomenological interpretation. By applying the phase space representation of squeezing, the following relation is easily derived

$$\boldsymbol{\sigma}_{\mu,r,\varphi} = \frac{1}{2\mu}\begin{pmatrix} \cosh(2r) - \sinh(2r)\cos(2\varphi) & \sinh(2r)\sin(2\varphi) \\ \sinh(2r)\sin(2\varphi) & \cosh(2r) + \sinh(2r)\cos(2\varphi) \end{pmatrix} , \tag{6}$$

where we used the relation $\mu = 1/(2\bar{n}+1)$ [4].

3 Evolution in Gaussian noisy channels

We now consider the evolution of a single-mode Gaussian state in a generic Gaussian environment. Let Γ be the coupling to the environment (being the inverse of the photon lifetime in the channel), the evolution of a quantum state in such a case is governed by the following master equation [3,4]

$$\dot{\varrho} = \frac{\Gamma}{2} N\, L[a^\dagger]\varrho + \frac{\Gamma}{2}(N+1)\, L[a]\varrho - \frac{\Gamma}{2}\left(M^*\, D[a]\varrho + M\, D[a^\dagger]\varrho\right) , \quad (7)$$

where the dot stands for time–derivative and the Lindblad superoperators are defined by $L[O]\varrho \equiv 2O\varrho O^\dagger - O^\dagger O\varrho - \varrho O^\dagger O$ and $D[O]\varrho \equiv 2O\varrho O - OO\varrho - \varrho OO$. The complex parameter M is the correlation function of bath (which is usually referred to as the 'squeezing' of the bath [5]), while N is a phenomenological parameter related to the purity of the asymptotic state. Positivity of the density matrix imposes the constraint $|M|^2 \le N(N+1)$. At thermal equilibrium, *i.e.* for $M = 0$, N coincides with the average number of thermal photons. Let $\varrho_\infty = S(r_\infty, \varphi_\infty)\nu_{\bar{n}_\infty} S(r_\infty, \varphi_\infty)^\dagger$ be the environmental Gaussian state. A more suitable parametrization of the quantum channel, endowed with a direct phenomenological interpretation, can be achieved by expressing N and M in terms of the three real variables μ_∞, r_∞ and φ_∞

$$\mu_\infty = \left[(2N+1)^2 - 4|M|^2\right]^{-\frac{1}{2}} , \quad (8)$$

$$\cosh(2r_\infty) = \left(1 + 4\mu_\infty^2 |M|^2\right)^{\frac{1}{2}} , \quad 2\varphi_\infty = -\mathrm{Arg}\, M . \quad (9)$$

With standard techniques, it can be shown that the master equation (7) corresponds to a Fokker-Planck equation for the Wigner function of the system. [3] In compact notation, one has

$$\dot{W}(X,t) = \frac{\Gamma}{2}\left[\partial_X \cdot X^T + \partial_X\, \boldsymbol{\sigma}_\infty\, \partial_X^T\right] W(X,t) , \quad (10)$$

with $\partial_X \equiv (\partial_x, \partial_p)$ and with a diffusion matrix $\boldsymbol{\sigma}_\infty = \boldsymbol{\sigma}_{\mu_\infty, r_\infty, \varphi_\infty}$ determined by Eqns. (6, 8, 9).

For an initial Gaussian state of the form (2), Eq. (10) preserves the Gaussianity and corresponds to a set of decoupled equations for the first and second moments that can be easily solved [4]. The drift term just damps the first statistical moments $\bar{X}$

$$\bar{X}(t) = \mathrm{e}^{-\frac{\Gamma}{2}t}\bar{X}(0) , \quad (11)$$

while the covariance matrix evolves as follows

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_\infty\left(1 - \mathrm{e}^{-\Gamma t}\right) + \boldsymbol{\sigma}(0)\,\mathrm{e}^{-\Gamma t} . \quad (12)$$

This is a simple Gaussian completely positive map. Notice that $\boldsymbol{\sigma}(t)$ satisfies the uncertainty relation (3) if and only if $\boldsymbol{\sigma}_\infty$ and $\boldsymbol{\sigma}_0$ themselves satisfy it. The compliance of $\boldsymbol{\sigma}_\infty$ with inequality (3) is equivalent to the condition $|M| \leq N(N+1)$. Obviously, the state of the environment coincides with the asymptotic state evolving in the channel. This state (together with Γ) fully characterize the channel, and it is asymptotic irrespective of the choice of the initial state (being the latter Gaussian or not).

4 Evolution of purity

4.1 Single-mode Gaussian states

Exploiting Eqns. (4) and (12), we are in a position to give a formula for the time evolution of purity as a function of the set of parameters μ_0, r_0 and φ_0 (characterizing the initial state), and Γ, μ_∞, r_∞ and φ_∞ (characterizing the channel)

$$\mu(t) = \mu_0 \Big[\frac{\mu_0^2}{\mu_\infty^2} \left(1 - \mathrm{e}^{-\Gamma t}\right)^2 + \mathrm{e}^{-2\Gamma t} + 2\frac{\mu_0}{\mu_\infty} \Big(\cosh(2r_\infty)\cosh(2r_0) + \sinh(2r_\infty)\sinh(2r_0)\big(\cos(2\varphi_\infty - 2\varphi_0)\big)\Big) \left(1 - \mathrm{e}^{-\Gamma t}\right) \mathrm{e}^{-\Gamma t} \Big]^{-1/2} \quad (13)$$

We see from Eq. (13) that $\mu(t)$ is a monotonically decreasing function of $\cos(2\varphi_\infty - 2\varphi_0)$, which gives the only dependence on the initial phase φ_0 of the squeezing. Thus, for any given φ_∞, $\varphi_0 = \varphi_\infty + \frac{\pi}{2}$ is the most favorable value of the initial angle of squeezing, i.e. the one which allows the maximum purity at a given time. For such a choice, $\mu(t)$ reduces to a decreasing function of the factor $\cosh(2r_\infty - 2r_0)$, so that the maximum value of the purity at a given time is achieved for the choice $r_0 = r_\infty$, and the evolution of the purity of a squeezed state in an equally squeezed channel is identical to the evolution of the purity of a non–squeezed state in a non–squeezed channel expressed by

$$\mu(t) = \frac{\mu_0 \mu_\infty}{\mu_0 + \mathrm{e}^{-\Gamma} t(\mu_\infty - \mu_0)} \, . \quad (14)$$

This is the optimal evolution of purity for a single-mode Gaussian state in a Gaussian channel, obtained for an initial state orthogonally squeezed to the channel with relative squeezing $\tilde{r} \equiv r_0 - r_\infty$ equal to 0. In Fig. 1 some relevant instances are shown. Notice that, for $\tilde{r} > 0$, a local minimum for purity can appear. [4]

4.2 Schrödinger cats

We will call a Schrödinger cat a coherent superposition of the kind

$$|\gamma\rangle = \frac{|\beta_0\rangle + |-\beta_0\rangle}{\sqrt{2 + 2\,\mathrm{e}^{-2|\beta_0|^2}}} \, , \quad (15)$$

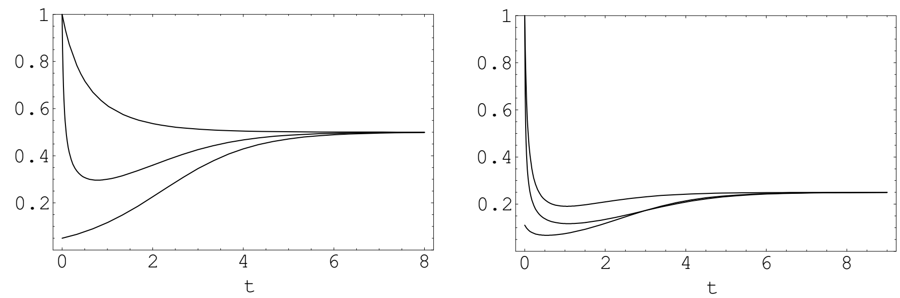

Figure 1. On the left, we show the purity of 1-mode Gaussian states evolving in a channel with $\mu_\infty = 0.5$. The squeezing angle is fixed to the optimal value ($\varphi_0 = \varphi_\infty + \pi/2$). The upper curve refers to an initial pure state with relative squeezing $\tilde{r} = 0$, the middle curve to an initial state with $\mu_0 = 1$ and $\tilde{r}_0 = 1.5$, and the lower curve to an initial state with $\tilde{r}_0 = 0$ and $\mu_0 = 0.05$. On the right, the purity evolution of 2-mode squeezed states is considered. The two modes evolve in identical channels with $\mu_\infty = 0.5$, while the two-mode squeezing parameter is always 1. The upper curve refers to an initial pure state in a non-squeezed bath, the middle curve to an initial pure state in squeezed baths ($r_\infty = 1$), and the lower curve to an initial squeezed thermal state (with $\mu_0 = 1/9$) in a thermal bath.

where $|\beta_0\rangle = S(r_0, \varphi_0)D(\beta_0)|0\rangle$ is a displaced squeezed state. Because of recent experimental developments [6], the study of the decoherence of such a state under dissipative evolutions arises increasing interest.

We point out that the Wigner function of $|\gamma\rangle$ is just the sum of four displaced Gaussian terms and that the evolution equation (10) is linear. Therefore, the four Gaussian terms evolve independently from each other and the exact evolution of an initial pure cat state $|\gamma\rangle$ through the Gaussian channel [described by Eq. (7)] is simply traced, using Eqns. (11, 12). Also the first moments of each term are relevant in this instance. Eventually, the purity of the evolving state results from a sum of Gaussian integrations in phase space

$$\mu(t) = \frac{1}{8(1+\mathrm{e}^{-(x_0^2+p_0^2)})^2\sqrt{\mathrm{Det}(\boldsymbol{\sigma}(t))}}\left[2\left(1+\mathrm{e}^{-\mathrm{e}^{-\gamma t}X_0^T\mathbf{R}\boldsymbol{\sigma}(t)^{-1}\mathbf{R}X_0}\right)\right.$$
$$\left. +\ 2\,\mathrm{e}^{-2(x_0^2+p_0^2)}\left(1+\mathrm{e}^{\mathrm{e}^{-\gamma t}\frac{X_0^T\mathbf{R}\boldsymbol{\sigma}(t)\mathbf{R}X_0}{\mathrm{Det}(\boldsymbol{\sigma}(t))}}\right)+8\,\mathrm{e}^{-(x_0^2+p_0^2)}A(t)\right], \qquad (16)$$

where, defining $\mathbf{R} \equiv \boldsymbol{\sigma}_{1,\frac{r_0}{2},\frac{\pi}{2}-\varphi}$ [see Eq. (6)], one has $\boldsymbol{\sigma}(t)$ given by Eq. (12) with initial condition $\boldsymbol{\sigma}(0) = 1/2\mathbf{R}^2$, and $A(t) \equiv \mathrm{Re}\left(\exp(-\mathrm{e}^{-\Gamma t}(x_0+ip_0)^2(M_{22}-M_{11}-2iM_{12})/4)\right)$, with $\mathbf{M} \equiv \mathbf{R}\boldsymbol{\sigma}(t)^{-1}\mathbf{R}$.

The numerical analysis shows that, for any choice of the bath, the purity encounters an initial steep fall, on a time scale of Γ^{-1} (characterizing the losses in the channel). Moreover, as one should expect, the purity decreases with increasing x_0 and p_0: the bigger the cat, the faster it decoheres. Further

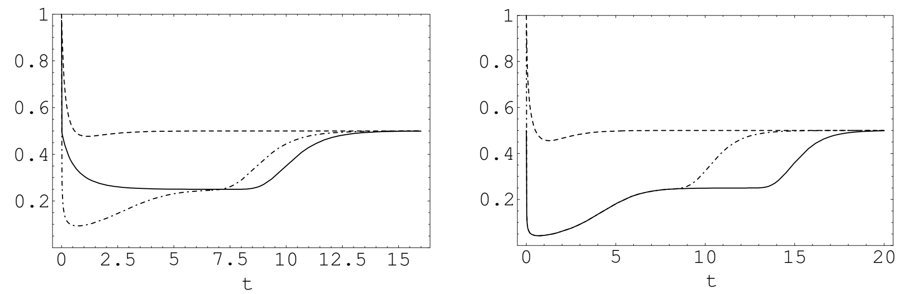

Figure 2. Purity of initial cat states. On the left a non squeezed channel with $\mu_\infty = 0.5$ is considered. The dashed line shows the behaviour of a cat with $x_0 = p_0 = 1$ and $r_0 = 0$; the dot–dashed line shows the behaviour o a cat with $x_0 = p_0 = 10$ and $r_0 = 2$; the continuous line shows the behaviour of a cat with $x_0 = p_0 = 100$ and $r_0 = 0$. On the right a squeezed channel with $M = 2+2i$ and $\mu_\infty = 0.5$ is considered. The dashed line shows the behaviour of a cat with $x_0 = p_0 = 1$ and $r_0 = 0$; the dot–dashed line shows the behaviour of a cat with $x_0 = p_0 = 10$ and $r_0 = 2$; the continuous line shows the behaviour of a cat with $x_0 = p_0 = 100$ and $r_0 = 2$.

details are shown in Fig. 2.

4.3 Perspectives

The generalization of this study to two-mode channels is interesting and quite straightforward. A detailed analysis of decoherence for general two-mode Gaussian states is actually in preparation. We just show a preview of this analysis in Fig. 1.

References

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