

Minimum error discrimination of Pauli channels

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Abstract

We solve the problem of discriminating with minimum error probability two given Pauli channels. We show that, unlike in the case of discrimination between unitary transformations, the use of entanglement with an ancillary system can strictly improve the discrimination, and any maximally entangled state allows one to achieve the optimal discrimination. We also provide a simple necessary and sufficient condition in terms of the structure of the channels for which the ultimate minimum error probability can be achieved without entanglement assistance. When such a condition is satisfied, the optimal input state is simply an eigenstate of one of the Pauli matrices.

Keywords: optimal discrimination, quantum operations, Pauli channels

1. Introduction

The concept of nonorthogonality of quantum states plays a relevant role in quantum computation and communication, cloning, and cryptography. Nonorthogonality is strongly related to the concept of distinguishability, and many measures have been introduced for comparing quantum states [1] and quantum processes [2]. Since the seminal work of Helstrom [3] on quantum hypothesis testing, the problem of discriminating nonorthogonal quantum states has received a lot of attention [4]. Not very much work, however, has been devoted to the problem of discriminating general quantum operations, apart from in the case of unitary transformations [5]. Quantum operations describe any physically allowed transformation of quantum states, including unitary evolutions of closed systems and non-unitary transformations of open quantum systems, such as systems interacting with a reservoir, or subjected to noise or measurements of any kind. The problem of discriminating quantum operations might be of great interest in quantum error correction [6], since knowing which error model is the proper one influences the choice of the coding strategy as well as the error estimation employed. Clearly, when a repeated use of the quantum operation is allowed, a full tomography can identify it. On the other hand, the minimum error discrimination approach can be useful when a restricted number of uses of the quantum operation is considered, as in quantum hypothesis testing [3].

In this paper we consider and solve the problem of discriminating with minimum error probability two given Pauli

channels. Pauli channels represent the most general unital channels for qubits (e.g. bit flip, phase flip, depolarizing channels). We show that, unlike in the case of unitary transformations [5], entanglement with an ancillary system at the input of the channel can strictly improve the discrimination. We prove that an arbitrary maximally entangled state is always an optimal input for the discrimination, and this holds true also for generalized Pauli channels in higher dimensional Hilbert spaces. However, the use of entanglement is not always needed to achieve optimality. In fact, we compare the strategies where either entangled or unentangled states are used at the input of the Pauli channels, and provide a necessary and sufficient condition in terms of the structure of the channels for which the ultimate minimum error probability can be achieved without the need for entanglement with an ancillary system. When such a condition is satisfied, the optimal input state is shown to be simply an eigenstate of one of the Pauli matrices.

The paper is organized as follows. In section 2 we briefly review the results for minimum error discrimination of two quantum states, and formulate the problem of discrimination of two quantum operations. In section 3 we consider the problem for generalized Pauli channels in the scenario where entanglement with an ancillary system is allowed at the input of the channels. We prove that in any dimension an arbitrary maximally entangled state is always an optimal input for the discrimination, and the corresponding optimal measurement is a degenerate Bell measurement. In section 4 we find the optimal strategy for minimum error discrimination of two Pauli channels without entanglement assistance. Finally, in section 5 we compare the two strategies and draw conclusions.

2. Discriminating quantum operations

In the problem of discrimination of two quantum states ρ_1 and ρ_2 , given with *a priori* probability p_1 and $p_2 = 1 - p_1$, respectively, one has to look for the two-value probability operator-valued measure (POVM) $\{P_i \geq 0, i = 1, 2\}$ with $P_1 + P_2 = I$ that minimizes the error probability:

$$p_E(P_1, P_2) = p_1 \text{Tr}[\rho_1 P_2] + p_2 \text{Tr}[\rho_2 P_1]. \quad (1)$$

We can rewrite this as

$$\begin{aligned} p_E(P_1, P_2) &= p_1 - \text{Tr}[(p_1 \rho_1 - p_2 \rho_2) P_1] \\ &= p_2 + \text{Tr}[(p_1 \rho_1 - p_2 \rho_2) P_2] \\ &= \frac{1}{2} (1 - \text{Tr}[(p_1 \rho_1 - p_2 \rho_2)(P_1 - P_2)]), \end{aligned} \quad (2)$$

where the third line can be obtained by summing and dividing the two lines above. The minimal error probability

$$p_E \equiv \min_{P_1, P_2} p_E(P_1, P_2) \quad (3)$$

can then be achieved by taking the orthogonal POVM $\{P_1, P_2\}$ made by the projectors on the support of the positive and negative parts of the Hermitian operator $p_1 \rho_1 - p_2 \rho_2$, respectively, and hence one has [3, 7]

$$p_E = \frac{1}{2} (1 - \|p_1 \rho_1 - p_2 \rho_2\|_1), \quad (4)$$

where $\|A\|_1$ denotes the trace norm of A . Equivalent expressions for the trace norm are the following [8]:

$$\|A\|_1 = \text{Tr} \sqrt{A^\dagger A} = \max_U |\text{Tr}[UA]| = \sum_n s_n(A), \quad (5)$$

where the maximum is taken over all unitary operators, and $\{s_n(A)\}$ denotes the singular values of A . In the case of equation (4), since the operator inside the norm is Hermitian, the singular values just correspond to the absolute values of the eigenvalues.

The problem of optimally discriminating two quantum operations \mathcal{E}_1 and \mathcal{E}_2 can be reformulated into the problem of finding in the input Hilbert space \mathcal{H} the state ρ such that the error probability in the discrimination of the output states $\mathcal{E}_1(\rho)$ and $\mathcal{E}_2(\rho)$ is minimal. We are interested in the possibility of exploiting entanglement with an ancillary system in order to increase the distinguishability of the output states. In this case the output states to be discriminated will be of the form $(\mathcal{E}_1 \otimes \mathcal{I}_{\mathcal{K}})\xi$ and $(\mathcal{E}_2 \otimes \mathcal{I}_{\mathcal{K}})\xi$, where the input ξ is generally a bipartite state of $\mathcal{H} \otimes \mathcal{K}$, and the quantum operations act just on the first party whereas the identity map $\mathcal{I} = \mathcal{I}_{\mathcal{K}}$ acts on the second.

In the following we will denote by p'_E the minimal error probability when a strategy without an ancillary system is adopted, and one has

$$p'_E = \frac{1}{2} \left(1 - \max_{\rho \in \mathcal{H}} \|p_1 \mathcal{E}_1(\rho) - p_2 \mathcal{E}_2(\rho)\|_1 \right). \quad (6)$$

On the other hand, by allowing the use of an ancillary system, we have

$$p_E = \frac{1}{2} \left(1 - \max_{\xi \in \mathcal{H} \otimes \mathcal{K}} \|p_1 (\mathcal{E}_1 \otimes \mathcal{I})\xi - p_2 (\mathcal{E}_2 \otimes \mathcal{I})\xi\|_1 \right). \quad (7)$$

The maximum of the trace norm in equation (7) is equivalent to the norm of complete boundedness [9] of the map $p_1 \mathcal{E}_1 - p_2 \mathcal{E}_2$, and in fact for a finite dimensional Hilbert space one can just consider $\dim(\mathcal{K}) = \dim(\mathcal{H})$ [9, 10]. Moreover, from the linearity of quantum operations and the convexity of the trace norm [8], it follows that in both equations (6) and (7) the maximum is achieved by pure states.

Of course, $p_E \leq p'_E$. In the case of discrimination between two unitary transformations [5], one has $p_E = p'_E$; that is, there is no need for entanglement with an ancillary system to achieve the ultimate minimum error probability.

3. Entanglement-assisted discrimination of generalized Pauli channels

When the quantum operations can be realized from the same set of orthogonal unitaries as random unitary transformations¹, namely

$$\mathcal{E}_i(\rho) = \sum_n q_n^{(i)} U_n \rho U_n^\dagger, \quad \sum_n q_n^{(i)} = 1, \quad (8)$$

with $\text{Tr}[U_m^\dagger U_n] = d \delta_{nm}$, and the use of an ancillary system is allowed, one can evaluate the minimum error probability as follows. Let us define $r_n = p_1 q_n^{(1)} - p_2 q_n^{(2)}$. One has

$$\begin{aligned} &\max_{|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}} \|p_1 (\mathcal{E}_1 \otimes \mathcal{I})|\psi\rangle\langle\psi| - p_2 (\mathcal{E}_2 \otimes \mathcal{I})|\psi\rangle\langle\psi|\|_1 \\ &= \max_{|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}} \left\| \sum_n r_n (U_n \otimes I) |\psi\rangle\langle\psi| (U_n^\dagger \otimes I) \right\|_1 \\ &\leq \sum_n |r_n| \max_{|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}} \|(U_n \otimes I) |\psi\rangle\langle\psi| (U_n^\dagger \otimes I)\|_1 \\ &= \sum_n |r_n|. \end{aligned} \quad (9)$$

The bound in equation (9) can be saturated by the maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |n\rangle|n\rangle \quad (10)$$

as we show in the following. Let us define

$$A \equiv \sum_n r_n (U_n \otimes I) |\Psi\rangle\langle\Psi| (U_n^\dagger \otimes I). \quad (11)$$

Notice that

$$\langle\Psi| (U_n^\dagger \otimes I) (U_m \otimes I) |\Psi\rangle = \frac{1}{d} \text{Tr}[U_m^\dagger U_n] = \delta_{nm}; \quad (12)$$

that is, A is diagonal on maximally entangled states. Then one has

$$\|A\|_1 = \text{Tr} \sqrt{A^\dagger A} = \sum_n |r_n|. \quad (13)$$

It follows that

$$p_E = \frac{1}{2} \left(1 - \sum_n |r_n| \right). \quad (14)$$

The corresponding measurement to be performed at the output of the channel is given by the projectors on the support of the

¹ This generalizes Pauli channels in arbitrary dimension. An example of such a set is operators of the form $X^k Z^l$ where X and Z act on the basis states $|0\rangle, \dots, |d-1\rangle$ as $X|j\rangle = |j \oplus 1\rangle$ and $Z|j\rangle = e^{2\pi i j/d} |j\rangle$, with \oplus denoting addition modulo d .

positive and negative parts of the operator A in equation (11); that is,

$$\begin{aligned} P_1 &= \sum_{n_+} (U_{n_+} \otimes I) |\Psi\rangle \langle \Psi| (U_{n_+}^\dagger \otimes I), \\ P_2 &= \sum_{n_-} (U_{n_-} \otimes I) |\Psi\rangle \langle \Psi| (U_{n_-}^\dagger \otimes I), \end{aligned} \quad (15)$$

where the index n_+ (n_-) is in correspondence with the positive (negative) elements of $\{r_n\}$. Notice that the set of projectors $\{(U_n \otimes I) |\Psi\rangle \langle \Psi| (U_n^\dagger \otimes I)\}$ are orthogonal maximally entangled states, and hence the measurement is a degenerate Bell measurement [11]. For the unitary invariance property of the trace norm [8], the minimal error probability can always be achieved by using an *arbitrary* maximally entangled state at the input, namely $(I \otimes V) |\Psi\rangle$ with V unitary. The corresponding optimal measurement will be $\{(I \otimes V) P_i (I \otimes V^\dagger)\}$, with $\{P_i\}$ as in equation (15).

By dropping the condition of orthogonality of the $\{U_n\}$, one just obtains the bounds

$$\frac{1}{2} \left(1 - \sum_n |r_n| \right) \leq p_E \leq \frac{1}{2} (1 - \|A\|_1), \quad (16)$$

since equation (9) gives the lower bound, whereas the upper bound is simply obtained by taking as input the maximally entangled state.

4. Discrimination of Pauli channels with no entanglement assistance

In this section we consider the case of discrimination of two Pauli channels for qubits, namely

$$\mathcal{E}^{(1)}(\rho) = \sum_{n=0}^3 q_n^{(1)} \sigma_n \rho \sigma_n, \quad \mathcal{E}^{(2)}(\rho) = \sum_{n=0}^3 q_n^{(2)} \sigma_n \rho \sigma_n, \quad (17)$$

where $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} = \{I, \sigma_x, \sigma_y, \sigma_z\}$ and $\sum_{n=0}^3 q_n^{(1)} = \sum_{n=0}^3 q_n^{(2)} = 1$. As shown in the previous section, the minimal error probability when entanglement with an ancillary system is used at the input is given by equation (14), where

$$r_n = p_1 q_n^{(1)} - p_2 q_n^{(2)}, \quad (18)$$

and p_1 and p_2 denote the *a priori* probabilities.

Here, we are interested in understanding when the entangled input strategy is really needed to achieve the optimal discrimination; hence we derive in the following the optimal strategy with no ancillary system. According to equation (6) the minimal error probability is given by

$$p'_E = \frac{1}{2} \left(1 - \max_{|\psi\rangle \in \mathcal{H}} \left\| \sum_{n=0}^3 r_n \sigma_n |\psi\rangle \langle \psi| \sigma_n \right\| \right). \quad (19)$$

By parameterizing the pure state of the qubit as

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle, \quad (20)$$

one has

$$\begin{aligned} \xi &\equiv \sum_{n=0}^3 r_n \sigma_n |\psi\rangle \langle \psi| \sigma_n \\ &= \begin{pmatrix} (r_0 + r_3) \cos^2 \frac{\theta}{2} + (r_1 + r_2) \sin^2 \frac{\theta}{2} \\ \frac{1}{2} \sin \theta [(r_0 - r_3) e^{-i\phi} + (r_1 - r_2) e^{i\phi}] \\ \frac{1}{2} \sin \theta [(r_0 - r_3) e^{i\phi} + (r_1 - r_2) e^{-i\phi}] \\ (r_0 + r_3) \sin^2 \frac{\theta}{2} + (r_1 + r_2) \cos^2 \frac{\theta}{2} \end{pmatrix}. \end{aligned} \quad (21)$$

The eigenvalues of ξ are given by

$$\begin{aligned} \lambda(\theta, \phi)_{1,2} &= \frac{1}{2} \{r_0 + r_1 + r_2 + r_3 \pm [\cos^2 \theta [r_0 + r_3 - r_1 - r_2]^2 \\ &\quad + \sin^2 \theta [(r_0 - r_3)^2 + (r_1 - r_2)^2 \\ &\quad + 2 \cos(2\phi)(r_0 - r_3)(r_1 - r_2)]^{1/2}\}. \end{aligned} \quad (22)$$

We then have

$$p'_E = \frac{1}{2} (1 - \max_{\theta, \phi} [|\lambda_1(\theta, \phi)| + |\lambda_2(\theta, \phi)|]). \quad (23)$$

Notice that the function $f(\theta, \phi) \equiv (|\lambda_1(\theta, \phi)| + |\lambda_2(\theta, \phi)|)$ can be rewritten as

$$\begin{aligned} f(\theta, \phi) &= \max\{|r_0 + r_1 + r_2 + r_3|, \{\cos^2 \theta [r_0 + r_3 - r_1 - r_2]^2 \\ &\quad + \sin^2 \theta [(r_1 - r_2)^2 + (r_0 - r_3)^2 \\ &\quad + 2 \cos(2\phi)(r_0 - r_3)(r_1 - r_2)]^{1/2}\}. \end{aligned} \quad (24)$$

The maximum over θ, ϕ in equation (23) can be found just by comparing the values of $f(\theta, \phi)$ at the stationary points, namely $\theta = k\pi$, and $\theta = \frac{(2k+1)\pi}{2}$, $\phi = l\frac{\pi}{2}$, with k, l integer. Since one has

$$\begin{aligned} &2[|\lambda_1(k\pi, \phi)| + |\lambda_2(k\pi, \phi)|] \\ &= |r_0 + r_1 + r_2 + r_3| + |r_0 + r_3 - r_1 - r_2| \\ &\quad + |r_0 + r_1 + r_2 + r_3 - |r_0 + r_3 - r_1 - r_2|| \\ &= 2(|r_0 + r_3| + |r_1 + r_2|); \end{aligned} \quad (25)$$

$$\begin{aligned} &2 \left[\left| \lambda_1 \left(\frac{(2k+1)\pi}{2}, l\pi \right) \right| + \left| \lambda_2 \left(\frac{(2k+1)\pi}{2}, l\pi \right) \right| \right] \\ &= |r_0 + r_1 + r_2 + r_3| + |r_0 - r_3 + r_1 - r_2| \\ &\quad + |r_0 + r_1 + r_2 + r_3 - |r_0 - r_3 + r_1 - r_2|| \\ &= 2(|r_0 + r_1| + |r_2 + r_3|); \end{aligned} \quad (26)$$

$$\begin{aligned} &2 \left[\left| \lambda_1 \left(\frac{(2k+1)\pi}{2}, \frac{l\pi}{2} \right) \right| + \left| \lambda_2 \left(\frac{(2k+1)\pi}{2}, \frac{l\pi}{2} \right) \right| \right] \\ &= |r_0 + r_1 + r_2 + r_3| + |r_0 - r_3 - r_1 + r_2| \\ &\quad + |r_0 + r_1 + r_2 + r_3 - |r_0 - r_3 - r_1 + r_2|| \\ &= 2(|r_0 + r_2| + |r_1 + r_3|), \end{aligned} \quad (27)$$

one finally obtains

$$p'_E = \frac{1}{2} (1 - M), \quad (28)$$

where

$$\begin{aligned} M &= \max\{|r_0 + r_3| + |r_1 + r_2|, |r_0 + r_1| + |r_2 + r_3|, \\ &\quad |r_0 + r_2| + |r_1 + r_3|\}. \end{aligned} \quad (29)$$

The three cases inside the brackets correspond to using an eigenstate of $\sigma_z, \sigma_x, \sigma_y$, respectively, as the input state of the unknown channel. The corresponding measurements to be performed at the output are of the three Pauli matrices themselves.

5. Conclusion

From comparing equations (28) and (29) with equation (14), one can see that entanglement with an ancillary system is not needed to achieve the ultimate minimal error probability in the discrimination of the two Pauli channels of equation (17) as long as $M = \sum_{n=0}^3 |r_n|$, with r_n given in equation (18). This happens when $\prod_{n=0}^3 r_n \geq 0$. Hence, entanglement assistance is necessary if and only if all $\{r_n\}$ are different from zero, with three of them with the same sign, and the remaining one with the opposite sign. Among these cases, there are striking examples where the channels can be perfectly discriminated only by means of entanglement. This is the case for two channels of the form

$$\mathcal{E}_1(\rho) = \sum_{n \neq m} q_n \sigma_n \rho \sigma_n, \quad \mathcal{E}_2(\rho) = \sigma_m \rho \sigma_m, \quad (30)$$

with $q_n \neq 0$, and arbitrary *a priori* probability. This example can be simply understood, since the entanglement-assisted strategy increases the dimension of the Hilbert space such that the two possible output states will have orthogonal support.

In conclusion, we considered the problem of discriminating two Pauli channels with minimal error probability. We showed that using maximally entangled states with an ancillary system at the input of the channel allows us to achieve the optimal discrimination, and this holds true also for generalized Pauli channels in higher dimensional Hilbert spaces. In the case of qubits, we also found the minimal error probability for the discrimination strategy with no entanglement assistance, and showed that the optimal input states are the eigenstates of one of the Pauli matrices. By comparison, we then characterized in a simple way the instances where the optimal discrimination can be achieved without the need for entanglement.

It could be interesting to look for similar conditions in the case of generalized Pauli channels in higher dimension. For this problem, one should translate the algebraic derivation in section 4 into a more geometrical picture. This could result in better physical insights into why in some cases entanglement is not required to achieve minimum error discrimination. As in the field of state discrimination, one could also study the problem of optimal unambiguous discrimination [12] of channels, where the absence of ambiguity is paid for by the possibility of getting inconclusive results from the measurement. Finally, an alternative approach is to consider the problem in the frequentistic scenario, instead of the Bayesian one, as has been recently studied for state discrimination [13]. In this case, one does not have *a priori*

probabilities and has to maximize the poorest probability of correct detection.

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