Optimal phase estimation for qubits in mixed states

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(Received 18 February 2005; published 26 October 2005)

We present the optimal phase estimation for qubits in mixed states, for an arbitrary number of qubits prepared in the same state.

DOI: 10.1103/PhysRevA.72.042327 PACS number(s): 03.67.−a, 03.65.Ta

I. INTRODUCTION

The problem of measuring the quantum phase has been a very long-standing one in quantum mechanics, since London's [1] and Dirac's first attempts [2] in the late 20's. One of the main motivations is that the estimation of the phase shift experienced by a quantum system is the only way of measuring time with high precision in quantum mechanics, since we lack a time observable. This posed the problem of quantum phase estimation naturally within the framework of frequency standards based on atomic clocks [3], and more generally, in high-precision measurements and interferometry, the typical scenario in which the sensitivity of phase estimation is profitably used.

More recently, the encoding of information into the relative phase of quantum systems has been exploited in quantum computation and communication. In fact, in quantum computing most of the existing quantum algorithms can be regarded as multiparticle interferometers, with the output of the computation encoded in the relative phase between different paths [4]. On the other hand, in some cryptographic communication protocols [e.g., the Bennett-Brassard protocol (BB84) [5]] information is encoded into phase properties.

The numerous applications above had focused a great deal of interest on the problem of optimal phase estimation, which has been widely studied in a thousand papers (see for example Ref. [6]) since the beginning of quantum theory [1,2]. The first satisfactory partial solution of the problem appeared in the late 70's (see Refs. [7,8] for reviews), and these works are generally regarded as one of the major successes of quantum estimation theory and covariant measurements, allowing a first consistent definition of phase, without the problems suffered by the original definition proposed by Dirac [2] in terms of an alleged observable conjugated to the number operator of the harmonic oscillator.

In the covariant treatment of Ref. [8], the estimated parameter is a phase shift resulting from the action of a circle group of unitary transformations, with generator a self-adjoint operator with purely integer spectrum. A generalization of this method to degenerate phase-shift generator has been presented in Ref. [9]. Such a general approach can be applied to any input pure state, along with a restricted class of mixed states, the so called *phase-pure states* [9,10].

The possibility of efficiently estimating the phase for mixed states is of fundamental interest for practical implementations, in the presence of unavoidable noise which generally turns pure states into mixed, and for estimation of local phase shift on entangled states. As a matter of fact, the freedom in the choice of the optimal measurement which results from degenerate shift operators [9] opens the problem of the stability of the quality of the estimation with increasing mixing of the shifted state. The problem of optimal phase estimation on mixed states is also very relevant conceptually, the phase being one of the most elusive quantum concepts. The main reason why the problem of optimal phase estimation on mixed states has never been addressed systematically so far is due to the intrinsic technical difficulties faced in any quantum estimation problem with mixed states. In this paper we derive an optimal measurement for phase estimation on qubits in mixed states, for an arbitrary number N of qubits prepared in the same state, using either the Uhlman fidelity or the periodicized variance as a figure of merit.

II. THEORETICAL DERIVATION

Let us consider a system of N identical qubits prepared in the same mixed state $\rho_{\vec{n}}=\frac{1}{2}(I+\vec{n}\cdot\vec{\sigma})$, where $|\vec{n}|\doteq r<1$ and σ_i are the three Pauli matrices. The total state of the N qubits is described by the density matrix $R_{\vec{n}}=\rho_{\vec{n}}^{\otimes N}$. The phase transformation U_ϕ is generated by the z component of the total angular momentum $J_z=\frac{1}{2}\sum_{k=1}^{N}\sigma_z^{(k)}$, namely

$$R_{\vec{n}}(\phi)=U_\phi R_{\vec{n}}U_\phi^\dagger=[e^{-i(\phi/2)\sigma_z}\rho_{\vec{n}}e^{i(\phi/2)\sigma_z}]^{\otimes N}. \quad (1)$$

The problem is now to estimate the unknown phase shift ϕ on the known state $R_{\vec{n}}$. We consider a covariant measurement, namely we require that the efficiency of the measurement procedure does not depend on the value of the phase to be estimated. In this case, without loss of generality we can assume that the initial state $\rho_{\vec{n}}$ has no component along σ_y, corresponding to real matrix $\rho_{\vec{n}}$ in the σ_z representation. The phase estimation problem then resorts to find the best Positive Operator Valued Measure (POVM) [7] $P(\mathrm{d}\phi)$ for determining the unknown parameter ϕ in Eq. (1). The fact that $P(\mathrm{d}\phi)$ is a POVM corresponds to the constraints

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$$P(\mathrm{d}\phi) \geq 0, \quad \int_0^{2\pi} P(\mathrm{d}\phi) = I. \tag{2}$$

In the quantum estimation approach the optimality is defined by maximizing the average of a given figure of merit $C(\phi,\phi')$, assuming a uniform prior distribution of the parameter ϕ

$$\langle C\rangle = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\int_0^{2\pi} C(\phi,\phi')\mathrm{Tr}[U_\phi R_{\vec{n}} U_\phi^\dagger P(\mathrm{d}\phi')], \tag{3}$$

where $C(\phi,\phi')=C(\phi-\phi')$. In Ref. [8] it was proved that the solution for an estimation problem covariant under a unitary group representation can be written as the group orbit under the same representation of a fixed positive operator ξ (called *seed* of the POVM), and for the present case one has

$$P(\mathrm{d}\phi') = U_{\phi'}\xi U_{\phi'}^\dagger \frac{\mathrm{d}\phi'}{2\pi}. \tag{4}$$

In the following we will denote by $|m,a\rangle$ an orthonormal basis, with m denoting the eigenvalues of $\frac{1}{2}J_z$, which label the equivalence classes of irreducible representations of the group $\{U_\phi\}$, while a is a degeneration index, corresponding to the multiplicity space of the representation m. The normalization condition (2) for the POVM $P(\mathrm{d}\phi)$ implies that $\langle m,a|\xi|m,b\rangle=0$ for $a\neq b$, whereas $\langle m,a|\xi|m,a\rangle=1$ for all a.

In quantum estimation theory the quantity to be minimized can be always written in the form of the expectation of a cost operator as follows:

$$\langle C\rangle = \int_0^{2\pi}\frac{\mathrm{d}\phi'}{2\pi}C(\phi')\mathrm{Tr}[R_{\vec{n}}U_{\phi'}\xi U_{\phi'}^\dagger]. \tag{5}$$

The choice of the cost function $C(\phi)$ depends on the estimation criterion. The most commonly adopted criteria are the periodicized variance

$$C(\phi) \equiv v(\phi) = 4\sin^2\frac{\phi}{2} = 2(1-\cos\phi), \tag{6}$$

or the (opposite of the) fidelity between the true and the estimated states, which for mixed states has the well-known Uhlman's form [11]

$$C(\phi) \equiv 1-F(\phi) = 1-[\mathrm{Tr}\sqrt{\sqrt{U_\phi\rho_{\vec{n}}U_\phi^\dagger}\rho_{\vec{n}}\sqrt{U_\phi\rho_{\vec{n}}U_\phi^\dagger}}]^2, \tag{7}$$

which for qubits simplifies as follows [12]:

$$1-F(\phi) = \frac{r^2}{2}(1-\cos\phi). \tag{8}$$

Both cost functions depend on ϕ only through its cosine; therefore, we need to maximize the averaged $\cos\phi$, namely

$$\langle c\rangle = \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}\phi'}{2\pi}(e^{i\phi'}+e^{-i\phi'})\mathrm{Tr}[R_{\vec{n}}U_{\phi'}\xi U_{\phi'}^\dagger]. \tag{9}$$

The evaluation of the integral in Eq. (9) leads to the following expression:

$$\langle c\rangle = \mathrm{Re}\sum_{m,a,b}\langle m,a|\xi|m+1,b\rangle\langle m+1,b|R_{\vec{n}}|m,a\rangle. \tag{10}$$

We now decompose $\rho^{\otimes N}$ into irreducible representations of SU(2), as shown in Ref. [13], recasting $R_{\vec{n}}$ into invariant block-diagonal form on the orthonormal basis $|j,m,\alpha\rangle_{\vec{b}} = U_{j,\alpha}|j,m,1\rangle_{\vec{b}}$ for the minimal invariant subspaces of the SU(2) representations, with $\vec{b}=\vec{n}/r$, and $U_{j,\alpha}$ denoting a suitable set of unitary operators

$$R_{\vec{n}} \doteq \rho_{\vec{n}}^{\otimes N} = \sum_{j=\langle\langle N/2\rangle\rangle}^{J}(r_+r_-)^J\sum_{\alpha=1}^{d_j}U_{j,\alpha}\tau_{j,1}U_{j,\alpha}^\dagger, \tag{11}$$

$$\tau_{j,1} = \sum_{m=-j}^{j}\left(\frac{r_+}{r_-}\right)^m|j,m,1\rangle_{\vec{b}}\langle j,m,1|, \tag{12}$$

$$|j,m,1\rangle_{\vec{b}} = |j,m\rangle_{\vec{b}}\otimes|\Psi_-\rangle^{\otimes J-j}, \tag{13}$$

where $r_\pm \doteq \frac{1}{2}(1\pm r)$, $\langle\langle x\rangle\rangle$ denotes the fractional part of x (i.e., $\langle\langle N/2\rangle\rangle=0$ for N even and $\langle\langle N/2\rangle\rangle=1/2$ for N odd), $J=N/2$, and d_j is the multiplicity of the jth irreducible representation of SU(2)

$$d_j = \binom{2J}{J-j} - \binom{2J}{J-j-1}, \tag{14}$$

whereas $|\Psi_-\rangle$ denotes the singlet state. This decomposition is useful since m,j,α label also the irreducible representations of $\{U_\phi\}$, m being the eigenvalue of J_z, and j,α becoming both degeneration indices. The block-diagonal form of $R_{\vec{n}}$ shows that the only coupling produced by the phase shift between irreducible components with m and $m+1$ can occur only between vectors in the same invariant subspace j,α of SU(2). Upon recasting $R_{\vec{n}}$ in the form of Eq. (11), the value of $\langle c\rangle$ in Eq. (10) involves only the following terms:

$$\langle c\rangle = \mathrm{Re}\sum_{m,j\alpha}\langle m,j\alpha|\xi|m+1,j\alpha\rangle\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle, \tag{15}$$

where we used the short notation $|m,j\alpha\rangle \doteq |j,m,\alpha\rangle_z$, since the subspaces j,α are invariant under any unitary in SU(2), and $|j,m,\alpha\rangle_{\vec{b}}=T^{(j)}(g)|j,m,\alpha\rangle_z$ for some $g\in$ SU(2). Now, the following bounding holds:

$$\begin{aligned}\langle c\rangle &\leq \left|\sum_{m,j\alpha}\langle m,j\alpha|\xi|m+1,j\alpha\rangle\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle\right| \\ &\leq \sum_{m,j\alpha}|\langle m,j\alpha|\xi|m+1,j\alpha\rangle\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle| \\ &\leq \sum_{m,j\alpha}|\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle|,\end{aligned} \tag{16}$$

where the last bound follows from positivity of ξ. We show now that all bounds can be achieved by a suitable choice of the operator ξ compatible with constraints (2). The first two bounds can indeed be achieved by choosing the phases of the matrix elements $\langle m,j,\alpha|\xi|m+1,j\alpha\rangle$ in such a way that they compensate the corresponding phases of $\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle$. The last bound is achieved by just taking the moduli of the matrix elements $\langle m,j\alpha|\xi|m+1,j\alpha\rangle$ to be 1. It remains to prove that these choices are compatible with positivity. In order to show this, let us write

$$\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle = |\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle| e^{i\chi(m+1,m,j\alpha)}. \tag{17}$$

Since only the elements on the first overdiagonal and underdiagonal are involved, one can write the phases $\chi(m+1,m,j\alpha)$ as the difference of two functions as follows:

$$\chi(m+1,m,j\alpha) = \gamma(m,j\alpha) - \gamma(m+1,j\alpha), \tag{18}$$

as the number of independent linear equations in Eq. (18) is 2^N-1 while the unknown phases are 2^N. Then, one can take

$$\xi = \sum_{j,\alpha} |e(j,\alpha)\rangle\langle e(j,\alpha)|, \tag{19}$$

where $|e(j,\alpha)\rangle$ is the generalized Susskind-Glogower vector

$$|e(j,\alpha)\rangle = \sum_{m=-j}^{j} e^{i\gamma(m,j\alpha)}|m,j,\alpha\rangle. \tag{20}$$

It is immediately clear that Eq. (19) represents a positive operator and by construction ξ produces a normalized POVM, while achieving the bounding in Eq. (16).

Specifically, for a collection of identically prepared mixed initial states, we have

$$\langle c\rangle = \sum_{m,j,\alpha} |\langle m+1,j\alpha|R_{\vec{n}}|m,j\alpha\rangle| = \sum_{m,j,\alpha} (r_+r_-)^J \times \left| \sum_n \left(\frac{r_+}{r_-}\right)^n \langle j,m+1,\alpha|j,n,\alpha\rangle_{\vec{b}}\langle j,n,\alpha|j,m,\alpha\rangle \right| = \sum_{j=\langle\langle N/2\rangle\rangle}^{J} \sum_{m=-j}^{j} d_j (r_+r_-)^J \left| \sum_n \left(\frac{r_+}{r_-}\right)^n T^{(j)}(g_{\vec{b}})_{m+1,n} T^{(j)}(g_{\vec{b}})^\dagger_{n,m} \right|. \tag{21}$$

Notice that, since we assumed that $\vec{n}$ has no component along the direction y, then $g_{\vec{b}}$ is just the rotation around the axis y connecting the oriented z axis with $\vec{b}$, namely $T(g_{\vec{b}})=e^{i\theta J_y}$ for some θ, with $J_y=\frac{1}{2}\sum_{k=1}^{N}\sigma_y^{(k)}$.

III. NUMERICAL RESULTS

The expression for the Wigner matrix elements $T^{(j)}(g_{\vec{b}})_{lk}$ is given by [14]

$$T^{(j)}(g_{\vec{b}})_{lk} = \sum_t (-1)^t \frac{\sqrt{(j+l)!(j-l)!(j+k)!(j-k)!}}{(j+l-t)!(j-k-t)!(t-l+k)!t!} \times \cos^{2j+l-k}\frac{\theta}{2}\sin^{2t-l+k}\frac{\theta}{2}. \tag{22}$$

The explicit expression of Eq. (21) is very lengthy, and has been evaluated using symbolic calculus for J up to 21/2, namely for a total number of spins equal to 21. The plot of the averaged cosine $\langle c\rangle$ as a function of θ and r is represented in Fig. 1 and exhibits two interesting intuitive features. The first is that the maximum versus θ occurs for $\theta=\pi/2$, namely for qubits lying in the equatorial plane. The second is the improving figure of merit versus the purity r. Equatorial pure qubits are optimal for phase detection; however, the figure of merit is quite stable around its maxima, still with $N=10$ copies.

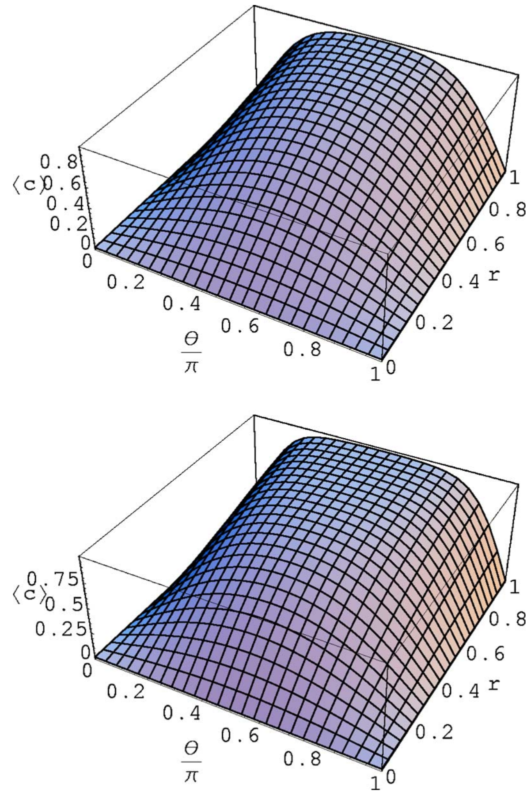

FIG. 1. (Color online) The plot of $\langle c\rangle$ as a function of θ and r. The plots correspond to systems of 10 qubits and 20 qubits, respectively.

Figure 2 shows the averaged cosine $\langle c\rangle$ versus the number of qubits N for equatorial states. Numerically, for $N\to\infty$ we find the asymptotic behavior $2(1-\langle c\rangle)\propto N^{-1}$. More precisely, for the Uhlman fidelity F in Eq. (7) we find an asymptotic behavior saturating the Cramer-Rao lower bound [15]. This gives a strict lower bound for variance $\Delta\phi^2$ valid for any estimate. For independent copies, one has [7]

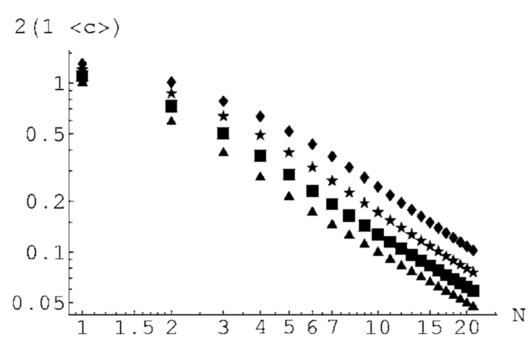

FIG. 2. The logarithmic plots represent $2(1-\langle c\rangle)$, where $\langle c\rangle$ is the averaged cosine, as a function of the number of spins N, for $\theta=\pi/2$ and for the following values of r: ◆ $r=0.7$, ★ $r=0.8$, ■ $r=0.9$, ▲ $r=1$.

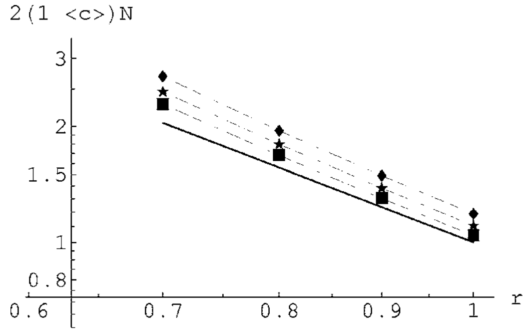

FIG. 3. The logarithmic plot of $2N(1-\langle c\rangle)$ vs r, for $N=16,18,20$ and $\theta=0$. The line on the bottom represents the bound given by the Cramer-Rao inequality, namely $1/r^2$.

$$\Delta\phi^2 \geqslant \frac{1}{N}\mathrm{Tr}[(\partial\rho/\partial\phi)\mathcal{L}]^{-1}, \tag{23}$$

where for each ϕ the operator $\mathcal{L}$ is defined by the identity

$$\partial\rho/\partial\phi \doteq \frac{1}{2}(\rho\mathcal{L}+\mathcal{L}\rho). \tag{24}$$

Notice that the bound holds for *any* estimate, regardless of the nature of the measurement (corresponding to either joint or separable POVMs). Since the estimation is covariant, we can just consider $\phi=0$. A simple evaluation shows that $\mathcal{L}=r\cos\theta\sigma_y$, and the bound is then given by $(1/N)\mathrm{Tr}[\rho_0\mathcal{L}^2]^{-1}=1/(Nr^2\cos^2\theta)$, namely

$$\Delta\phi^2 \geqslant \frac{1}{N}\frac{1}{r^2\cos^2\theta}. \tag{25}$$

For small $\Delta\phi^2$, using the Taylor expansion of the cosine one has $\Delta\phi^2\simeq 2(1-\langle c\rangle)$. In Fig. 3 we plot $2(1-\langle c\rangle)N$ of our optimal estimation for $\theta=0$ versus r for $N=16,18,20$, against the Cramer-Rao bound $1/r^2$. From the comparison we see that our estimation approaches the Cramer-Rao bound for large N. Notice that, according to recent studies of theoretical statistics [16], there should exist a separable strategy (such as an adaptive scheme) which is not necessarily covariant; nevertheless, it would be able to achieve the same Cramer-Rao bound asymptotically: such noncovariant schemes, e.g., homodyne-based estimation of the phase, will be the subject of further studies.

IV. CONCLUSIONS

In conclusion, we have presented an optimal measurement for phase estimation on N qubits all prepared in the same arbitrary mixed state. The Uhlman fidelity saturates the Cramer-Rao bound for this problem, confirming the optimality of the measurement. An optimal estimation is achieved for equatorial qubits and generally the fidelity is improving with purity. The specific form of the optimal POVM in terms of the generalized Susskind-Glogower vector in Eqs. (20) and (13) suggests possible physical implementations in terms of a generalized multipartite Bell measurement.

ACKNOWLEDGMENTS

We are grateful to R. Gill for pointing us the asymptotic behavior in terms of the Fisher information. This work has been cofounded by the EC under the program ATESIT (Contract No. IST-2000-29681) and the MIUR cofinanziamento 2003. P.P. acknowledges support from the INFM under Project PRA-2002-CLON. G.M.D. acknowledges partial support by the MURI program administered by the U.S. Army Research Office under Grant No. DAAD19-00-1-0177.

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