

Entanglement production by quantum error correction in the presence of correlated environment

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Abstract. – We analyze the effect of a quantum error correcting code on the entanglement of encoded logical qubits in the presence of a dephasing interaction with a correlated environment. Such correlated reservoir introduces entanglement between physical qubits. We show that for short times the quantum error correction interprets such entanglement as errors and suppresses it. However, for longer time, although quantum error correction is no longer able to correct errors, it enhances the rate of entanglement production due to the interaction with the environment.

Quantum error correction (QEC) [1,2] has been introduced to perform quantum information processing [3] in the presence of noise due to the interaction between quantum bits and environment. The basic idea is to encode each logical qubit in a redundant way on a set of physical qubits and to periodically acquire information on the errors that affected the system but not on the quantum state of the system itself. Such techniques have been developed to deal with independent errors on individual physical qubits due to the interaction of each physical qubit with its own reservoir. In several physical situations, however, the presence of correlated reservoirs [4] can result in non-trivial effects. For example, it has been shown that the interaction of two subsystems with a finite-temperature common bath of harmonic oscillators can, for short times, induce entanglement between the two subsystems initially in a product state. This is possible when the environment has some spatial correlations [5], as often occurs in solid-state physics, leading to an effective interaction between the two subsystems. This effect is also present for noisy baths in the Markovian regime [6]. The dynamics of the entanglement rate in the presence of decoherence was also studied [7].

Quantum error correction has been analyzed in the presence of correlated environments in ref. [8]. In the present work, we address the issue of the effects of QEC on the entanglement

between *logical* qubits. It is reasonable to expect that the entanglement induced by the correlated bath between physical qubits will modify the encoded state in a way that is interpreted by the QEC procedure as error and therefore corrected. However, when such entanglement becomes sufficiently large, the protocol may be not able to correct it. It is therefore interesting to study how entanglement is modified by the application of QEC. In the following we will show that, although QEC is unable to correct such errors, it can enhance the generation of entanglement in a pair of *logical* qubits with respect to the entanglement induced by the environment on a pair of *physical* qubits.

The model we consider in this work, the same as in ref. [4], consists of a register of quantum bits interacting with a common environment, modelled as a bath of harmonic oscillators. The bath-qubit interaction is described by the following Hamiltonian:

$$H = \sum_j \sigma_z^j \xi_j(t), \quad (1)$$

where $\xi_j(t) = \sum_{m,\omega} [\lambda_{j,m}(\omega)a_{m,\omega} + hc]$. In the previous expression $\lambda_{j,m}(\omega)$ denote the coupling constants between the j -th qubit and the oscillator at frequency ω in the m -th bath with corresponding annihilation (creation) operator $a_{m,\omega}$ ($a_{m,\omega}^\dagger$).

In the following we will concentrate our interest on the register dynamics, *i.e.* on the reduced density operator $\rho(t) = \text{Tr}_E[U(t)\rho(0)U^\dagger(t)]$, where Tr_E denotes the partial trace performed on the environment degrees of freedom. The resulting density matrix can be written in a compact form as [8]

$$\rho(t) = \$[U_r \rho(0) U_r^\dagger], \quad (2)$$

where $\$$ is a map (a super-operator) defined as

$$\$ = \exp \left[-\frac{1}{2} \sum_{j,j'} \Gamma_{jj'} (\sigma_z^j - \bar{\sigma}_z^j) (\sigma_z^{j'} - \bar{\sigma}_z^{j'}) \right]. \quad (3)$$

In the above expression we used the convention that a bar over an operator means that it acts on the density matrix from the left and $\Gamma_{jj'}(t) = \langle \phi_j(t) \phi_{j'}(t) \rangle$, with $\phi_j(t) = \int_0^t \xi_j(t') dt'$, involves correlations of the bath at different times. Finally, the unitary evolution U_r is given by

$$U_r = \exp \left[i \sum_{jj'} V_{jj'} t \sigma_z^j \sigma_z^{j'} \right], \quad (4)$$

where the quantity $V_{jj'} = 2 \text{Re} \sum_{m,\omega} \lambda_{m,j}(\omega) \lambda_{m,j'}^*(\omega) / \omega$ is non-zero only if the same reservoir is coupled to different qubits. As we will see in the following, the unitary operator U_r is responsible for the creation of entanglement, while the super-operator $\$$ describes the dephasing of the off-diagonal elements and is present also without spatial correlations of the environment.

As a measure of entanglement between two qubits we will use the concurrence defined by Wootters [9]. If ρ is the density matrix of the global system of two qubits, let us define $\tilde{\rho} \doteq \sigma^y \otimes \sigma^y \rho^* \sigma^y \otimes \sigma^y$ and $R = \rho \tilde{\rho}$. The concurrence is then defined as $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i are the square roots of the eigenvalues of R labelled in decreasing order. Using this definition we obtain for Bell states $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle) / \sqrt{2}$ the result $C(t) = e^{-4(\Gamma_{11} + 2\Gamma_{12} + \Gamma_{22})}$, while for $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle) / \sqrt{2}$ the concurrence decays as $C(t) = e^{-4(\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})}$. It is easy to see that if the environment has special symmetries, for instance if Γ_{ij} is equal for different i and j , then the subspace spanned by $|\psi^\pm\rangle$ is decoherence free [4, 10]. Note that the typical decoherence times are of the order of $dt/d\Gamma_i$. If we choose $|ij\rangle_x$, eigenstates of σ_x as initial states,

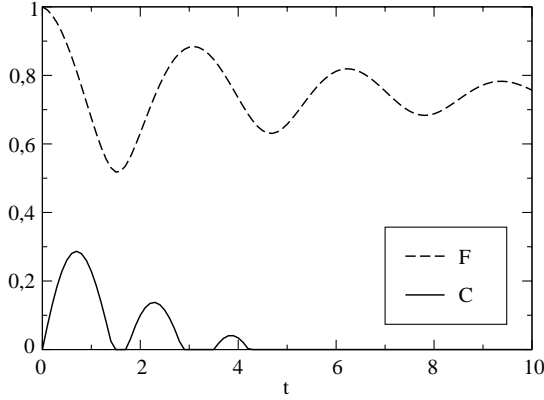


Fig. 1 – Time evolution of fidelity and concurrence between physical qubits, averaged over initial states, in the absence of QEC. The time scale is chosen so that $V = 1$ and $\Gamma(t) \simeq 0.1t$.

it is possible to see that the qubits become entangled if $V_{12} > d\Gamma_i/dt$. In fact the entanglement oscillates with a frequency $4V_{12}$ and is damped with a rate proportional to $d\Gamma_i/dt$ (see fig. 1).

Let us now introduce a QEC protocol. In the following we will encode each logical qubit into three physical qubits as follows: $|\tilde{0}\rangle \rightarrow |000\rangle_x$ and $|\tilde{1}\rangle \rightarrow |111\rangle_x$ where $|0\rangle_x$ and $|1\rangle_x$ are eigenstates of σ_x . This code will protect the logical qubits against one single phase error on individual qubits [1]. When VT , $\Gamma(T) \ll 1$, where T is the time interval between two quantum error corrections, following the same approach of [8] we can write the following master equation for the continuous evolution of the density matrix of the two logical qubits:

$$\frac{d\rho}{dt} = -(\gamma_1 + \gamma_2)\rho + \gamma_1\sigma_x^1\rho\sigma_x^1 + \gamma_2\sigma_x^2\rho\sigma_x^2, \quad (5)$$

where

$$\gamma_i = \frac{2}{T} \sum_{j>j'} (T^2 V_{jj'}^2 + 2\Gamma_{jj'}^2(T) + \Gamma_{jj}(T)\Gamma_{j'j'}(T)), \quad (6)$$

as in ref. [8]. In the above equation we have to take $j = 1, 2, 3$ for $i = 1$ and $j = 4, 5, 6$ for $i = 2$. The rates $\gamma_i \ll d\Gamma_{ij}/dt$ are the same found in [8] for the single-qubit case.

Let us now turn our attention to the time evolution of the entanglement between logical qubits in the presence of QEC in the regime in which the above master equation holds. As a first example, we consider the initial state $|\tilde{0}\tilde{0}\rangle$. In this case, the only non-vanishing elements at time t are those in the main diagonal, and they decay exponentially with decay constants equal to γ_1, γ_2 and $\gamma_1 + \gamma_2$. If instead we start with a Bell state $|\phi^+\rangle$ of logical qubits, the non-zero elements at time t are those on the two diagonals. Such state evolves towards an incoherent superposition of two Bell states whose concurrence decays as $C(t) = e^{-2(\gamma_1 + \gamma_2)t}$ and the fidelity with respect to the initial state is $F(t) = \frac{1}{2}(1 + e^{-2(\gamma_1 + \gamma_2)t})$, where, for an initially pure state $|\psi(t=0)\rangle$, the fidelity is defined as $F(t) = \langle\psi(0)|\rho(t)|\psi(0)\rangle$. From these results it is evident that the fidelity and the concurrence, in the presence of QEC, decay with the same rate. It is interesting also to see what happens to the state $|\tilde{i}\tilde{j}\rangle_x$. In this case, the density operator does not evolve: $\rho(t) = \rho(0)$. This does not follow from the decoherence free subspace phenomenon but rather from the fact that QEC freezes the unitary evolution by means of repeated measurement, as in the quantum Zeno effect [11], and conditional dynamics depending on the measurement outcome.

In order to obtain information about the global properties of the map, we will consider states with fixed initial concurrence, which are of the form

$$|\Psi\rangle = \cos\vartheta |\tilde{0}_{n1}\tilde{0}_{n2}\rangle + \sin\vartheta |\tilde{1}_{n1}\tilde{1}_{n2}\rangle \quad (7)$$

with

$$|\tilde{0}_{ni}\rangle = \cos\frac{\vartheta_i}{2} |\tilde{0}\rangle + \sin\frac{\vartheta_i}{2} e^{i\varphi_i} |\tilde{1}\rangle, \quad (8)$$

$$|\tilde{1}_{ni}\rangle = \cos\frac{\vartheta_i}{2} |\tilde{1}\rangle - \sin\frac{\vartheta_i}{2} e^{-i\varphi_i} |\tilde{0}\rangle. \quad (9)$$

The concurrence of $|\Psi\rangle$ is $\sin 2\vartheta$. In the following analysis we will average over $\vartheta_1, \vartheta_2, \varphi_1, \varphi_2$. We can see that averaging over all initial product states $\vartheta = 0, \pi/2$, one finds that the concurrence is always zero and the fidelity decays with a rate proportional to γ_i . For partially or maximally entangled states, the entanglement decays monotonically again with decay constants $\gamma_1 + \gamma_2$. This means that QEC suppresses the effective interaction between logical qubits due to the presence of a correlated environment. In other words, the bath does not induce entanglement between logical qubits and the entanglement initially presents decays. This implies that for small T there is not creation of entanglement since the QEC protocol destroys all the correlations between physical qubits of different logical qubits. The reason for this can be seen in a qualitative way: for a state $|\tilde{0}\tilde{0}\rangle = |000\rangle_x |000\rangle_x$ the operator $U_r = \mathbf{1} + i \sum_{ij} V_{ij} t \sigma_z^i \sigma_z^j$ up to first order creates superpositions like $|000\rangle_x |000\rangle_x + |110\rangle_x |000\rangle_x + \dots$ which is entangled. After the corrections, one gets a mixture of $|000\rangle_x |000\rangle_x$ and $|111\rangle_x |000\rangle_x$ which is no longer entangled. Not surprisingly then, QEC inhibits the production of entanglement as this is seen by the protocol as an error.

This may no longer be true if the time T is comparable to the period of oscillation of entanglement without QEC. In this case, the environment may have time to create enough entanglement to be interpreted as a property of the initial state and amplified by QEC. If the time T between two corrections is not short, the approximations which lead to the master equation (5) are no longer valid and we must use the exact map which links $\rho(t)$ and $\rho(t+T)$ after the free time evolution and the QEC. In terms of Kraus operators such map can be written as

$$\rho_C(t+T) = \sum_k M_k \rho(t) M_k^\dagger, \quad (10)$$

where $\sum_k M_k^\dagger M_k = \mathbf{1}$. In our case we found the following seven Kraus operators:

$$\begin{aligned} M_0 &= m_0 \mathbf{1}, \\ M_1 &= m_1 \sigma_{x1}, \\ M_2 &= m_2 \sigma_{x2}, \\ M_3 &= m_3 \sigma_{x1} \sigma_{x2}, \\ M_4 &= m_4 (\sigma_{x1} + \sigma_{x2}), \\ M_5 &= m_5 (\mathbf{1} + \sigma_{x1} \sigma_{x2}), \\ M_6 &= |m_6| (\mathbf{1} \pm i \sigma_{x1} \sigma_{x2}). \end{aligned} \quad (11)$$

Such Kraus operators are linked to the $\Gamma_{ij}(T)$ rates, however the expressions which explicit such dependence are lengthy and not of immediate reading. A more straightforward physical picture of the action of the map is instead gained by considering the plot of the coefficients

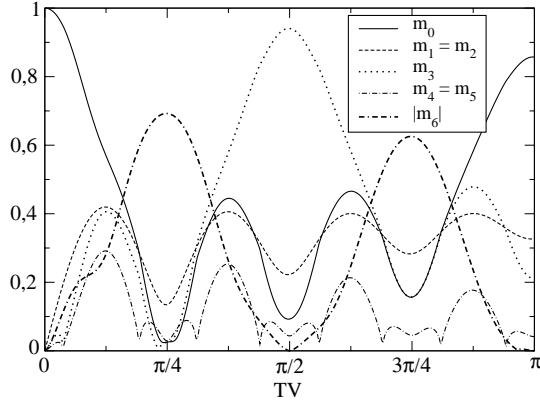


Fig. 2 – Relative weights of the Kraus operators, *i.e.* the quantities m_i , as functions of T . For simplicity, we assume $\Gamma_{ij}(T) = \Gamma(T) \simeq 0.1T$. From this it follows that $m_1 = m_2$.

m_i as functions of T (see fig. 2). We supposed that $\Gamma_{ij}(T) = \Gamma(T) \simeq 0.1T$ from which it follows that $m_1 = m_2$. Notice that in general $m_4 = m_5$ and that the sign in M_6 is plus or minus depending on the sign of $\text{Im}(m_6)$. Figure 2 shows clearly various regimes for the evolution of the logical qubits. Indeed for $VT \ll 1$ only m_0 and m_1 are non-zero, leading to the dissipative master equation (5). As we have seen, in this regime there is no creation of entanglement because $M_{1,2}$ are single-qubit operators. When T is close to the value $\pi/8V$ there are also contributions from the two-qubit operators $M_{4,5,6}$, but the dissipative effect of $M_{1,2,3}$ is predominant and again we verified that the entanglement production is zero. The interesting regime is for $T \simeq \frac{\pi}{4V}$, when the main contribution comes from M_6 . As expected in this case there is creation of entanglement that oscillates between zero and a maximum value (which will gradually decay). Indeed for the particular choice $T = \frac{\pi}{4V}$ (see fig. 3) the created entanglement is maximum and the QEC amplifies it. The other non-zero dissipative contribution is M_1 and it is present only if Γ_i is non-zero. On the other hand, if Γ_i is negligible, M_6 is

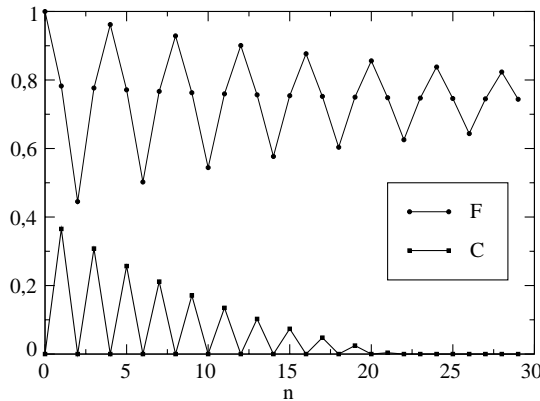


Fig. 3 – Mean fidelity and concurrence between logical qubits in the presence of QEC as functions of the number of applications of the map, for $T = \pi/4V$, $V = 1$ and $\Gamma(t) \simeq 0.1t$, the same parameters used in fig. 1.

the only Kraus operator, and therefore the evolution is unitary even in the presence of QEC. Under this condition this evolution induces the transformation $|\tilde{0}\rangle \rightarrow |\tilde{1}\rangle$ and vice versa. For larger T one again finds regimes with no entanglement production.

When the map (10) is applied (for $\Gamma_i = 0$ and $T = \frac{\pi}{4V}$) to the state $|\tilde{0}\tilde{0}\rangle$, then one sees that it makes a sort of periodic oscillation:

$$|\tilde{0}\tilde{0}\rangle \longrightarrow |\tilde{0}\tilde{0}\rangle + i|\tilde{1}\tilde{1}\rangle \longrightarrow |\tilde{1}\tilde{1}\rangle \longrightarrow |\tilde{0}\tilde{0}\rangle - i|\tilde{1}\tilde{1}\rangle \longrightarrow |\tilde{0}\tilde{0}\rangle. \quad (12)$$

Of course, if Γ is not zero then the other Kraus operators also contribute and so this oscillation is damped. This type of oscillation is not restricted to initial product states but also to partially entangled states. As a consequence, it is possible to create entanglement that can be distilled to obtain maximally entangled states or even extracted from the system.

Figure 3 shows that the maximum average concurrence achievable is around 0.4. It is important to underline that this is an average value: there are states, like $|\tilde{0}\tilde{0}\rangle_x$, that do not evolve and so there is no production of entanglement, while there are states for which the created entanglement is more than the average value. This is the case of initial product states of the computational basis for which the production of entanglement is maximum. For example, let us compare the initial states that lead to the maximum production of entanglement with and without QEC. In the presence of QEC the best case is $|\tilde{0}\tilde{0}\rangle$: the entanglement is 0.92 after just one application of QEC. On the other hand, the maximum entanglement reached in the case without QEC is only 0.74 for the state $|00\rangle_x$. These results are insensitive to the change $0 \rightarrow 1$.

In summary, we studied the effect of QEC on the entanglement between logical qubits in the presence of correlated noise. We found that when the time interval T between consecutive applications of QEC is small, QEC is helpful in preserving the state of the qubits and indeed entanglement decays with a smaller rate. As expected, in the absence of QEC there is production of entanglement between physical qubits, due to the correlations introduced by the environment, while in the presence of QEC this phenomenon is inhibited. The interesting new effect we have discovered is that when T is not small then entanglement may be generated with a bigger rate than in the absence of QEC. In this scenario we are in a situation in which the entanglement induced by the unitary dynamics generated by the correlated bath is enhanced by the QEC protocol which prevents the dephasing effects of the coupling with the bath. Note that such enhancement in the production rate of entanglement is achieved by means of local measurements and conditional local unitary operations on the logical qubits, in other words the entanglement is not induced by joint measurements on the pair of logical qubits.

Our results suggest a new strategy to enhance the rate of entanglement production for interacting qubits in the presence of decoherence in a more general scenario. Consider a system of interacting subsystems in the presence of a decohering dynamics. Such situation might be due either to a coupling of the subsystem to a common reservoir, as in our case considered above, or to a direct mutual coupling between subsystems in the presence of independent environments. The entanglement induced between subsystems by the unitary dynamics is destroyed by the dissipative one. By enlarging the Hilbert space, in our case by increasing the number of qubits, it is possible to identify separate larger subsystems such that by means of local measurements and local conditional unitary operations on such subsystems the amount of entanglement due to the mutual interaction is increased with respect to the entanglement that would have been generated between the original subsystems. In other words, we conjecture that the strategy outlined above is not restricted to QEC but is a more general technique. The issue is, given an effective interaction in the presence of decoherence, to find optimal subsystems and projection protocols in order to maximize the production of entanglement.

In some sense our protocol can be described as a generalization of the one proposed in ref. [12]. In ref. [12] the entanglement production is optimized, in the presence of a direct interaction and in the absence of decoherence, by means of local operations and ancillas. In our QEC protocol, we use some sort of ancillary system to enlarge the Hilbert space, although in this case there is not a sharp distinction between qubits and ancillas. Furthermore, we make use also of local projections on the enlarged subsystems and conditional dynamics.

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