International Journal of Quantum Information
Vol. 3, No. 1 (2005) 207–211

QUANTUM ERROR CORRECTION DRIVEN ENTANGLEMENT DYNAMICS IN THE PRESENCE OF CORRELATED NOISE

GABRIELE DE CHIARA and ROSARIO FAZIO
NEST-INFM & Scuola Normale Superiore, piazza dei Cavalieri 7, I-56126 Pisa, Italy

CHIARA MACCHIAVELLO
INFM & Dipartimento di Fisica A. Volta,
Universita' degli Studi di Pavia, Via Bassi 6, I-27100 Pavia, Italy

G. MASSIMO PALMA
NEST-INFM & Dipartimento di Tecnologie dell'Informazione,
Universita' degli Studi di Milano, via Bramante 65, I-26013 Crema (CR), Italy

Received 27 July 2004

The effects of quantum error correction (QEC) on the dynamics of entanglement between logical qubits in the presence of a dephasing interaction with a correlated environment is investigated. Such a correlated reservoir introduces entanglement between physical qubits which, for short times, is interpreted as error and suppressed by the QEC routine. However for longer times, although QEC is no longer able to correct errors, it enhances the rate of entanglement production due to the interaction with the environment.

Keywords: Entanglement; quantum error correction; correlated quantum noise.

1. Introduction

The dynamics of a multipartite quantum system interacting with a correlated reservoir has features which differ strongly from the ones characteristic of the dynamics of composite open quantum systems interacting with independent local reservoirs.[1] For instance, depending on the symmetries of the interaction Hamiltonian, decoherence free subspaces may appear,[1,2] or instead, for some particular initial conditions, the decoherence processes may be enhanced. Furthermore, when separate qubits interact with a common reservoir they can become entangled.[3,4] Such features open the question of the effects of quantum error correction (QEC) procedures, which have been introduced to stabilize quantum information in the presence of independent noise acting on separate physical qubits,[5–7] when the qubits interact with correlated noise. This problem has been investigated in Ref. 8 for the case of collective phase noise. Here we will look at a related problem, namely the effects of QEC on the entanglement between logical qubits in the presence of correlated noise. Physical intuition suggests that the entanglement induced by the correlated bath between physical qubits modifies the encoded state in a way that is interpreted by the QEC procedure as error, and therefore corrected. However, when the induced entanglement becomes sufficiently large, the protocol may not be able to correct it. In the following we will show that, although in this regime QEC is unable to correct such errors, it can enhance the generation of entanglement in a pair of *logical* qubits with respect to the entanglement induced by the environment on a pair of *physical* qubits.[9]

2. Entanglement between two *Physical* Qubits in the Presence of a Correlated Reservoir

The model we consider is the same as in Ref. 1 and consists of a register of quantum bits interacting with a common environment, modeled as a bath of harmonic oscillators. The bath–qubit interaction is described by the following Hamiltonian:

$$H = \sum_j \sigma_z^j \xi_j(t), \tag{1}$$

where $\xi_j(t) = \sum_{m,\omega}[\lambda_{j,m}(\omega)a_{m,\omega} + \text{h.c.}]$. In the previous expression $\lambda_{j,m}(\omega)$ denotes the coupling constants between the jth qubit and the oscillator at frequency ω in the mth bath with corresponding annihilation (creation) operator $a_{m,\omega}(a^\dagger_{m,\omega})$.

When the reduced dynamics of the qubits, register is studied it emerges that the effect of the coupling with the environment can be decomposed in terms of a dissipative map acting on the qubits, which evolve unitarily under the action of an effective Hamiltonian. The effect of the effective Hamiltonian, obtained after eliminating the bath degrees of freedom, is to couple the register qubits and, according to the initial conditions, to induce entanglement between them. The dissipative map, on the other hand is responsible of the decoherence process.

Figure 1 shows the time evolution, averaged over the initial condition, of the fidelity and of the entanglement of two qubits under such an evolution. The fidelity is defined as usual as $F(t) = \langle\psi(0)|\varrho(t)|\psi(0)\rangle$ where $|\psi(0)\rangle$ is the initial state of the

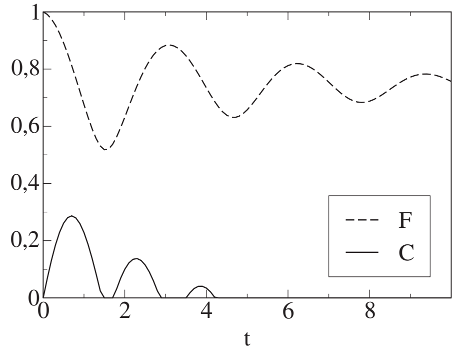

Fig. 1. Time evolution of fidelity and concurrence between physical qubits, averaged over initial states, in the absence of QEC for a particular choice of the Hamiltonian parameters.

two qubits while $\varrho(t)$ is their reduced density operator at time t. As a measure of entanglement between two qubits we will use the concurrence defined by Wootters.[11] If ρ is the density matrix of the global system of two qubits, let us define $\tilde{\rho} \doteq \sigma^y \otimes \sigma^y \rho^* \sigma^y \otimes \sigma^y$ and $R = \rho\tilde{\rho}$. The concurrence is then defined as $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i are the square roots of the eigenvalues of R labeled in decreasing order.

The presence of oscillations in the entanglement induced by the interaction with the environment are clear. After a while the dissipative dynamics washes out the effects of the effective interaction between the qubits, and the average entanglement decays exponentially to zero.

3. Effects of QEC on the Entanglement between two *Logical* Qubits in the Presence of a Correlated Reservoir

In a QEC code which protects the logical qubits against one single phase error on individual qubits,[5] each logical qubit is encoded into three physical qubits as follows $|\tilde{0}\rangle \rightarrow |000\rangle_x$ and $|\tilde{1}\rangle \rightarrow |111\rangle_x$, where $|0\rangle_x$ and $|1\rangle_x$ are eigenstates of σ_x. At time intervals T the presence of phase errors in the physical qubits is detected and corrected accordingly, however no knowledge of the state of the logical qubits is gained.

In order to obtain information about the average properties of the time evolution of the entanglement we will consider states of two logical qubits with fixed initial concurrence between logical qubits, which are of the form:

$$|\Psi\rangle = \cos\vartheta|\tilde{0}_{n1}\tilde{0}_{n2}\rangle + \sin\vartheta|\tilde{1}_{n1}\tilde{1}_{n2}\rangle \tag{2}$$

with

$$|\tilde{0}_{ni}\rangle = \cos\frac{\vartheta_i}{2}|\tilde{0}\rangle + \sin\frac{\vartheta_i}{2}e^{i\varphi_i}|\tilde{1}\rangle; \quad |\tilde{1}_{ni}\rangle = \cos\frac{\vartheta_i}{2}|\tilde{1}\rangle - \sin\frac{\vartheta_i}{2}e^{-i\varphi_i}|\tilde{0}\rangle. \tag{3}$$

The concurrence of $|\Psi\rangle$ is $\sin 2\vartheta$. In the following analysis we will average over $\vartheta_1, \vartheta_2, \varphi_1, \varphi_2$. Let us consider first the case of short T, i.e. of frequent QEC. Averaging over all initial product states $\vartheta = 0, \pi/2$ one finds that the concurrence is always zero while the fidelity decays exponentially at a rate much slower than in the absence of any QEC. For partially or maximally entangled states the entanglement also decays exponentially. This means that QEC suppresses the effective interaction between logical qubits due to the presence of a correlated environment, i.e. for small T there is no creation of entanglement since the QEC protocol destroys all the correlations between physical qubits of different logical qubits. The reason for this can be seen in a qualitative way: for a state $|\tilde{0}\tilde{0}\rangle = |000\rangle_x|000\rangle_x$ the effective unitary evolution $U_r = \mathbb{1} + i\sum_{ij} V_{ij} t \sigma_z^i \sigma_z^j$ up to first order creates superposition like $|000\rangle_x|000\rangle_x + |110\rangle_x|000\rangle_x + \cdots$, which are entangled. After the corrections one gets a mixture of $|000\rangle_x|000\rangle_x$ and $|111\rangle_x|000\rangle_x$, which is no longer entangled. Not surprisingly then QEC inhibits the production of entanglement as this is seen by the protocol as an error.

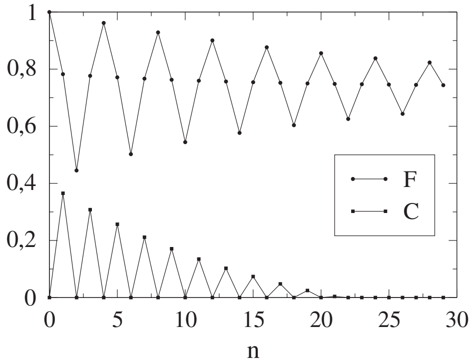

Fig. 2. Mean fidelity and concurrence between logical qubits in the presence of QEC as functions of the number of applications of the map, for the same parameters used in Fig. 1.

This may no longer be true if the time T is comparable to the period of oscillation of entanglement without QEC. In this case the environment may have time to create enough entanglement to be interpreted as a property of the initial state and amplified by QEC.

Figure 2 shows that the maximum average concurrence achievable is around 0.4. It is important to underline that this is an average value: there are states that do not evolve and so there is no production of entanglement while there are states for which the created entanglement is more than the average value. This is the case of initial product states of the computational basis for which the production of entanglement is maximum.

4. Conclusions

In summary, we have studied the effect of QEC on the entanglement between logical qubits in the presence of correlated noise. We have found that when the time interval T between consecutive applications of QEC is small, QEC is helpful in preserving the state of the qubits and indeed the fidelity decays at a smaller rate. As expected, in the absence of QEC there is production of entanglement between physical qubits, due to the correlations introduced by the environment, while in the presence of QEC this phenomenon is inhibited. The interesting new effect we have discovered is that when T is not small then entanglement may be generated with a bigger rate than in absence of QEC. In this scenario we are in a situation in which the entanglement induced by the unitary dynamics generated by the correlated bath is enhanced by the QEC protocol which prevents the dephasing effects of the coupling with the bath. Note that such enhancement in the production rate of entanglement is achieved by means of local measurements and conditional local unitary operations on the logical qubits, in other words the entanglement is not induced by joint measurements on the pair of logical qubits. In some sense our protocol can be described as a generalization of that proposed in Ref. 12, where the entanglement production is optimized, in the presence of a direct interaction and in the absence of decoherence, by means of local operations and ancillas. In our QEC protocol we use some sort of ancillary system to enlarge the Hilbert space, although in this case there is no sharp distinction between qubits and ancillas. Furthermore, we make use also of local projections on the enlarged subsystems and conditional dynamics.

Acknowledgments

This work was supported by the EU (IST-SQUBIT, HPRN-CT-2002-00144; QUPRODIS: IST-2002-38877).

References

1. G. M. Palma, K.-A. Suominen and A. K. Ekert, *Proc. R. Soc.* London, Ser. **A452**, 567 (1996).
2. P. Zanardi and M. Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997).
3. D. Braun, *Phys. Rev. Lett.* **89**, 277901 (2002).
4. M. A. Cirone, G. Compagno, G. M. Palma, R. Passante and F. S. Persico, quant-ph/0407032.
5. P. Shor, *Phys. Rev.* **A52**, R2493 (1995).
6. A. M. Steane, *Phys. Rev. Lett.* **77**, 793 (1996).
7. For recent reviews, see M. Nielsen and I. Chuang, *Quantum Computation and Quantum information* (Cambridge University Press, 2000); D. Bouwmeester, A. Ekert and A. Zeilinger (eds.), *The Physics of Quantum Information* (Springer, 2000).
8. D. V. Averin and R. Fazio, *JETP Lett.* **78**, 664 (2003); also cond-mat/0212127.
9. G. De Chiara, R. Fazio, C. Macchiavello and G. M. Palma, *Europhys. Lett.* **67**, 714 (2004).
10. X. X. Yi, H. T. Cui and X. G. Wang, *Phys. Lett.* **A306**, 285 (2003).
11. W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
12. W. Dür, G. Vidal, J. I. Cirac, N. Linden and S. Popescu, *Phys. Rev. Lett.* **87**, 137901 (2001).