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Properties of entanglement in interacting spin systems

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We briefly review recent work on entanglement with particular emphasis on its behaviour in interacting spin systems. As already pointed out by several authors a general understanding of the properties of entanglement in condensed matter systems may lead to a deeper comprehension of both dynamics and thermodynamics of many-body systems. After a brief resume of known means to identify and quantify entanglement, we take interacting spins on a network as an example. We briefly discuss critical properties of entanglement, if the system undergoes a quantum phase transition, and some aspects of its dynamical behaviour.

1 Introduction

A quantum mechanical system possesses additional correlations that do not have a classical counterpart. Entanglement, originally named *Verschränkung* by Erwin Schrödinger [1], is probably one of the most peculiar features of quantum mechanics. Soon after quantum mechanics was born it was realized that the very existence of entangled states would require to reconsider well established concepts in classical physics. Albert Einstein, who dubbed it as a *spooky action at distance*, understood that the concept of entanglement was in contrast with the requirement of any physical theory to be local. In his famous 1935 paper Einstein, together with Podolsky and Rosen, [2] exploited several consequences deriving from the existence of entangled states and arrived to the conclusion that quantum mechanics was an incomplete theory. The long debate on the possible existence of hidden variables was brought to an experimental test with the paper of John Bell [3] who showed that quantum mechanics and a local theory would lead to different outcomes of certain physical observables (the predictions of quantum mechanics were confirmed by numerous experiments conducted so far) (see [4] and references therein).

Besides its fundamental interest, a great deal of attention in the understanding of entanglement has been brought forth by its role in quantum information [5]. Entanglement is believed to be the main ingredient to realize new schemes of computation and communications. Two of the most celebrated examples are quantum teleportation [6] and entanglement-based quantum key distribution protocols [7]. Moreover, entanglement has proved to be a precious resource also in many other tasks, such as for example to increase the precision of frequency standards [8] and to enhance the transmission of classical information over quantum channels in the presence of correlated noise [9]. Viewed as a resource, it is crucial to find suitable methods to identify and quantify it. This program has been carried on satisfactorily only for bipartite systems, multipartite entanglement is still under intense investigation though several important steps have been achieved in the last years.

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An interesting aspect, which is acquiring increasing attention, is the activity devoted to the understanding of interconnection between quantum information and quantum statistical mechanics. In many-body systems entangled states naturally appear. The ground state of condensed matter systems (superconductors, Fractional quantum Hall states, ferromagnets just to mention few examples) are all entangled. Many-body systems might become useful for the development of new quantum information protocols. It is therefore desirable to analyze entanglement properties in those systems. In addition the study of the interconnection between quantum statistical mechanics and quantum information may lead to a better characterization of condensed matter states. To realize such a program, tools of quantum information theory should be applied for the study of many-body systems [10].

In the following we give a brief introductory overview to the concept of entanglement, which gives an idea of the main achievements and open problems in this area. It is not meant to be exhaustive: for more advanced and complete reviews on this topic see for example [11, 12]. As a natural application to the main concepts introduced in the Sects. 2–4 we then review entanglement properties of interacting spin systems.

The paper is organized as follows. In Sect. 2 we briefly review the main quantities that characterize bipartite entanglement. Further properties of entangled states are discussed in Sect. 3 where questions related to distillability of entanglement and the concept of bound entanglement are introduced. Multipartite entanglement is discussed in Sect. 4, where examples like GHZ- and W-states are presented. In the last Section we apply the concepts introduced earlier to the case of a one-dimensional model of interacting spins, we show how scaling properties typical of a quantum phase transition emerge in the entanglement measures, we discuss the relation between entanglement witnesses and correlation functions and with thermodynamical properties. Finally we briefly discuss dynamics of entanglement and how it is possible to generate in these systems multipartite entangled states. In the concluding remarks we try to highlight some issues which are likely to be investigated in the next future.

2 Bipartite entanglement

2.1 Pure states

The concept of entanglement for bipartite pure states can be easily defined as follows. A system of two particles A and B is described by a quantum state $|\psi\rangle_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ that can be generally written as $|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B$, where $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are bases for particles A and B respectively, and c_{ij} are normalised complex coefficients. Entangled pure states of two parties are defined as states that cannot be written in the factorised form

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B = \left(\sum_i c_i^{(A)} |i\rangle_A\right) \otimes \left(\sum_j c_j^{(B)} |j\rangle_A\right) \tag{1}$$

(for simplicity in the following we will usually omit the tensor product symbol $\otimes$). Therefore, if we are given a bipartite pure state, we can quickly test whether it is entangled or not by looking at the reduced density operator of subsystem A or B and checking whether it is a pure state or not.

In the case of bipartite pure states there is also an easy way to quantify the amount of entanglement. This is given by the von Neumann entropy of the reduced density operator describing the state of either system A or B, namely

$$E(|\psi\rangle) = -\operatorname{Tr}[\rho_A \log \rho_A], \tag{2}$$

where $\rho_A = \operatorname{Tr}_B[|\psi\rangle\langle\psi|]$ denotes the reduced density operator describing system A (notice that the same value of $E(|\psi\rangle)$ is obtained even if we consider ρ_B in place of ρ_A in eq. (2)). States of the form $|\psi\rangle = \sum_{i=0}^{d-1} |ii\rangle/\sqrt{d}$, where $d = \min[d_A, d_B]$ and $d_{A,B} = \dim \mathcal{H}_{A,B}$, are accordingly called maximally entangled states.

2.2 Mixed states

A mixed bipartite state described by the density operator ρ_{AB} is separable if it can be written as

$$\rho_{AB} = \sum_i p_i |\,\phi_i\rangle\langle\phi_i\,|_A \otimes |\,\chi_i\rangle\langle\chi_i\,|_B\,, \tag{3}$$

where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. If the density operator cannot be expressed in the above form the system is entangled. The decomposition of a density operator is not unique, and in general for a given ρ_{AB} it is not easy to state whether it can be expressed in the form (3). It is therefore important to find other conditions that allow to check whether a state is separable or not.

In the following we will briefly review some separability criteria, which give us conditions to state whether a density operator is entangled or not. The aspect of quantifying entanglement for mixed states is not as easy as for pure states, and several entanglement measures have been proposed. We will not review this quantitative aspect in this paper, but just mention in the following a very popular measure for bipartite systems, namely the concurrence.

2.2.1 Positive partial transpose (PPT) criterion

A very useful criterion for separability, proposed in [13], is based on the partial transpose. To define the partial transpose of a bipartite density operator ρ consider its matrix elements in a product basis $\{|\,n\rangle \otimes |\,\nu\rangle\}$, where the Latin letters refer to a basis for system A and the Greek ones for system B. The partial transposition of ρ with respect to the second system B is then defined as

$$(\rho_{m\mu,n\nu})^{T_B} \equiv \rho_{m\nu,n\mu}\,. \tag{4}$$

A density operator whose partial transpose is positive is said to be PPT, otherwise it is NPT (Non-positive Partial Transpose).

The idea behind this criterion can be understood by considering the case of separable states

$$(\rho_{\rm sep})^{T_B} = \sum_i p_i |\,\phi_i\rangle\langle\phi_i\,|_A \otimes (|\,\chi_i\rangle\langle\chi_i\,|_B)^{T_B}\,. \tag{5}$$

Since a positive operator remains positive under transposition, $(|\,\chi_i\rangle\langle\chi_i\,|_B)^{T_B}$ is still an allowed density operator for system B and therefore $(\rho_{\rm sep})^{T_B}$ is positive. The same holds if $\rho_{\rm sep}$ is partially transposed with respect to system A. Therefore, all separable states are PPT.

However, it is not generally true that if $(\rho_{AB})^{T_B}$ is positive then the state is separable. The PPT criterion was proved to be a necessary and sufficient condition for separability for bipartite systems with dimension 2×2 and 2×3 [14]. In all the other cases it gives only a necessary condition for separability.

For example, for a pair of qubits described by a density operator of the Werner form

$$\rho_p = p|\,\psi_-\rangle\langle\psi_-\,| + \frac{1-p}{4}\,\mathbb{1}\,, \tag{6}$$

where $0 \leq p \leq 1$, $|\,\psi_-\rangle = (|\,01\rangle - |\,10\rangle)/\sqrt{2}$ is the singlet state and $\mathbb{1}$ is the 4×4 identity operator, we can see by applying the PPT criterion that the state is separable for $p \leq 1/3$, otherwise it is entangled.

We just mention here that other interesting criteria that are easy to test, weaker than PPT [15] or independent of it [16], were also proposed.

2.2.2 Positive maps

A necessary and sufficient criterion for separability in any dimension, proposed in [14], is based on positive maps. It states that *ρ is separable if and only if for any positive map Λ*

$$(\mathbb{1} \otimes \Lambda)\rho \geq 0\,, \tag{7}$$

where $\mathbb{1}$ is the identity operator for system A. Notice that $(\mathbb{1}\otimes\Lambda)(\rho_A\otimes\rho_B)\geq 0$, namely factorised density operators are transformed into factorised density operators and therefore separable states are transformed into positive operators. Moreover, the above criterion says that if the state ρ is entangled then there exists a positive map M such that $(\mathbb{1}\otimes M)\rho$ is non positive.

The criterion is strong because it gives a necessary and sufficient condition for positivity, but it is not easily applicable because a complete characterisation of positive maps in arbitrary dimension is not known.

2.2.3 Entanglement witnesses

Another criterion, which gives a necessary and sufficient condition for separability, is based on the so called entanglement witnesses [14,17]. According to this criterion, *a density matrix ρ is entangled if and only if there exists a Hermitian operator W such that* $\mathrm{Tr}(W\rho)<0$ *and* $\mathrm{Tr}(W\rho_{\mathrm{sep}})\geq 0$ *for any separable density operator* ρ_{sep}. The operator W is called entanglement witness [18].

Also this criterion, similary to the one discussed in Sect. 2.2.2, in general cannot be tested by a simple procedure. Nevertheless, the entanglement witnesses criterion is at the roots of a recently proposed general method to detect entanglement with few local mesurements [19]. This method is based on suitable decompositions of the witness operators in terms of pseudomixtures of local projectors and can be applied when partial knowledge about the density operator to be tested is available. As an example we consider that case in which the class of states to be tested is known to be of the Werner form, see eq. (6). In this case a suitable witness operator is given by $W=(|\,e_-\rangle\langle e_-\,|)^{T_B}$, where $|\,e_-\rangle=(|\,00\rangle+|\,11\rangle)/\sqrt{2}$. This can be decomposed as a linear combination of local projectors as follows [19]

$$\begin{aligned} W=\tfrac{1}{2}\big(&|\,z^+z^+\rangle\langle z^+z^+\,|+|\,z^-z^-\rangle\langle z^-z^-\,|-|\,x^+x^+\rangle\langle x^+x^+\,|-|\,x^-x^-\rangle\langle x^-x^-\,|\\ &+|\,y^+y^-\rangle\langle y^+y^-\,|+|\,y^-y^+\rangle\langle y^-y^+\,|\big)\,;, \end{aligned} \tag{8}$$

where $|\,z^+\rangle=|\,0\rangle, |\,z^-\rangle=|\,1\rangle, |\,x^\pm\rangle=\frac{1}{\sqrt{2}}(|\,0\rangle\pm|\,1\rangle)$ and $|\,y^\pm\rangle=\frac{1}{\sqrt{2}}(|\,0\rangle\pm i|\,1\rangle)$ are the eigenstates of the Pauli operators. Therefore just local measurements along three different directions are sufficient to test the value of $\mathrm{Tr}[W\rho_p]$. Recently this method was exerimentally tested on quantum optical systems both for bipartite [20] and multipartite states [21].

Before concluding this discussion on methods to test whether a bipartite density operator is separable or not, we want to mention that recently interesting algorithms to test separability in a finite number of steps have been proposed [22,23].

2.2.4 Concurrence

The criteria defined above, Sects. 2.2.1–2.2.3, do allow to find out if a given state is entangled or not. However they do not provide quantitative measures of the amount of entanglement. Differently from the case of pure states, there is not a unique measure for entanglement for bipartite mixed states. We introduce here one possible measure, which is easy to evaluate and will be used later for the case of spin systems, named concurrence [24]. The concurrence C is defined as

$$C[\rho_{AB}]:=\max\left\{0,2\lambda_{\max}-\mathrm{tr}\,\sqrt{R}\right\}, \tag{9}$$

$$R:=\rho_{AB}\sigma_y\otimes\sigma_y\rho^*_{AB}\sigma_y\otimes\sigma_y \tag{10}$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $\sqrt{R}$, which is the product of ρ_{AB} with its time-reversed; σ_y is a Pauli matrix. The concurrence is related to the entanglement of formation [25], however the great advantage of the concurrence is that it is directly defined in terms of the density matrix ρ_{AB} without minimization procedure and hence much simpler to evaluate.

3 Distillability and bound entanglement

In this Section we discuss the problem of distillability of entanglement, namely whether from a collection of mixed entangled states it is possible to distill a smaller number of pairs with a higher degree of entanglement by means of local operations and classical communication. This problem was posed for the first time, in the context of quantum teleportation, in [26] where a scheme was proposed to distill entangled states from a collection of mixed states. The aim was to perform teleportation efficiently even in the presence of noise along the transmission channel used to distribute maximally entangled states. In [27] an alternative distillation protocol was proposed in the context of quantum cryptography, and it was shown that it allows to achieve secure key distribution by means of entangled states even in the presence of noise.

In general, the distillability problem deals with the possibility of extracting maximally entangled states from a collection of identical copies of mixed states and to characterise the set of entangled states that can be distilled. In the simplest case of a pair of qubits the problem is solved: it was proven that all entangled pairs of qubits can be distilled [28]. In higher dimension the problem is still open, and it is not true in general that an entangled state can be distilled.

A necessary and sufficient condition for distillability was provided in [29]. According to this condition, *a state ρ is distillable if and only if there exists a state $|\chi\rangle = \alpha|a_1\rangle|b_1\rangle + \beta|a_2\rangle|b_2\rangle$, and a number n, such that $\langle\chi|(\rho^{T_B})^{\otimes n}|\chi\rangle \leq 0$,* where $\langle a_1|a_2\rangle = \langle b_1|b_2\rangle = 0$. As a consequence of the above theorem a PPT state cannot be distilled.

In dimension higher than 2×2 and 2×3 it is possible to have entangled states that are PPT and therefore not distillable. These states are called bound entangled states [29]. In higher dimension we have then a more complicated structure of entanglement: there are free entangled states, which can be distilled and therefore employed in quantum information tasks, such as quantum teleportation [6], and there are bound entangled states, that cannot be distilled. A natural question is then to ask whether bound entangled states contain useful entanglement, or in other words whether they lead to a better performance of quantum information tasks with respect to separable states. This aspect has been recently analysed in some quantum information protocols and it was shown that bound entanglement is not useful for teleportation [30] and for dense coding [31], but it is a useful resource for quantum cryptography [32]. The issue of how to generate bound entangled states is in general still open. Only recently a method to experimentally generate two families of bound entangled states of qubits has been proposed [33].

Another natural question to ask, which is still unsolved at the moment, is whether all NPT states are distillable or there exist undistillable NPT states. When one of the subsystems has dimension two it was proved that all NPT states are distillable [34]. However, it is conjectured that this is not true in general [34,35].

4 Multipartite entanglement

What we discussed so far is related to entanglement in bipartite systems. Obviously this cannot exhaust the classification of entangled states. In this Section we introduce the concept of entanglement among more than two parties for the case of pure states. The characterisation of entanglement for many parties is more complex than for the bipartite states. For multipartite states there are different kinds of entanglement depending on the number of parties that are correlated. For example, for three parties we can have genuine entanglement among A, B and C, or otherwise we can have just bipartite entanglement between one of the parties with the remaining two.

More rigorously, a tripartite state is separable if it can be written as

$$|\psi\rangle_{ABC} = |\phi\rangle_A \otimes |\chi\rangle_B \otimes |\varphi\rangle_C \,. \tag{11}$$

It is biseparable if there are correlations between one of the parties, say C, and the other two, A and B. In this case the state has the form

$$|\psi\rangle_{ABC} = |\psi\rangle_{AB} \otimes |\phi\rangle_C \,, \tag{12}$$

where $|\psi\rangle_{AB}$ is an entangled bipartite state. A genuine tripartite state cannot be factorised in none of the forms (11) and (12).

In contrast to the case of bipartite systems, in the case of three qubits there exist two non equivalent classes of genuine tripartite entangled states [36]. Two states belong to the same class if they can be transformed one into the other by means of local operations and classical communication with non vanishing probability. The representative states of the two classes are called GHZ and W states, and take the form

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\,, \tag{13}$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)\,. \tag{14}$$

It can be easily seen that a W-state is still maximally entangled if we trace over one of the three systems, while the GHZ-sate is not. We want to point out that for pure states of three qubits there is a simple criterion to establish whether an entangled state belongs to the GHZ class or to the W class, namely the 3-tangle [37].

We just mention that a classification of mixed states of three qubits has been presented in [38], where four classes have been identified, namely separable, biseparable, W-type and GHZ-type states. In general, however, for more than three qubits or for multi-party higher dimensional systems the classification problem for mixed states is more complex and not fully solved. Useful tools, such as entanglement witnesses, and concepts like distillation, introduced originally for bipartite systems, can still be applied [38,39].

5 Entanglement in interacting spin networks

A natural arena where to test all the concepts introduced before is the characterization of interacting spin systems. A number of interesting results in this direction has been already obtained. O'Connors and Wootters have shown that in the Heisenberg chain the maximization of entanglement at zero temperature is related to the energy minimization [40] while at finite temperatures violation of Bell's inequality can be directly related to the properties of the internal energy [41]. Temperature and/or magnetic field can *increase* the entanglement of the systems as shown for the Ising and Heisenberg model in [42,43]. More recently, it was shown that W-states of an arbitrary number of qubits can be obtained as ground states of XY isotropic ferromagnetic spin chains with transverse magnetic field [44], and multipartite entanglement is proved to exist in these systems even at finite temperatures. Moreover at finite temperature, macroscopic thermodynamical properties can detect quantum entanglement. Brukner and Vedral [45] have identified an entanglement witness for spin system and related it to thermodynamic quantities.

It is natural to suspect that quantum correlations become increasingly important if the system behaves collectively. Close to a quantum phase transition entanglement can been classified in the framework of scaling theory [46–51]. Nevertheless it has been demonstrated that there is a profound difference between non-local quantum and classical correlations.

Interest was not only confined to the equilibrium properties but also attention has been devoted to the dynamical properties of entanglement, see [52,53] and references therein. Dynamical aspects aquire additional importance as the intrinsic dynamics of the spin system leads to the production of multipartite entangled states. Wang has demonstrated that W states of three and four qubits can be generated in a one dimensional XY model [54].

There is no space to give a broad overview of the whole activity of the field. We choose to highlight few aspects which may help to get the flavour of the main issues of the field and at the same time we make connections with the general concepts reviewed in the previous Sections. We confine the discussion to one-dimensional spin systems.

The system we consider is a spin-1/2 ferromagnetic chain with an exchange coupling λ in a transverse magnetic field. The Hamiltonian (in units of the magnetic field strength) is

$$H = -\lambda \sum_{i=1}^{N} (1+\gamma) S_i^x S_{i+1}^x + (1-\gamma) S_i^y S_{i+1}^y - \sum_{i=1}^{N} S_i^z \tag{15}$$

where S^a are the spin-1/2 matrices ($a = x, y, z$) and N is the number of sites. We assume periodic boundary conditions. The anisotropy parameter γ connects the quantum Ising model for $\gamma = 1$ with the isotropic XY model for $\gamma = 0$. In the interval $0 < \gamma \leq 1$ the model belongs to the Ising universality class and for $N = \infty$ it undergoes a quantum phase transition at the critical value $\lambda_c = 1$. In equilibrium the order parameter is the magnetization in x-direction, $\langle S^x \rangle$, which is different from zero for $\lambda > 1$ and vanishes at and below the transition. On the contrary the magnetization along the z-direction, $\langle S^z \rangle$, is different from zero for any value of λ.

5.1 Entanglement close to a quantum phase transition

Close to the quantum critical point the system is characterized by long-range correlations which manifest themselves in the singular behavior of different physical observables. Both classical and quantum critical points are governed by a diverging correlation length. However, as a quantum mechanical system possesses additional correlations that do not have a classical counterpart, one would expect peculiar features due to entanglement if the system is close to a critical point. Indeeed, since at a quantum phase transition the ground state wave-function undergoes qualitative changes, it is important to understand how its genuine quantum features evolve throughout the transition. Will entanglement between distant subsystems be extended over macroscopic regions, as correlations are? Will it carry distinct features of the transition itself and show scaling behavior?

As the many-body wave-function contains, in general, multi-partite entanglement, it is impossible to provide a complete answer to these questions. A possible strategy would be a study of certain types of entanglement. In [46,47] the focus was on bipartite entanglement and the concurrence (defined in Sect. 2.2.4) was chosen as a measure of the entanglement. It turns out that the range (in space) over which the concurrence is different from zero depends strongly on the parameter γ. Surprisingly for $\gamma = 1$ (Ising model) the maximum distance between two spins for which the concurrence is different from zero is two lattice sites. Even at the critical point, where spin-spin correlations extend over a long range (the correlation length is diverging for an infinite system), the concurrence vanishes unless the two sites are at most next-nearest neighbors. In order to quantify the change of the many-body wave-function when the system crosses the critical point, one can look at the derivatives of the concurrence as a function of λ. Intuitively one expects that the phase transition is characterized by changes in the wave-function and therefore derivatives of the concurrences as a function of λ should be very sensitive observables to this aim. The derivative of the concurrence between nearest neighbor sites (at $\gamma = 0$) for systems of different size (including the thermodynamic limit) are presented in Fig. 1. For the infinite chain $\partial_\lambda C(1)$ diverges on approaching the critical value as

$$\partial_\lambda C(1) = \frac{8}{3\pi^2} \ln|\lambda - 1| + const. \tag{16}$$

The vanishing of the concurrence between distant sites does not mean that two sites that are far apart are not entangled. Different indicators of entanglement capture different aspects of the complicate nature of a many-body quantum state close to a critical point. Starting from the work by Vidal et al. [48] several groups (see [50,51] and references therein) studied the problem from a different perspective. They analyzed the entropy of entanglement between a block of L adjacent spins with the rest of the chain. The intuitive idea behind this choice is that if the length L would exceed the correlation lenght ξ the entropy would saturate. One expects a different situation at the critical point where the correlation lenght diverges. As

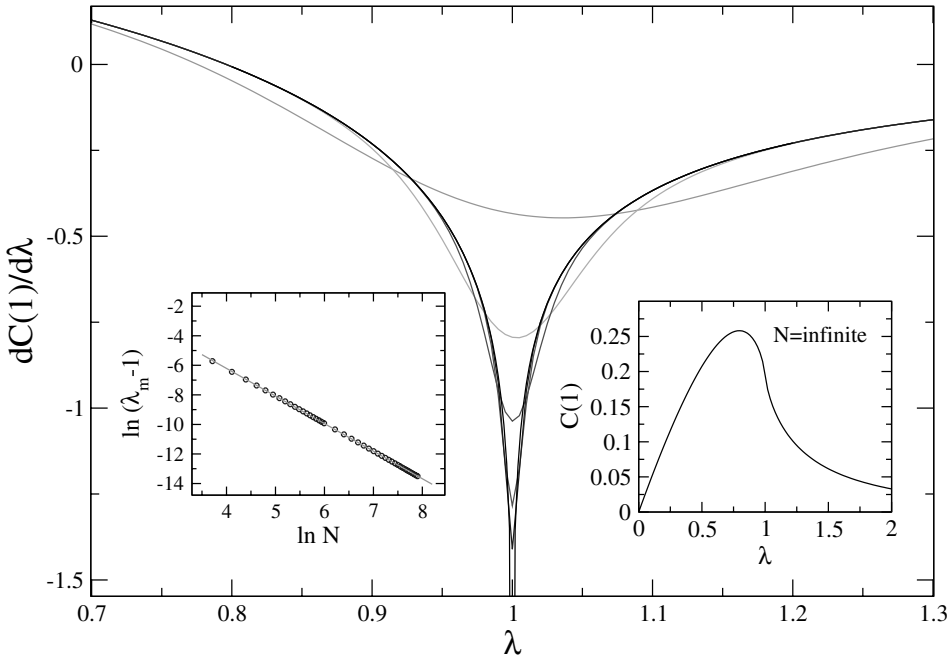

Fig. 1 (online colour at: www.ann-phys.org) The change in the ground state wave-function in the critical region is analyzed considering $\partial_\lambda C(1)$ ($C(1)$ is the concurrence between nearest-neighbour sites) as a function of the reduced coupling strength λ. The curves correspond to different lattice sizes $N = 11, 41, 101, 251, 401, \infty$. In the left inset the behaviour of the minimum as a function of the number of sites is shown while on the right inset the behaviour of the concurrence as a function of the coupling λ. It is evident from the figure that the critical properties of the entanglement manifest in the derivatives of the concurrence and not in the quantity itself.

originally proven numerically and then put on firm basis by means of conformal field theory, it turns out that at the critical point (again we consider for simplicity the case of the Ising model at $\gamma = 1$) the entropy of entanglement diverges logarithmically with the lenght of the block

$$S^{(L)} = \frac{c + \bar{c}}{6} \log_2 L + k \tag{17}$$

and, most inportant, the coefficients c,$\bar{c}$ are the central charges of the conformal field theory that describes the universal properties of the systems at the critical point.

In a different approach, by Verstraete et al. [49], the concept of localizable entanglement has been introduced. In this case one assumes that it is possible to perform local operations on each site of the chain and then asks what is the maximum amount of entanglement one can achieve between two different sites after proper local operations have been performed. In [49] it was shown that this procedure amounts in a maximization over all possible two-site correlation functions. As a consequence it is possible to introduce an entanglement lenght ξ_E which diverges at a quantum phase transition.

5.2 Entanglement witnesses for spin chains

As discussed in the Sect. 2.2.3, the entanglement witness is a properly defined operator whose expectation value obeys a given inequality if evaluated over an entangled state. This topic has been recently discussed for spin chains in a paper by Brukner and Vedral [45] where it was shown that certain thermodynamic quantities

can be used as entanglement witnesses. In order to keep the presentation self-contained we consider the Hamiltonian defined in eq. (15) although the discussion of [45] has been performed for a more general class of models.

For the model defined in eq. (15) with $\gamma = 0$, an entanglement witness is defined by a combination of the expectation values of the energy and the magnetization as

$$\frac{\mid U + M \mid}{\lambda N} \geq 1 \tag{18}$$

where $U = \langle H \rangle$ is the internal energy and $M = \langle S^z \rangle$ is the average magnetization. The inequality is satisfied when the chain is in an entangled state. Two simple limits are the case of infinite and zero temperature. In the first case both the magnetization and internal energy are zero saying, as expected, that the thermal state at infinite temperature is separable. In the opposite limit, the averages have to be performed over the ground state. In the limit of zero magnetic field the left hand side of eq. (18) is 2 thus implying that the ground state is entangled.

5.3 Dynamics of entanglement and generation of W-states

Besides the equilibrium properties also dynamical properties of entanglement are very interesting. In condensed matter physics it is customary to analyze the dynamical behaviour of the system by creating an excitation in a given point and studying its evolution in time. A similar approach can be followed to examine entanglement dynamics. The idea is to create some localized entangled state (typically a Bell state) and follow the evolution of entanglement generated by the dynamics of the spin chain. There are a number of questions that can be addressed in this way. In particular one would like to see if there is a well defined velocity of propagation of entanglement, how it is related to the propagation of the elementary excitations of the spin system and what are the time scales for damping the entanglement created at the initial time [52]. All these aspects can be important if one attempts to use collective degrees of freedom to encode information or if entanglement needs to be stored and then swapped in distant parts of a register (the spin chain itself).

As an example of entanglement propagation we consider the case $\gamma = 0$ where the Hamiltonian of eq. (15) reduces to the XY model [55]. We consider the case in which the initial state is a singlet where the description is amenable to a simple analytical solution. In this case the Hilbert space relevant to describe the dynamics is spanned by the states

$$|\boldsymbol{j}\rangle \equiv |\downarrow\downarrow \ldots \downarrow\uparrow\downarrow \ldots \downarrow\rangle\,, \tag{19}$$

which for $j = 1, \ldots, N$ represents a state of the chain where the jth spin is prepared in $|\uparrow\rangle$ and the others $N - 1$ ones in $|\downarrow\rangle$.

We consider the case of a chain initially prepared in a maximally entangled singlet-like state on sites i and j

$$|\psi, t = 0\rangle = \frac{1}{\sqrt{2}}(|\boldsymbol{i}\rangle + |\boldsymbol{j}\rangle).$$

The evolution of the state at later times is given by

$$|\psi, t\rangle = \sum_l w_l(t)\,|\boldsymbol{l}\rangle \tag{20}$$

with

$$w_l(t) = \frac{1}{N\sqrt{2}} \sum_k \left[\mathrm{e}^{-\frac{2\pi i k}{N}(i-l)} + \mathrm{e}^{-\frac{2\pi i k}{N}(j-l)}\right] \mathrm{e}^{-i\Lambda_k t}\,. \tag{21}$$

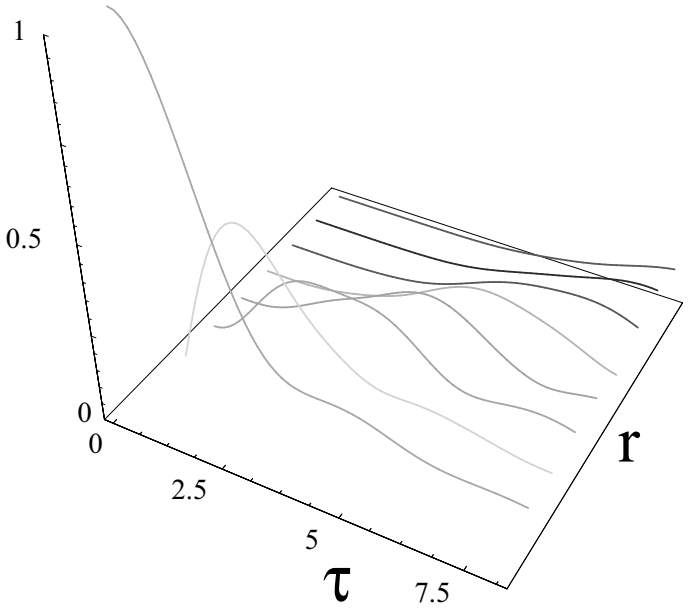

Fig. 2 (online colour at: www.ann-phys.org) Concurrence between sites at a distance r (namely, $n = i$, and $n = i + r$ in eq. (22)), for the case of an initial state shared by two nearest neighbor sites, $i = 0, j = 1$. The various plots correspond to $r = 1, \ldots, 8$. The time is expressed in units of the exchange coupling $\tau = \lambda t$.

The concurrence between two selected sites n and m is given by

$$C(n, m, t) = 2\big|w_n(t)w_m^*(t)\big| \,. \tag{22}$$

An example of the propagation is plotted in Fig. 2. Supposing that the initially selected sites are nearest neighbors, $j = i + 1$, we look at the concurrence between one of these two (namely, i) and another site of the chain at distance r (i.e. $n = i$ and $m = i + r$ in eq. (22)). The result is shown in Fig. 2, where one can see the propagation of the maximum with velocity related to the exchange coupling. The time evolution dictated by the Hamiltonian, amounts to a simultaneous spin flip between sites l and $l \pm 1$ or a propagation of the single flipped spin in either directions. The time scale is set up by the interaction, so that the information exchange or "entanglement propagation" over a distance of d lattice spacings, approximately takes a time $t \sim d/\lambda$.

For a deeper investigation of the entanglement propagation observed for the concurrence, we evaluate the entropy $S_{n,m}$ of the reduced density matrix $\rho_{n,m}$ (in the notation of the Sect. 2 the two sites n and m correspond to the two subsystems A and B). Since the chain is in a pure state, this entropy gives a measure of the entanglement of the two sites with the rest of the chain. In particular, if $S_{n,m} = 0$, then the state of the two sites is pure and no entanglement exists with the rest of the chain, while $S_{n,m} \neq 0$ means that $\rho_{n,m}$ is a mixed state, and the pair (n, m) consequently is entangled with the rest of the system.

From the one-site reduced density matrix for this case we deduce $S_{n,m}(t) = -(1-p)\log_2(1-p) - p\log_2 p\,,$ with $p = |w_n(t)|^2 + |w_m(t)|^2$. $S_{-x,x+1}$ is shown in Fig. 3 unveiling the same qualitative structure found for the concurrence. Initially, this entropy is zero and for $x > 0$ it remains zero until the "entanglement wave" has arrived. This is because a pure state $|\uparrow\rangle_n|\uparrow\rangle_m$ is found until the "entanglement wave" arrives. Then the pair of spins becomes entangled among themselves but also with the rest of the system. This is understood in the sense of entanglement sharing from the fact that also the nearest neighbors x and $x + 1$ become entangled at that time. After that, the two sites are left partially entangled with the rest of the chain. The above discussion can be extended to the von Neumann entropy for an N-site subsystem at positions $\vec{j} := (j_1, \ \ldots\ , \ j_N)$: Since the reduced density matrix has still rank 2 we still have $S_{\vec{j}}^{(N)} = -p\log_2 p - (1-p)\log_2 1 - p$ but with $p = \sum_{i=1}^{N} |w_{j_i}|^2$.

The $XY-$Model on a one-dimensional chain can also be employed to generate (a slight generalization of) W-states defined in eq. (14). This has been discussed by Wang [54]. Starting from an initial state $|\,\psi, t =$

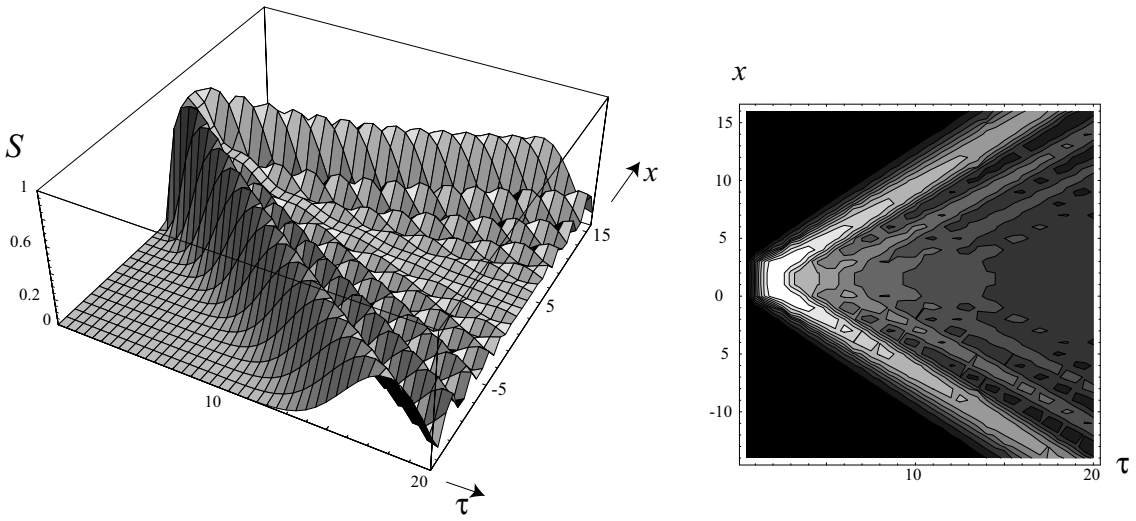

Fig. 3 (online colour at: www.ann-phys.org) Entropy $S_{(-x,x+1)}(\tau = \lambda t)$ for pairs of sites symmetrically displaced with respect to the initial singlet position at $(a, b) = (0, 1)$. The initial state was chosen to be (up to the normalization factor) $(|\mathbf{0}\rangle - |\mathbf{1}\rangle)$.

$0\rangle = |\boldsymbol{j}\rangle$, it can be shown that at given times the probabilities for any terms of the superposition of the state of the basis set becomes equal. If one considers, for example, a four site chain ($N = 4$) then at times

$$t_W = \frac{\pi}{2} + 2n\,\pi$$

the state of the system is

$$|\,\psi, t = 0\rangle = \tfrac{1}{2}|\mathbf{1}\rangle - i|\mathbf{2}\rangle - |\mathbf{3}\rangle - i|\mathbf{4}\rangle\,. \tag{23}$$

6 Conclusions

The "spooky action at distance" already noted by Albert Einstein in the early days of quantum mechanics, over the years has been identified as one of the crucial ingredients for the outperformance of quantum information as compared to the classical one. In this paper we touched very briefly on the various issues which are presently at the center of the investigation in the theory of quantum information. We put emphasis on some aspects of the interconnection between quantum information and statistical mechanics that have been emerging in the last couple of years. Even in this respect the review is by far not complete and in these conclusions we just give few additional examples. An analysis of the entanglement was not confined to spin systems. Attention has also been devoted to the BCS model [56, 57], quantum Hall systems [58] and Boson systems [59]. The link between statistical mechanics and quantum information theory has been carried ahead in refining established methods for examining many body interacting systems. An important example is the new approach to Density Matrix Renormalization Group discussed in [60]. Interesting connections also emerged in connection to Renormalization Group theory, in [61] the entanglement loss along renormalization group trajectories was analyzed.

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