

## Multipartite entanglement in quantum spin chains

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We study the occurrence of multipartite entanglement in spin chains. We show that certain genuine multipartite entangled states, namely  $W$  states, can be obtained as ground states of simple XX type ferromagnetic spin chains in a transverse magnetic field, for any number of sites. Moreover, multipartite entanglement is proven to exist even at finite temperatures. A transition from a product state to a multipartite entangled state occurs when decreasing the magnetic field to a critical value. Adiabatic passage through this point can thus lead to the generation of multipartite entanglement.

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Quantum entanglement is a valuable resource with potential applications including quantum frequency standards [1], quantum cryptography [2] and quantum teleportation [3]. Recent progress in experimental techniques allows to generate and to control *multipartite* entanglement [4,5]. This motivates studies of quantum entanglement in more complex physical systems, for example chains of interacting spins. Here we show how passage through avoided crossings in spin chains can be used to generate multipartite entanglement.

Quantum spin chains have been extensively studied in the context of quantum information science, in particular their use as quantum wires [6] and as simple quantum processors [7]. Bipartite entanglement in spin chains has been also thoroughly investigated [8–15]. In contrast, very little is known about multipartite entanglement in such systems. A notable exception is work by Wang, who performed a numerical analysis of multipartite entanglement in the Heisenberg model [16], and by Stelmachovic and Buzek, who showed that in the limit of infinite coupling strength the ground state of the Ising chain is locally unitarily equivalent to the  $N$ -partite generalization of the GHZ state [17]. More recently Santos showed that the presence of defects in a spin chain governed by the XXZ Hamiltonian could be exploited to generate bipartite and tripartite entangled states between selected spins [18].

Here we analyze a quantum spin chain with an arbitrary number of sites and demonstrate the existence of multipartite entanglement of the  $W$  type [see Eq. (10) below] in the ground state of the XX Hamiltonian with suitable transverse magnetic field. We find the critical value of the magnetic field at which the ground state undergoes a transition from a product state to this multipartite entangled state. Our work therefore provides a method for generating a multipartite  $W$ -type entangled state by tuning a *single external* parameter, namely the global magnetic field, while the interspin interaction remains constant. We refer to spins as qubits and use  $|0\rangle$  and  $|1\rangle$  to denote spin-up and spin-down states, respectively.

Let us start with a simple example, where the transition

from a product state to a multipartite entangled state can be easily seen. We consider the XXZ model for a chain of 3 spins in an external magnetic field and with periodic boundary conditions. The corresponding Hamiltonian is given by

$$H_{\text{XXZ}} = \sum_{i=1}^3 (J\vec{\sigma}_i \otimes \vec{\sigma}_{i+1} + \Delta\sigma_i^z \otimes \sigma_{i+1}^z + B\sigma_i^z), \quad (1)$$

where  $J$  is the coupling in the  $x$  and  $y$  directions and  $J+\Delta$  is the coupling in the  $z$  direction. The parameter  $\Delta$  quantifies the anisotropy in the interaction; for the Heisenberg interaction  $\Delta=0$ . N.B. this model exhibits the same symmetry as the XX model with  $N$  sites which will be discussed later. In both cases the  $z$  component of the total spin,  $\sigma_{\text{tot}}^z = \sum_i \sigma_i^z$ , commutes with the Hamiltonian  $H$ . Thus the eigenstates of  $H$  are superpositions of states with a fixed number of up-spins.

In Eq. (1) the term proportional to  $\Delta$  commutes with the terms proportional to  $J$ . Thus, the eigenstates of the Hamiltonian  $H_{\text{XXZ}}$  coincide with those of the isotropic Heisenberg model. There are four nondegenerate eigenstates, given by  $|111\rangle$ ,  $|\bar{W}\rangle = (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}$ ,  $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ , and  $|000\rangle$ . The corresponding eigenvalues are  $E_{111} = 3J + 3B + 3\Delta$ ,  $E_{\bar{W}} = 3J + B - \Delta$ ,  $E_W = 3J - B - \Delta$ , and  $E_{000} = 3J - 3B + 3\Delta$ . The remaining eigenstates are doubly degenerate and are given by  $|\bar{W}^{(k)}\rangle = (|110\rangle + e^{2\pi ik/3}|101\rangle + e^{-2\pi ik/3}|011\rangle)/\sqrt{3}$  with  $k=1, -1$  and eigenvalue  $E_{\bar{W}^{(k)}} = J + B - \Delta$ , and  $|W^{(k)}\rangle = (|001\rangle + e^{2\pi ik/3}|010\rangle + e^{-2\pi ik/3}|100\rangle)/\sqrt{3}$  with eigenvalue  $E_{W^{(k)}} = J - B - \Delta$ .

The relative values of the parameters  $J$ ,  $B$  and  $\Delta$  determine which of these states is the ground state. Let us first consider the isotropic case,  $\Delta=0$ . When the value  $J$  is negative, i.e., in the ferromagnetic case, the ground state is always a product state, regardless of the value of  $B$ . Changing the sign of the magnetic field  $B$  simply leads to the relabeling  $0 \leftrightarrow 1$  of each qubit. In the following, we will always consider  $B$  to be positive. For positive  $J$ , i.e., the antiferromagnetic case, and for  $B > J$  the ground state is also a product

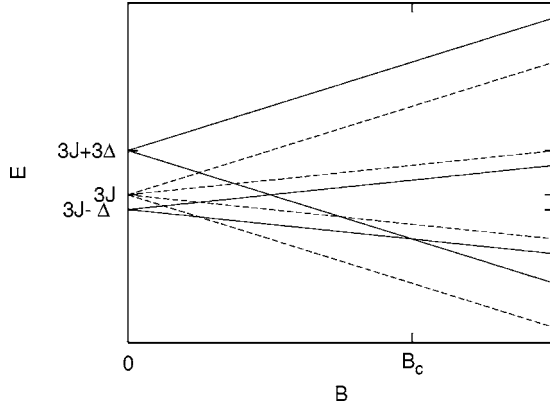


FIG. 1. Role of the anisotropy: Energy levels for the ferromagnetic XXZ model with anisotropy  $\Delta$  as a function of the magnetic field  $B$ , for the case  $N=3$ . Dashed lines are for  $\Delta=0$  and solid lines for  $\Delta>0$ . For  $B>B_c$  the levels are ordered from bottom to top as  $E_{000}, E_W, E_{\bar{W}}, E_{111}$  (see main text for the definition), both for  $\Delta=0$  and  $\Delta>0$ .

state. However, for  $B<J$  the ground state lies in the subspace spanned by  $|W_k\rangle$ , with  $k=1, -1$ . This subspace contains states with  $W$ -type tripartite entanglement and biseparable states, i.e., states in which one qubit is unentangled.

Let us now turn to the anisotropic case,  $\Delta \neq 0$ . For negative values of  $J$  and negative  $\Delta$ , the ground state is again given by a product state, regardless of the value of  $B$ . However, the case of negative  $J$  and positive  $\Delta$  leads to interesting results. For high values of  $B$ , the spins are aligned, and the ground state is  $|000\rangle$ , i.e., a product state. When one decreases the value of  $B$ , at  $B_c=2\Delta$  the ground state changes to  $|W\rangle$ , namely a genuinely tripartite entangled state. The essential role of the nonvanishing anisotropy  $\Delta$  for the existence of the ground state transition from a product state to a tripartite entangled one is illustrated in Fig. 1.

The above considerations hold if the spin chain is at zero temperature. However, at finite but low temperatures, the state of this three-spin chain can be shown to be still genuinely tripartite entangled. To see this, one expands the density matrix of the system at a temperature  $T$ , which is given by its Gibbs state, as follows:

$$\rho \propto e^{-\beta E_{000}}|000\rangle\langle 000| + e^{-\beta E_W}|W\rangle\langle W| + e^{-\beta E_{\bar{W}}}|{\bar{W}}\rangle\langle {\bar{W}}| + e^{-\beta E_{111}}|111\rangle\langle 111| + \dots, \quad (2)$$

where  $\beta=1/kT$  and  $k$  is the Boltzmann constant. In the vicinity of  $B_c=2\Delta$ , the two lowest eigenvalues are  $E_{000}$  and  $E_W$ . Hence by retaining only the two leading terms of the expansion (2), which is a good approximation for sufficiently low temperatures, the state of the system can be approximated as

$$\varrho = p|W\rangle\langle W| + (1-p)|000\rangle\langle 000|, \quad (3)$$

where  $p=1/(1+\exp[\mp 2\beta(2\Delta-B)])$  for  $B_c<2\Delta$  and  $B_c>2\Delta$ , respectively. This state was first studied in the context of approximate quantum cloning [19]. It is a genuinely

tripartite entangled mixture, which cannot be written as a mixture of biseparable states for any value of  $p \neq 0$ . To prove this, let us assume the contrary, namely that  $\varrho$  is biseparable. A biseparable state  $\varrho$  of a tripartite system  $ABC$  can be decomposed as [20]

$$\varrho = \sum_{ijk} (p_i^C \tau_i^{AB} \eta_i^C + p_j^B \tau_j^{AC} \eta_j^B + p_k^A \tau_k^{BC} \eta_k^A), \quad (4)$$

where  $p_i^C, p_j^B, p_k^A$  are probabilities with  $\sum_{ijk}(p_i^C + p_j^B + p_k^A)=1$ . In the first term on the right-hand side (RHS) of (4),  $\tau_i^{AB}$  denotes the joint density operator of the subsystems  $A$  and  $B$ , while  $\eta_i^C$  denotes the density operator of the subsystem  $C$  (and analogously for the two subsequent terms). The decomposition (4) implies that there exists a biseparable state of the form  $|\psi_i^{AB}\rangle|\phi_i^C\rangle$  in the range of  $\varrho$ . This in turn implies that there should exist nonzero coefficients  $\alpha$  and  $\beta$  such that  $|\psi_i^{AB}\rangle|\phi_i^C\rangle = \alpha|W\rangle + \beta|000\rangle$ . However, it is straightforward to see that there are no biseparable vectors in the subspace spanned by  $|W\rangle$  and  $|000\rangle$ , apart from the vector  $|000\rangle$ , which however corresponds to the trivial case  $\alpha=0$ . Hence one arrives at a contradiction. Let us already note here that this argument can be generalised in a straightforward way to  $N$  parties: any mixture of an  $N$ -party  $W$  state and the product state  $|00\dots 0\rangle$  is genuinely multipartite entangled.

Let us now turn to the general case of a spin chain with  $N$  sites. Here we will focus on the choice  $\Delta=-J$ , namely the XX model, with periodic boundary conditions, described by the Hamiltonian

$$H_{XX} = \sum_{i=1}^N [J(\sigma_i^x \otimes \sigma_{i+1}^x + \sigma_i^y \otimes \sigma_{i+1}^y) + B\sigma_i^z]. \quad (5)$$

Again, we consider the ferromagnetic case  $J<0$ , and  $B>0$ . As in the case of the XXZ model with three sites,  $[H_{XX}, \sigma_{tot}^z]=0$  holds, where  $\sigma_{tot}^z = \sum_{i=1}^N \sigma_i^z$ . Hence, the Hilbert space decomposes into invariant subspaces, each corresponding to a distinct eigenvalue of  $\sigma_{tot}^z$  which we denote as  $m$ . It represents the total number of excitations (down-spins) in the chain. There are  $N+1$  such subspaces, corresponding to spin configurations with  $m=0, 1, 2, \dots, N$ . We refer to the subspace corresponding to a particular  $m$  as the  $m$ -excitation subspace. The subspaces for which  $m=0$  and  $m=N$  are one-dimensional. The eigenstates of the Hamiltonian  $H$  in these subspaces are given by the product states  $|000\dots 0\rangle$  and  $|111\dots 1\rangle$ , respectively. The corresponding eigenvalues are

$$E^{(0)} = -NB \text{ and } E^{(N)} = +NB. \quad (6)$$

The energy eigenvalues for the  $m$ -excitation subspace can be obtained using the standard fermionization technique, introduced by Lieb *et al.* and Katsura [21]. The eigenvalues are labeled by  $m$  distinct quantum numbers  $k_1, k_2, \dots, k_m \in \{1, 2, \dots, N\}$  and can be expressed as

$$E_{k_1, k_2, \dots, k_m}^{(m)} = 4J \left[ \cos\left(\frac{2\pi k_1}{N}\right) + \cos\left(\frac{2\pi k_2}{N}\right) + \dots + \cos\left(\frac{2\pi k_m}{N}\right) \right] - (N-2m)B. \quad (7)$$

In particular, for the single excitation subspace,

$$E_k^{(1)} = 4J \cos\left(\frac{2\pi k}{N}\right) - (N-2)B, \quad (8)$$

where  $k=1, \dots, N$ . The corresponding energy eigenstates are given by

$$|\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{2\pi i k n/N} |n\rangle, \quad (9)$$

where the states  $|n\rangle$ , with  $n=1, 2, \dots, N$ , correspond to spin configurations in which all spins are up, apart from the spin at the site  $n$  which is down. For example, for  $N=3$  we have  $|1\rangle=|100\rangle$ ,  $|2\rangle=|010\rangle$  and  $|3\rangle=|001\rangle$ . These states form a complete basis in the single excitation subspace.

Let us now look at the ground state of the chain as a function of the magnetic field  $B$ . If the chain is held in a very strong magnetic field  $B \gg J$  then its ground state is the product state of the form  $|000\dots 0\rangle$  and energy  $E^{(0)} = -NB$ . However, when we decrease  $B$ , keeping  $J$  fixed, then the lowest energy single excitation eigenstate  $|\phi_N\rangle$ , which from now on we will label as  $|W_N\rangle$ ,

$$|W_N\rangle = \frac{1}{\sqrt{N}} [ |1\rangle + |2\rangle + |3\rangle + \dots + |N\rangle ], \quad (10)$$

becomes the new ground state with energy  $E_N^{(1)} = 4J - (N-2)B$ . The crossover from the product state  $|000\dots 0\rangle$  to the entangled state  $|W_N\rangle$  occurs at the critical value of the field  $B_c = -2J$ , which is independent of  $N$ . The new ground state  $|W_N\rangle$  is a generalization of the tripartite entangled  $W$  state to  $N$  spins and is genuinely  $N$ -partite entangled.

This is the first crossover because in our particular case of  $J < 0$  the lowest energy in the  $m$ -excitation subspace (for  $m \geq 2$ ) is lower bounded by  $4Jm - (N-2m)B$  [see Eq. (7)], which can be written as  $E_N^{(1)} + 2(m-1)(2J+B)$  and is certainly greater than  $E_N^{(1)}$  for all values of  $B \geq -2J$ .

Note that the value  $B_c = -2J$  corresponds to the critical point for the Bose Hubbard model with infinite on-site repulsion energy of the bosons [22], namely to the quantum phase transition from the Mott insulator to superfluid phase. We also point out that, as explained below Eq. (3), at finite temperatures and near  $B_c = -2J$  there remains genuine multipartite entanglement.

In order to generate the state  $|W_N\rangle$  from the state  $|00\dots 0\rangle$  by lowering the magnetic field  $B$  from an initial large value, it is necessary to turn the level crossing at  $B_c = -2J$  into an avoided crossing. An avoided crossing can be realized by adding a small perturbation to the Hamiltonian  $H_{XX}$ , e.g., a term  $B' \sum_{i=1}^N \sigma_i^x$ , with  $B' \ll B$ . In the presence of an avoided crossing, a transition from  $(00\dots 0)$  to  $|W_N\rangle$  can be achieved by lowering the magnetic field slowly enough, so that the ground state adiabatically follows [23]. The required rate of change of  $B$  depends inversely on the gap of the avoided crossing. Using degenerate perturbation theory, this gap is found in first order to be equal to  $2B'\sqrt{N}$ .

Before concluding, let us make a brief remark about the ground state for values of the magnetic field in the range  $B$

$< -2J$ . It turns out that the ground state changes successively from one excitation number to the next-higher one when decreasing the value of the magnetic field  $B$ . To show this, let us calculate the crossings of the energy levels for different excitation numbers (for large  $N$ ). It is straightforward to see that the lowest energy in the  $m$ -excitation subspace, for *odd* values of  $m$ , is given by

$$E_{\text{odd}}^{(m)} = 4J \left[ 1 + \sum_{j=1}^{[m/2]} 2 \cos\left(\frac{2\pi j}{N}\right) \right] - (N-2m)B, \quad (11)$$

whereas, for *even* values of  $m$ , it is given by

$$E_{\text{ev}}^{(m)} = 4J \left[ 1 + \sum_{j=1}^{m/2-1} 2 \cos\left(\frac{2\pi j}{N}\right) + \cos\left(\frac{\pi m}{N}\right) \right] - (N-2m)B. \quad (12)$$

From this it follows that the crossing  $E^{(\tilde{m}+1)} = E^{(\tilde{m})}$  occurs at  $B_c^{(m)} = -2J [1 - (2m^2 \pi^2)/N^2]$ . Thus, with the addition of a suitable small perturbation, and by adiabatically decreasing the magnetic field, transitions through a cascade of ground states with increasing excitation numbers can be obtained.

In conclusion, we have shown that a multipartite entangled state of the  $W$ -type occurs naturally as a ground state in the ferromagnetic XX spin chain with  $N$  sites in an external magnetic field.  $W$  states are a useful resource for quantum information processing tasks, e.g., quantum teleportation and dense coding [24]. Our analysis suggests a new method of generating an  $N$ -partite entangled  $W$  state, namely by driving the chain adiabatically through an avoided level crossing. This amounts to preparation of the initial product state  $|000\dots 0\rangle$  in a strong magnetic field,  $B \gg J$  and then slowly reducing the strength of the field, in the presence of a small perturbation, until it passes through the value  $B_c = -2J$  and generates the  $N$ -partite entangled  $W$  state. (Let us mention in passing that the concurrence, which measures entanglement between any two qubits in the chain, has the value  $2/N$ . In this sense the  $W$  state also carries bipartite long-range entanglement.) This method does not require any dynamical control over the couplings in the spin chain. Only one external global parameter, namely the magnetic field  $B$ , has to be modified. Thus, our result opens new possibilities for the creation of multipartite entanglement in condensed matter physics. Moreover, if each qubit in the chain can be controlled separately then this transition can be achieved by local operations, i.e., by reducing the field locally at each on the  $N$  sites of the chain. This does not imply that entanglement can be created by local operations as qubits do interact with each other, but it opens new possibilities of manipulating multipartite quantum entanglement.

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