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How the First Partial Transpose was Written*

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We tell the tale of the first writing of a partial transpose, without guaranteeing historical authenticity.

KEY WORDS: quantum entanglement; quantum information; time reversal.

In the high and far-off times nobody knew how to distinguish between entangled and separable quantum states. Many people were interested in the phenomenon of entanglement,(1) however it was regarded a topic where Nature presents its mysterious side. But there was the Tiger's Child(2) who was full of 'satiable curiosity, and that means he asked ever so many questions. (The name of the Tiger's Child begins with "A", but in order to protect his anonymity his full name will not be given here(3).)

One fine morning this 'satiable Tiger's Child asked a new fine question that he had never asked before. He asked: "Given a quantum state, how can I find out whether it is entangled?" What was the reason for this question? Usually, when the Tiger's Child strolled around to explore the world, his mother supplied him with a set of entangled qubits. Entangled quantum states can be used as direction indicators.(4) In this way his mother was sure that he would not get lost. In addition, if he found some interesting quantum state in his explorations he could teleport the state to her.(5) Of course, he already knew that unfortunately he could not use the entangled states to communicate superluminally with his mother.(6)

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* Dedicated to Prof. Asher Peres, Haifa, on the occasion of his 70th birthday.

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One fine morning, however, walking towards the banks of the great grey-green Limpopo River, the Tiger's Child suddenly started to worry about his entangled states and whether they might loose their entanglement due to interaction with the environment. Of course, he knew the tool of Bell inequalities[7] as a test of entanglement properties, but they were not sufficient because they do not detect all entangled states: due to his 'satiable curiosity he wanted to know more. He asked everybody on his way the above-mentioned question, but nobody knew a satisfactory answer.

Another fine morning, sitting again by the Limpopo river and watching thoughtfully his mirror image in the still waters, the Tiger's Child had an idea: "what happens in a system of two spins $\frac{1}{2}$, shared between two parties, if one party, let's say "A" (i.e. me), reflects its spin components at a plane, i.e. takes the mirror image of the Bloch vector? If the two subsystems are not entangled, and their density matrix can thus be written as a mixture of projectors onto product states, this partial operation will again lead to some valid density matrix of the whole system. However, if the spins are entangled, it might matter whether A uses a left-handed coordinate system to describe the Bloch vector, and the other party, let's say "B", uses a right-handed one, or whether both use the same type of coordinate system."

He had already learned quantum mechanics, so he had the general shape of a density matrix of two spins $\frac{1}{2}$ clearly in his mind:

$$\varrho^{AB}=\frac{1}{4}\begin{pmatrix} (s_z^A+s_z^B+t_{zz}+1) & (s_x^B+is_y^B+t_{zx}+it_{zy}) & (s_x^A+is_y^A+t_{xz}+it_{yz}) & (t_{xx}-t_{yy}+it_{xy}+it_{yx}) \\ (s_x^B-is_y^B+t_{zx}-it_{zy}) & (s_z^A-s_z^B-t_{zz}+1) & (t_{xx}+t_{yy}-it_{xy}+it_{yx}) & (s_x^A+is_y^A-t_{xz}-it_{yz}) \\ (s_x^A-is_y^A+t_{xz}-it_{yz}) & (t_{xx}+t_{yy}+it_{xy}-it_{yx}) & (s_z^B-s_z^A-t_{zz}+1) & (s_x^B+is_y^B-t_{zx}-it_{zy}) \\ (t_{xx}-t_{yy}-it_{xy}-it_{yx}) & (s_x^A-is_y^A-t_{xz}+it_{yz}) & (s_x^B-is_y^B-t_{zx}+it_{zy}) & (-s_z^A-s_z^B+t_{zz}+1) \end{pmatrix},$$

where $s_i^{A,B}$ are the components of the Bloch vectors of A and B, and t_{ij} denotes the components of the correlation tensor.

From here, it was pretty easy to deduce what happens if only A reflects his spin components at a plane, or if only B does so: partial reflection at the yz-plane corresponds to reversing the sign of the coefficients in front of the Pauli matrix σ_x for one of the two parties. Reflection on A's side thus means $s_x^A \to -s_x^A$ and $t_{xi} \to -t_{xi}$ in the density matrix ϱ_{AB}. The Tiger's Child took a stick and started to draw symbols in the sand of the bank of the Limpopo river. His drawings looked like in Figs. 1 and 2.

Here, the entries "circle" stand for matrix elements that remain unchanged. The entries "crosses" have to be exchanged as indicated by the arrows. The minus sign next to the arrows means that not only the entries have to be exchanged, but they also gain a minus sign.

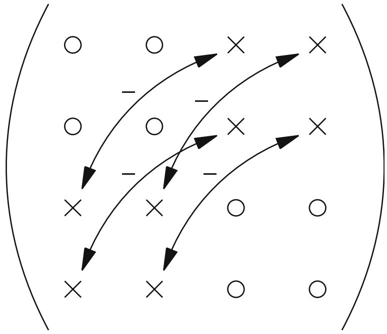

Fig. 1. Spin reflection at the yz-plane for A.

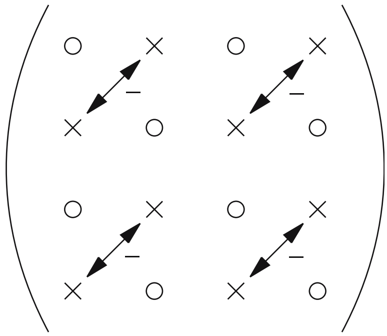

Fig. 2. Spin reflection at the yz-plane for B.

The Tiger's Child looked at the drawing and was not very happy: it looked rather complicated. He especially did not like the minus signs. One can easily make mistakes with minus signs. Therefore he tried reflection at the xy-plane, i.e. reversing the z-direction of the spin, in the same way. (He also did not like imaginary numbers, and therefore wanted to keep things real.)

For this type of reflection he arrived at Figs. 3 and 4. Again he was not completely happy, because every mixed density matrix has at least two entries on the diagonal, so these partial reflections would involve some effort in any case. Therefore, finally he tried a transformation of the y-direction, i.e. reflection at the xz-plane, and was drawing Figs. 5 and 6 in the sand. This was aesthetically pleasing, and a simple recipe to remember as a separability check: if there was no entanglement in the system, after this reflection the density matrix should still remain a proper density matrix. Any negative eigenvalue would indicate entanglement.

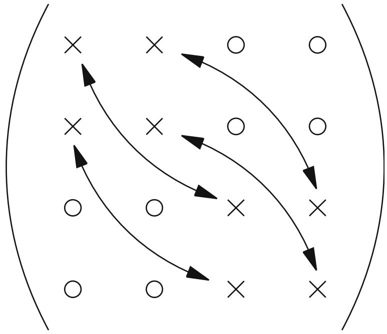

Fig. 3. Spin reflection at the xy-plane for A.

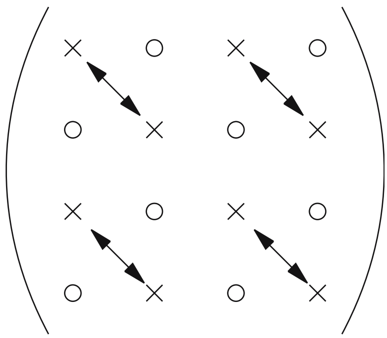

Fig. 4. Spin reflection at the xy-plane for B.

Looking again thoughtfully across the Limpopo river and watching a branch drifting along in the steady stream, he played with the thought of what would happen if the river would flow backwards. Or what might happen if one could reverse the direction of time, just for a little while. Suddenly the Tiger's Child realised that all the transformations that he had studied so far had something to do with time reversal:[(8)] they were all antiunitary. Naturally, he remembered Wigner's theorem[(9)] which says that a probability preserving transformation is either unitary or antinunitary. Any antiunitary transformation can be written as a product of a unitary transformation and time reversal.[(10)] Thus, it did not matter which of the three possibilities he used. He suddenly started to like the last transformation of the y-direction, because it could be expressed in a very simple way as a transpose. And if it was applied to only one of the subsystems, it was a *partial transpose*. The first partial transpose was written.[(11)]

The Tiger's Child went home to share his answer with everybody. Many people found it very useful and started to think more about it. At

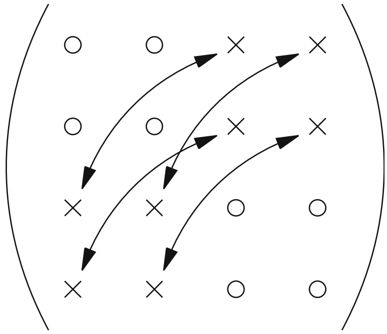

Fig. 5. Spin reflection at the xz-plane for A.

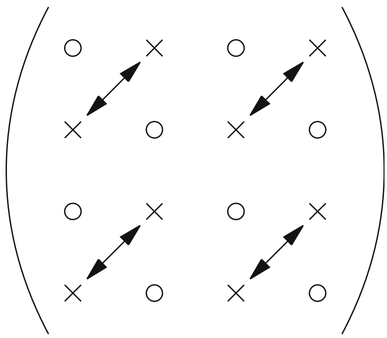

Fig. 6. Spin reflection at the xz-plane for B.

last things grew so exciting that his dear friends went off one by one in a hurry to the banks of the great grey–green Limpopo River, to find new answers to new questions.

A major interest was to understand what happens when the dimension of the quantum systems is higher than two. The Tiger's Child and his friends actually knew that playing with systems with higher dimensions gives some advantages, for example it allows to increase the robustness of secret communication methods among them.(12) Therefore, starting from the discoveries of the Tiger's Child, some friends realised that the partial transpose operation is a necessary and sufficient condition for entanglement only when the system is composed of two qubits or one qubit and one qutrit.(13) They also discovered that in higher dimensions Nature is more complicated, and new exciting aspects of entanglement were found: for example, there are states that are entangled but cannot be distilled,(14) and they were called bound entangled states. The Tiger's Child and his friends played a bit with these particular entangled states and scribbled on the sands of the bank of the Limpopo river some interesting forms of them.[15]

Ever since that day, all the density matrices that one is interested in, besides those ones one is not interested in, can be tested for separability with the partial transpose, precisely like the 'satiable Tiger's Child did.

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Note added: The Tiger's child was so much distracted by the separability question that he completely forgot to ask the Crocodile what it has for dinner. This explains why the Tiger does not have a trunk.

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