

BINARY COMMUNICATION IN ENTANGLED CHANNELS

MATTEO G A PARIS

Quantum Optics & Information Group, INFN Unità di Pavia, Italy

E-mail: paris@unipv.it, URL: www.qubit.it/~paris

We analyze optical binary communication assisted by entanglement and show that: i) ideal entangled channels have smaller error probability than ideal single-mode coherent channels if the photon number of the channel is larger than one; ii) realistic entangled channels with heterodyne receivers have smaller error probability than ideal single-mode coherent channels if the photon number of the channel is larger than a threshold of about five photons.

1 Introduction

In order to convey classical information to a receiver using quantum channels a transmitter prepares a quantum state drawn from a collection of known states. The receiver detects the information by measuring the channel, such to determine the state prepared. Since the given states are generally not orthogonal, then no measurement will allow the receiver to distinguish perfectly between them. The problem is therefore to construct a measurement optimized to distinguish between nonorthogonal quantum states, and to find realistic signals that minimize the error probability at fixed energy of the channel.

In optical binary communication, say amplitude modulation keyed (AMK) ^a, information is conveyed by two quantum states $\varrho_j = |\psi_j\rangle\langle\psi_j|$, $j = 1, 2$ with $|\psi_1\rangle = |\psi_0\rangle$ and $|\psi_2\rangle = D(\alpha)|\psi_0\rangle$ where $|\psi_0\rangle$ is a given reference state, usually taken as the vacuum. The amplitude a may be taken as real without loss of generality, and $D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a)$ is the displacement operator. If we consider equal *a priori* probabilities for the two signals, the optimal quantum measurement to discriminate the $|\psi\rangle$'s with minimum error probability ¹ is the POVM $\{M_j\}_{j=1,2}$, $M_1 + M_2 = I$ corresponding to the so-called square-root measurement ². We have $M_j = |\mu_j\rangle\langle\mu_j|$ with $|\mu_j\rangle = \sum_k \mu_{kj} |k\rangle$, $\mu_{kj} = [\Psi(\Psi^\dagger\Psi)^{-1/2}]_{kj}$. $[\Psi]_{ij} = \psi_{kj}$ is the matrix of the coefficients of the two signals $|\psi_j\rangle = \sum_k \psi_{kj} |k\rangle$ in a given basis $\{|k\rangle\}$. The error probability is given by

$$P_e = \frac{1}{2} \text{Tr}[M_1\varrho_2 + M_2\varrho_1] = \frac{1}{2} \left[1 - \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2} \right].$$

For AMK binary communication we have $|\langle\psi_1|\psi_2\rangle|^2 = \exp(-2N)$, where N is the average number of photons in the channel *per use* $N = \frac{1}{2} \text{Tr}[a^\dagger a(\varrho_1 + \varrho_2)] = \frac{1}{2} |\alpha|^2$. In the following we will refer to this quantity as *the photon number of the channel*.

^aAn equivalent analysis may be performed for phase-shift keyed signals.

In this communication we describe how binary communication can be improved by using realistic sources of entanglement, either considering ideal or heterodyne receivers for state discrimination. The corresponding error probabilities are denoted by Q_e and R_e respectively. We find that entanglement is convenient unless the photon number of the channel is very small.

2 Binary communication in entangled channels

Binary optical communication assisted by entanglement may be implemented using as a reference state the so-called twin-beam (TWB) state of two modes of radiation $|\psi_0\rangle = |\lambda\rangle = \sqrt{1-\lambda^2} \sum \lambda^p |p\rangle|p\rangle$, $|\lambda| < 1$. TWBs are produced by spontaneous downconversion in a nondegenerate optical parametric amplifier. The TWB parameter is given by $\lambda = \tanh G$, G being the effective gain of the amplifier. The two states to be discriminated are now given by $|\psi_1\rangle = |\lambda\rangle$ and $|\psi_2\rangle = D(\alpha)|\lambda\rangle$, where we consider the displacement performed on the beam a . Since the average number of photons of TWB is given by $N_\lambda = 2\lambda^2/(1-\lambda^2)$, the photon number of TWB channels is given by $N = N_\lambda + \frac{1}{2}|\alpha|^2$. The error probability for the ideal discrimination of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ reads as follows $Q_e = \frac{1}{2} \left[1 - \sqrt{1 - \exp[-2N(1-\beta)(1+\beta N)]} \right]$, where $\beta = N_\lambda/N$ is the fraction of the photon number of the channel used to establish entanglement between the two modes. We have that $Q_e < P_e$ for $N > (1-\beta)^{-1}$ *i.e.* entanglement is always convenient if the photon number of the channel is larger than one. The optimal entanglement fraction is given by $\beta_{opt} = (N-1)/(2N)$, corresponding to an error probability given by $Q_e = P_e$ if $N < 1$ and

$$Q_e = \frac{1}{2} \left[1 - \sqrt{1 - \exp[-\frac{1}{2}(1+N)^2]} \right] \quad N \geq 1.$$

In assessing the usefulness of entanglement in binary communication it should be taken into account that for the single-mode coherent channels the performances of the ideal detector can be in principle achieved by means of the Dolinar's receiver³. On the other hand, it has not been so far suggested how to achieve ideal performances for TWB entangled channels. Therefore, a question arises whether or not entanglement could be practically used to improve binary communication.

In order to answer to this question we consider a receiver measuring the real and the imaginary part of the complex operator $Z = a + b^\dagger$, a and b being the two modes of the TWB. The measurement of $\text{Re}[Z]$ and $\text{Im}[Z]$ can be experimentally implemented, and corresponds to multipoint homodyne detection if the two involved modes have the same frequencies⁴, or to heterodyne detection otherwise⁵. The outcome of each Z measurement is a complex number z , and the POVM of the measurement is given by $\Pi_z = |z\rangle\langle z|$, with $|z\rangle = D_j(z) \sum_n |n\rangle|n\rangle$ and either $j = a$ or $j = b$.

As inference rule we adopt the condition " $\text{Re}[z] > \Lambda \rightarrow |\psi_2\rangle\rangle$ ", where Λ is a threshold value, to be determined such to minimize the probability of error R_e . Since the heterodyne distribution, conditioned to a displacement α , is given by $p(z|\alpha) = |\langle z|D(\alpha)|\lambda\rangle|^2 = 1/\Delta_\lambda \exp\{-|\alpha - z|^2/\Delta_\lambda^2\}$ with $\Delta_\lambda^2 = (1 - \lambda)/(1 + \lambda) = (\sqrt{N\lambda} + 2 - \sqrt{N\lambda})/(\sqrt{N\lambda} + 2 + \sqrt{N\lambda})$, we have $R_{12} = \int_0^{\text{Re}[z] > \Lambda} d^2z p(z|\alpha)$ and $R_{21} = \int_{\text{Re}[z] > \Lambda}^\infty d^2z p(z|0)$ such that

$$R_e = \frac{1}{2}(R_{12} + R_{21}) = 1 - \frac{1}{2} \left\{ \text{Erf} \left[\frac{\Lambda}{\Delta_\lambda} \right] + \text{Erf} \left[\frac{\alpha - \Lambda}{\Delta_\lambda} \right] \right\},$$

where $\text{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}$ denotes the error function. R_e is minimized by the choice $\Lambda = \alpha/2$, and therefore the minimum probability of error in a realistic entangled channel with heterodyne receiver is given by $R_e = \frac{1}{2}[1 - \text{Erf}(\frac{1}{2}\alpha/\Delta_\lambda)]$. Error probability R_e can be further minimized by tuning the entanglement fraction β . Substituting in R_e the expression of $\alpha = \sqrt{2N(1 - \beta)}$ and Δ_λ we obtain

$$R_e = \frac{1}{2} \left\{ 1 - \text{Erf} \left[\frac{1}{2} \sqrt{\frac{2N(1 - \beta)(\sqrt{\beta N} + 2 - \sqrt{\beta N})}{(\sqrt{\beta N} + 2 + \sqrt{\beta N})}} \right] \right\}.$$

The optimal entanglement fraction is given by $\beta_{opt} = \frac{1}{2}N/(1 + N)$ which maximizes the argument of Erf, and thus minimizes R_e . We do not report the resulting error probability, whose expression is rather cumbersome. Rather, in Fig. 1 we report the error probabilities as a function of the photon number of the channel. We have that $R_e < P_e$ for N larger than the threshold value $N \simeq 5.2$. As it is apparent from the plot, although TWB entangled channels with heterodyne receiver do not approach the ideal performances, they definitely show smaller error probability than single-mode coherent channels

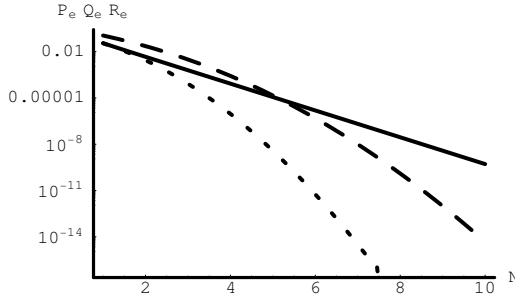


Figure 1. Logarithmic plot of the error probabilities as a function of the photon number of the channel. The solid line is the error probability P_e for a single-mode coherent channel, dotted line is Q_e for ideal entangled channels, and dashed line for R_e of heterodyne entangled channels. $R_e < P_e$ for N larger than the threshold $N \simeq 5.2$.

Asymptotically, for large photon number of the channel, we have $P_e \simeq \frac{1}{4}e^{-2N}$, $Q_e \simeq \frac{1}{4}e^{-\frac{1}{2}(1+N)^2}$ and $R_e \simeq (\sqrt{2\pi}N)^{-1}e^{-\frac{1}{8}N^2}$.

3 Conclusions

We have analyzed binary communication in TWB-based entangled channels, and compared their performances with ideal single-mode coherent channels. We have found that ideal entangled channels show smaller error probability than single-mode ones if the photon number of the channel is larger than one, whereas realistic entangled channels, *i.e.* channels equipped with heterodyne receivers, show smaller error probability than single-mode ones if the photon number of the channel is larger than a threshold of about five photons. We conclude the TWB binary communication represents a realistic alternative to single-mode coherent channels.

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