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INFORMATION-DISTURBANCE TRADEOFF IN COVARIANT QUANTUM STATE ESTIMATION

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We provide a general framework for quantifying the optimal tradeoff between the information retrieved by a quantum measurement and the disturbance on the quantum state in covariant quantum state estimation.

There exists a precise tradeoff between the amount of information extracted from a quantum measurement and the amount of disturbance caused on the system, analogous to Heisenberg relations holding in the preparation procedure of a quantum state. The study of such a tradeoff is relevant for both foundation and its enormous relevance in practice, in the realm of quantum key distribution and quantum cryptography. Quantitative derivations of such a tradeoff have been obtained in the scenario of quantum state estimation, in estimating a single copy of a pure state [1], many copies of identically prepared pure qubits [2], a single copy of a pure state generated by independent phase-shifts [3], an unknown maximally entangled state [4], coherent state [5], and spin-coherent state [6], and in state discrimination of two pure states [7].

In this paper, we provide a unified framework to study the optimal tradeoff between information and disturbance for set of states with given symmetry. Our results will be obtained by exploiting the group symmetry of the problem, which allows us to restrict our analysis on *covariant measurement instruments*. In fact, the property of covariance generally leads to a striking simplification of problems that may look intractable, and has been thoroughly used in the context of state and parameter estimation.

The problem of the tradeoff in covariant state estimation is the following. One performs a measurement on a quantum state picked randomly from a known set, and evaluates the retrieved information along with the disturbance caused on the state. To quantify the tradeoff between information and disturbance, one can adopt two mean fidelities [1]: the estimation fidelity G, which evaluates on average the best guess we can do of the original state on the basis of the measurement outcome, and the operation fidelity F, which measures the average resemblance of the state of the system after the measurement to the original one. The solution of the optimal tradeoff provides a set of minimum-disturbing measurements, that for fixed value of G maximizes F.

A measurement process on a quantum state ρ with outcomes $\{r\}$ is described by an *instrument*, namely a set of trace-decreasing completely positive (CP) maps $\{\mathcal{E}_r\}$. Each map can then be written in the Kraus form

$$\mathcal{E}_r(\rho) = \sum_\mu A_{r\mu} \rho A^\dagger_{r\mu} \;, \tag{1}$$

and provides the state after the measurement $\rho_r = \frac{\mathcal{E}_r(\rho)}{\mathrm{Tr}[\mathcal{E}_r(\rho)]}$, along with the probability of outcome $p_r = \mathrm{Tr}[\mathcal{E}_r(\rho)] = \mathrm{Tr}\left[\sum_\mu A^\dagger_{r\mu} A_{r\mu} \rho\right]$.

The set of positive operators $\{\Pi_r = \sum_\mu A^\dagger_{r\mu} A_{r\mu}\}$ is known as positive operator-valued measure (POVM), and normalization requires the completeness relation $\sum_r \Pi_r = I$. This is equivalent to require that the map $\sum_r \mathcal{E}_r$ is trace-preserving.

We are interested in covariant state estimation, a problem where we want to estimate a quantum state that belongs to a covariant set of states

$$\{|\psi_g\rangle = U_g|\psi_0\rangle\}\,, \tag{2}$$

where $|\psi_0\rangle$ is a fixed reference state of a Hilbert space $\mathcal{H}$ with finite dimension $\dim(\mathcal{H}) = d$, and U_g, $g \in \mathbf{G}$, is unitary representation of a group $\mathbf{G}$ on $\mathcal{H}$. We will consider only compact groups (which then admit normalizable invariant Haar measure dg), and for convenience we will take the normalized invariant Haar measure over the group, i.e. $\int_{\mathbf{G}} dg = 1$. The unknown state is then randomly distributed according to dg.

The operation fidelity F evaluates on average how much the state after the measurement resembles the original one, in terms of the squared modulus of the scalar product. Hence, for a measurement described by (1) one has

$$F = \int dg \sum_{r\mu} |\langle\psi_g|A_{r\mu}|\psi_g\rangle|^2\,. \tag{3}$$

By adopting a guess function f, for each measurement outcome r one guesses a spin coherent states $|\psi_{f(r)}\rangle$, and the corresponding average estimation fidelity is given by

$$G = \int dg \sum_{r\mu} \langle\psi_g|A^\dagger_{r\mu} A_{r\mu}|\psi_g\rangle\, |\langle\psi_{f(r)}|\psi_g\rangle|^2\,. \tag{4}$$

We are interested in the optimal tradeoff between F and G, and without loss of generality we can restrict our attention to *covariant* instruments, that satisfy

$$\mathcal{E}_h(U_g\,\rho\, U_g^\dagger) = U_g \mathcal{E}_{g^{-1}h}(\rho) U_g^\dagger\,. \tag{5}$$

In fact, for any instrument $\{A_{r\mu}\}$ and guess function f the covariant instrument

$$\mathcal{E}_h(\rho) = \sum_{r\mu} U_h U^\dagger_{f(r)} A_{r\mu} U_{f(r)} U_h^\dagger\, \rho\, U_h U^\dagger_{f(r)} A^\dagger_{r\mu} U_{f(r)} U_h^\dagger \tag{6}$$

with continuous outcome $h \in \mathbf{G}$, along with the guess $|\psi_h\rangle$, provides the same values of F and G as the original instrument. Moreover, for covariant instruments the optimal guess function automatically turns out to be the identity function.

It is useful now to consider the Jamiołkowski representation that gives a one-to-one correspondence between a CP map $\mathcal{E}$ from $\mathcal{H}_{in}$ to $\mathcal{H}_{out}$ and a positive operator R on $\mathcal{H}_{in} \otimes \mathcal{H}_{out}$ through the equations

$$\begin{aligned} \mathcal{E}(\rho) &= \mathrm{Tr}_{in}[(\rho^\tau \otimes I_{out})R]\,, \\ R &= (I_{in} \otimes \mathcal{E})|\Phi\rangle\langle\Phi|\,, \end{aligned} \tag{7}$$

where $|\Phi\rangle = \sum_{i=1}^d |i\rangle \otimes |i\rangle$ represents the unnormalized maximally entangled vector of $\mathcal{H}_{in}^{\otimes 2}$, and τ denotes the transposition on the fixed basis. When $\mathcal{E}$ is trace preserving, correspondingly one has $\mathrm{Tr}_{out}[R] = I_{in}$.

For covariant instruments $\mathcal{E}_g$ the operator R_g has the form

$$R_g = U_g^* \otimes U_g \, R_0 \, U_g^\tau \otimes U_g^\dagger \ , \tag{8}$$

where $*$ denotes the complex conjugation, and the trace-preserving condition is given by

$$\int dg \, \mathrm{Tr}_{out}[R_g] = \int dg \, U_g^* \, \mathrm{Tr}_{out}[R_0] \, U_g^\tau = I_{in} \ . \tag{9}$$

The fidelities F and G in Eqs. (3) and (4) can be rewritten as follows

$$F = \int dg \int dh \, \langle\psi_g|\mathcal{E}_h(|\psi_g\rangle\langle\psi_g|)|\psi_g\rangle = \int dg \, \langle\psi_0|\mathcal{E}_g(|\psi_0\rangle\langle\psi_0|)|\psi_0\rangle \ , \tag{10}$$

$$G = \int dg \int dh \, |\langle\psi_g|\psi_h\rangle|^2 \, \mathrm{Tr}[\mathcal{E}_h(|\psi_g\rangle\langle\psi_g|)] = \int dg \, |\langle\psi_0|U_g|\psi_0\rangle|^2 \mathrm{Tr}[\mathcal{E}_g(|\psi_0\rangle\langle\psi_0|)] \ , \tag{11}$$

where the covariance property (5) and the invariance of the Haar measure have been used. Using the isomorphism (7), we can write F and G as $F = \mathrm{Tr}[R_F R_0]$ and $G = \mathrm{Tr}[R_G R_0]$, where R_F and R_G are the following positive operators

$$R_F = \int dg \, |\psi_g\rangle\langle\psi_g|^\tau \otimes |\psi_g\rangle\langle\psi_g| \ , \tag{12}$$

$$R_G = \mathrm{Tr}_{out}[(I_{in} \otimes |\psi_0\rangle\langle\psi_0|)R_F] \otimes I_{out} \ . \tag{13}$$

The optimal tradeoff between F and G can be found by maximizing the operation fidelity $F = \mathrm{Tr}[R_F R_0]$ versus R_0, for fixed value of the estimation fidelity $G = \mathrm{Tr}[R_G R_0]$ and under the constraint (9).

The evaluation of the constraint (9) and the operator R_F (and hence of R_G) needs group averages with the representations $V_g = U_g^*$ and $V_g = U_g^* \otimes U_g$, respectively. These can be obtained as follows. The Hilbert space can be decomposed into orthogonal subspaces

$$\mathcal{H} \equiv \bigoplus_{\mu \in S} \mathcal{H}_\mu \otimes \mathbb{C}^{m_\mu} \ , \tag{14}$$

where the sum runs over the set S of irreducible representations that appear in the Clebsch-Gordan decomposition of V_g. The action of the group is irreducible in each representation space $\mathcal{H}_\mu$, while it is trivial in the multiplicity space $\mathbb{C}^{m_\mu}$, which takes into account the presence of equivalent representations.

From Schur's lemma one has

$$\int dg \, V_g \, Y \, V_g^\dagger = \sum_\mu I_{d_\mu} \otimes \frac{\mathrm{Tr}_{\mathcal{H}_\mu}[Y P_\mu]}{d_\mu} \ , \tag{15}$$

where $\{P_\mu\}$ are the orthogonal projectors on the invariant subspaces $\mathcal{H}_\mu \otimes \mathbb{C}^{m_\mu}$, and I_{d_μ} denotes the identity operator on $\mathcal{H}_\mu$ (which has dimension d_μ).

When the irreducible representations of V_g are all inequivalent, notice that all m_μ are equal to one. This is the case of all examples given in the following.

i) For unknown pure state with $\dim(\mathcal{H}) = d$, one has $|\psi_g\rangle = U_g|\psi_0\rangle$, with $|\psi_0\rangle$ arbitrary, $U_g \in SU(d)$ and

$$R_F = \frac{1}{d(d+1)}(I+\mathcal{I}) \ , \tag{16}$$

$$R_G = \frac{1}{d(d+1)}(I+|\psi_0\rangle\langle\psi_0|^\tau)\otimes I \ , \tag{17}$$

where $\mathcal{I} = (\sum_{n=1}^{d} |n\rangle \otimes |n\rangle)\,(\sum_{m=1}^{d}\langle m| \otimes \langle m|)$.

ii) For an unknown maximally entangled state of $\mathcal{H}^{\otimes 2}$ with $\dim(\mathcal{H}) = d$, one has $\{|\psi_g\rangle = (U_g \otimes I)|\psi_0\rangle\}$ with $|\psi_0\rangle = \frac{1}{\sqrt{d}}\sum_{n=1}^{d}|n\rangle \otimes |n\rangle$, $U_g \in SU(d)$ and [4]

$$R_F = \frac{1}{d^2(d^2-1)}\left[I + \mathcal{I}^{(13)} \otimes \mathcal{I}^{(24)} - \frac{1}{d}(I^{(13)} \otimes \mathcal{I}^{(24)} + \mathcal{I}^{(13)} \otimes I^{(24)})\right] \ , \tag{18}$$

$$R_G = \frac{1}{d^2(d^2-1)}\left[\left(1-\frac{2}{d^2}\right) I + \frac{1}{d} I^{(12)} \otimes \mathcal{I}^{(34)}\right] \ . \tag{19}$$

where $\mathcal{I}^{(ij)} = (\sum_{n=1}^{d} |n\rangle_i \otimes |n\rangle_j)(\sum_{m=1}^{d} {}_i\langle m| \otimes {}_j\langle m|)$.

iii) For spin-j coherent states, $\{|\psi_g\rangle = U_g|-j\rangle\}$, where $|-j\rangle$ is the eigenvector of J_z with minimum eigenvalue, U_g is a $(2j+1)$-dimensional irreducible representation of $SU(2)$, and [6]

$$R_F = \frac{1}{4j+1} P_{2j}^{\theta} \ , \tag{20}$$

$$R_G = \frac{1}{4j+1} \mathrm{Tr}_2[(I \otimes |-j\rangle\langle -j|)P_{2j}] \otimes I \ , \tag{21}$$

where θ denotes the partial transpose on the first Hilbert space, and P_l represents the projector on the subspace of $\mathcal{H} \otimes \mathcal{H}$ with total spin l.

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