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Second harmonic generation: the solution for an amplitude-modulated initial pulse

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Abstract

We address the initial value problem for one-dimensional second harmonic generation starting from a purely amplitude-modulated fundamental wave. A general method to solve the problem in terms of a Schrödinger equation is presented, in which the initial pulse-shape is taken as a potential. Several examples with the complete solution given in analytical form are discussed. A much broader class of solutions can be found with the help of a single numerical integration. In particular, solutions with incident pulses approximating a sech^2-shape have been obtained. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In 1961 Franken et al. [1] found that a ruby laser beam in crystalline quartz generates a very weak beam of UV radiation, its frequency being two times that of the ruby laser. This event marked the beginning of nonlinear optics. Shortly afterwards, Terhune et al. [2] achieved a conversion efficiency of 20%, opening a wide range of possibilities for the experimental investigation of this phenomenon. The process of second harmonic generation (SHG) for a cw beam was theoretically studied by Armstrong et al. [3]. Later on, the laser pulses became shorter and shorter, and the walk-off of the pulses at fundamental and harmonic frequencies became important. In fact, with the advent of ultrashort pulses, non-stationary effects like group velocity mismatch have become a problem of interest nowadays [4,5]. Actually, even in a one-dimensional theory, which we have here in mind exclusively, nonlinear partial differential equations have to be solved.

In the case of a purely amplitude-modulated fundamental wave, the problem is governed by a second-order equation named after Liouville [6] (which should not be confused with the Liouville equation in Statistical Mechanics). Even though this connection has been known since a quarter of a century [7], to the best of our knowledge no attention has been paid to it in current text books on nonlinear optics, or in articles on SHG. This is rather

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surprising since the Liouville equation is one of the few examples of nonlinear equations for which the general solution can be written down explicitly. Such a solvable theory is very useful, and even under conditions where the inherent idealizations are only approximately fulfilled could be used as a zeroth-order approximation. Despite the general solvability it is not quite obvious how to select, among the solutions of the Liouville equation, the physically relevant SHG solutions. Therefore, it is a matter of interest to consider the following two questions, whose answers characterize the purpose of the present paper.

1. How to solve the initial value problem for an incident pulse of arbitrary shape at the fundamental frequency, under the condition that there is no incident harmonic wave?
2. Are there examples for which this problem has a fully analytical solution?

Both problems are of physical relevance and experimental interest [8–11,5,12], and to our knowledge the prescribed initial value problem has never been systematically studied. A particular type of analytical solutions with the initial pulse being of squared Lorentzian shape was found by Akhmanov et al. [13], cf., Section 4.3 in the present paper.

Throughout this work, we will discuss a more general approach and study this problem systematically. We will give several new analytical solutions for experimentally relevant initial pulses.

Our manuscript is organized as follows: In Section 2 we briefly review the basic equations describing one-dimensional SHG in the slowly varying amplitude approximation and establish the connection to the Liouville equation. Similarity transformations of evolution equations, which are useful in the subsequent analysis, are discussed in subsection 2.2. In subsection 2.3 we define Goursat and Cauchy problems. Section 3 is the main part of the paper. Here we show how the physically relevant initial value problem is reduced to the Schrödinger equation, where the initial pulse shape plays the role of a repulsive potential. In Section 4 several examples are analyzed in detail. Analytical solutions are given for many realistic cases and a broad class of exponentially decaying initial pulses is found with the help of a single numerical integration. In Section 5 we state our conclusions.

2. Second harmonic generation in one space dimension

2.1. SHG and the Liouville equation

In a second-order nonlinear medium, the interaction between two quasi-monochromatic plane electromagnetic waves, with slowly-varying complex electric field amplitudes $A_1(x,t), A_2(x,t)$ and respective frequencies ω_1 and $\omega_2 = 2\omega_1$, is described by the two differential equations [14]

$$\left(\partial_x + \frac{1}{v_1}\partial_t\right) A_1 = -\mathrm{i}\gamma A_2 A_1^* ,$$
$$\left(\partial_x + \frac{1}{v_2}\partial_t\right) A_2 = -\mathrm{i}\gamma A_1^2 . \qquad (1)$$

Here we assumed that the wave-numbers of both carrier waves fulfil the phase-matching condition $k_2 = 2k_1$. The star denotes complex conjugation. x,t are laboratory space and time coordinates. The coupling constant γ is expressed as

$$\gamma = \frac{2\pi\chi^{(2)}\omega_1^2}{k_1 c^2} . \qquad (2)$$

In order to write the equations of motion in a convenient form we introduce characteristic coordinates

$$\chi = \nu(-t + x/v_2) = -\nu\tau_2 , \quad \tau = \nu(t - x/v_1) = \nu\tau_1 , \qquad (3)$$

where the parameter ν describing the group velocity mismatch is given by

$$\nu = (1/v_2 - 1/v_1)^{-1} , \qquad (4)$$

and introduce new amplitudes q_1, q_2 by taking

$$A_1 = \sqrt{2}\,\gamma^{-1} q_1 , \quad A_2 = 2\mathrm{i}\gamma^{-1} q_2 . \qquad (5)$$

The inverse transformation of (3) is

$$x = \chi + \tau , \quad t = \chi/v_1 + \tau/v_2 , \qquad (6)$$

and the derivatives are transformed according to

$$\partial_\chi = \partial_x + \frac{1}{v_1}\partial_t , \quad \partial_\tau = \partial_x + \frac{1}{v_2}\partial_t . \qquad (7)$$

In this way we arrive at the differential equations

$$\left(\partial_x + \frac{1}{v_1}\partial_t\right) q_1 = \partial_\chi q_1 = -2 q_2 q_1^* ,$$
$$\left(\partial_x + \frac{1}{v_2}\partial_t\right) q_2 = \partial_\tau q_2 = q_1^2 . \qquad (8)$$

The scaling of the amplitudes is such that, up to a common scaling factor, $|q_1|^2$, $|q_2|^2$ may be interpreted as photon current densities. The physical conditions included in the present model can be summarized as follows:

1. Applicability of the slowly-varying amplitude approximation (SVA), i.e. the pulses should still be long compared to the wavelengths.
2. One-dimensionality in space, i.e. the transverse structure can be neglected.
3. The phase matching condition $2k_1 = k_2$ for the wave numbers $k_{1,2}$ are fulfilled exactly for the two carrier waves.
4. The group velocities v_1 and v_2 do not coincide, but
5. the dispersion within either of the two pulses at frequencies ω_0 and $2\omega_0$ can be neglected.

According to theoretical estimations [14] and experiments [5] these presumptions are quite realistic. E.g., for 100 fs pulses with intensities above 100 GW/cm^2 and long-focus conditions in KDP or $LiIO_3$ crystals, high conversion efficiencies can be achieved over crystal lengths of several mm. Both pump depletion and group velocity dispersion then become important and a non-stationary approach to the problem is required. One may also think of a realization in a planar optical waveguide.

The basis for applying the inverse scattering transform method to Eqs. (8) with complex amplitudes was established by Kaup [15]. Its full development, however, met some particular difficulties which have not yet been solved. Recently, Hamiltonian structures and particular solutions, not reducible to real ones, of (8) were established [16]. Here we focus our attention on purely amplitude-modulated signals. Thus the amplitudes q_1 and q_2 are real, and the stars in Eqs. (8) can be omitted. Clearly, this further approximation excludes important physical effects due to phase mismatch and related applications [17].

Let us now consider causality. In the $1+1$ dimensional world of the present SHG model, propagation occurs only with velocities v_1 and v_2. For definiteness we will assume $v_1 > v_2$, *normal group dispersion*, such that ν defined by (4) is positive. One should notice, however, that our results are easily transferred to $v_1 < v_2$, i.e., anomalous group dispersion. The *cone of future* from $x = t = 0$ is given by the region

$$v_2 t < x < v_1 t, \quad \text{i.e.,} \quad \chi > 0, \tau > 0. \tag{9}$$

The signs of the characteristic coordinates χ, τ are such that causal action always occurs in the direction of increasing coordinates.

It was found by Bass and Sinitsyn [7] that in the case of real amplitudes, the SHG problem of Eqs. (8) is solvable. Actually, it is ''*C*-integrable'' [18], which means integrable by change of variables. Indeed, one can see that from Eqs. (8) with real waves q_1, q_2 we may eliminate q_2 arriving at the Liouville equation

$$\partial_\chi \partial_\tau \ln\left(4 q_1^2\right) = -4 q_1^2. \tag{10}$$

The general solution is well known [6] and it is given by

$$q_1^2 = -\frac{1}{2} \frac{F'(\chi) G'(\tau)}{\left(F(\chi) + G(\tau)\right)^2}. \tag{11}$$

In Eq. (11) $F(\chi)$ and $G(\tau)$ are arbitrary functions, which depend only on χ and τ respectively. The primes denote differentiations. The solution is completed by substitution of Eq. (11) in Eqs. (8), leading to

$$q_2 = -\frac{1}{4} \frac{F''(\chi)}{F'(\chi)} + \frac{1}{2} \frac{F'(\chi)}{F(\chi) + G(\tau)}. \tag{12}$$

2.2. Similarity transformations

Given any solution $q_1(\chi,\tau)$, $q_2(\chi,\tau)$ of the equations of motion, a two-parameter manifold of solutions can be found by use of similarity (scale) transformations. Eqs. (8), in fact, are invariant under the two following scale transformations:

(i) The conformal transformation

$$\chi = a\tilde{\chi}, \quad \tau = a\tilde{\tau}, \quad \tilde{q}_1(\tilde{\chi},\tilde{\tau}) = a q_1(a\tilde{\chi}, a\tilde{\tau}),$$
$$\tilde{q}_2(\tilde{\chi},\tilde{\tau}) = a q_2(a\tilde{\chi}, a\tilde{\tau}). \tag{13}$$

(ii) The τ-dilatation

$$\tau = b^2\tilde{\tau}, \quad \tilde{q}_1(\tilde{\chi},\tilde{\tau}) = b q_1\left(\tilde{\chi}, b^2\tilde{\tau}\right), \tag{14}$$

and the general similarity transformation is obtained as a combination of both these types. Here a and b are real numbers. Under conformal transformations all the four quantities

$$\int \mathrm{d}\chi\, q_k(\chi,\tau), \quad \int \mathrm{d}\tau\, q_k(\chi,\tau), \quad k = 1,2, \tag{15}$$

are invariant while, on the other hand, the τ-dilatation does not change the integral

$$\int \mathrm{d}\tau\, q_1^2(\chi,\tau). \tag{16}$$

In the context of the Cauchy problem discussed below, it will be of interest to use only invariance transformations that map the set of straight lines $x \equiv \chi + \tau = \text{const}$ to themselves. This restriction is fulfilled by the conformal transformation, but not by the τ-dilatation. Thus any particular solution of Eqs. (8) represents a one-parameter family of solutions.

2.3. Goursat and Cauchy problems

To such a type of partial differential equations (8) one may relate two typical initial value problems:

(i) The *Goursat problem* in which initial values are given at characteristics

$$q_1(0,\tau) = q_{10}(\tau), \quad \tau > 0;$$
$$q_2(\chi,0) = q_{20}(\chi), \quad \chi > 0. \tag{17}$$

After substitution of the general solution (11), (12) in Eqs. (17) the functions $F(\chi)$ and $G(\tau)$ can be determined by quadratures.

(ii) The *Cauchy problem* in which the initial values are given on some line that is *not* a characteristic. This is usually the case, as from the physical point of view, it is natural to give both fields q_1 and q_2 for $x = 0$, i.e., according to Eqs. (3) and (6), for $\tau = -\chi$ and $t = \chi/\nu$,

$$q_1(-\tau,\tau) = q_{10}(\tau), \quad q_2(-\tau,\tau) = q_{20}(\tau). \tag{18}$$

In a common situation encountered in experiments [19] there is an incident ground wave, q_1, with no incident harmonic wave, q_2. This is the Cauchy problem specified by $q_{20} = 0$. In the following we will be concerned with this problem, and we will refer to it as the *restricted Cauchy problem.*

3. The restricted Cauchy problem

3.1. Solution

Let us write the restricted Cauchy problem in the form

$$q_1^2(-\tau,\tau)=I_1(\tau), \quad q_2(-\tau,\tau)=0. \tag{19}$$

Thus, starting from the general solution of Eqs. (11), (12), we have to determine the functions $F(\chi)$ and $G(\tau)$. Upon defining

$$K(\tau)\equiv F(-\tau), \tag{20}$$

we get

$$2I_1(\tau)=\frac{K'(\tau)G'(\tau)}{(K+G)^2}, \quad \varrho(\tau)=\frac{1}{2}\frac{K''}{K'}=\frac{K'}{K+G}. \tag{21}$$

By elimination of G in (21), we find

$$\{K,\tau\}\equiv\frac{K'''}{K'}-\frac{3}{2}\left(\frac{K''}{K'}\right)^2=-4I_1, \tag{22}$$

where the curly bracket denotes the Schwarzian derivative [20]. The function $\varrho(\tau)$, defined in the second of Eqs. (21), fulfils the Riccati equation

$$\varrho'=\varrho^2-2I_1, \tag{23}$$

which by taking

$$\varrho=-\phi'/\phi, \tag{24}$$

is connected to the Schrödinger-type equation

$$\phi''=2I_1\phi. \tag{25}$$

By comparison of Eqs. (24) and (21) we find

$$\frac{\phi'}{\phi}=-\frac{1}{2}\frac{K''}{K'}, \tag{26}$$

and, by integration,

$$\phi=\frac{1}{\sqrt{K'}}. \tag{27}$$

Summarizing we may formulate the following ‘‘recipe’’ for solving the restricted Cauchy problem :

1. Given an initial pulse shape $I_1(\tau)$ one first has to solve the second order differential equation (25), which can be viewed as a Schrödinger equation with repulsive potential $2I_1(\tau)$ and eigenvalue 0. This is also known as Hill’s equation [21]. We require ϕ to be a real function.
2. By means of Eq. (27), the function F can be evaluated as

$$K(\tau)=\int_0^\tau\frac{d\tau'}{\phi^2(\tau')}, \quad F(\chi)=K(-\chi). \tag{28}$$

3. Upon substitution in Eqs. (21) and (24) we arrive at the function G,

$$G(\tau)=-K(\tau)+\frac{2K'^2}{K''}=-F(-\tau)-\frac{1}{\phi(\tau)\phi'(\tau)}. \tag{29}$$

4. Eventually, the solution $q_1(\chi,\tau)$, $q_2(\chi,\tau)$ is found by substituting $F(\chi)$ and $G(\tau)$ in Eqs. (11) and (12),

$$q_1^2(\chi,\tau)=\frac{\phi''(\tau)\phi(\tau)}{2\phi^2(-\chi)}\times\left[\frac{1}{1+\phi(\tau)\phi'(\tau)[F(\tau)-F(-\chi)]}\right]^2, \tag{30}$$

$$q_2(\chi,\tau)=-\frac{1}{2\phi(-\chi)}\left[\phi'(-\chi)-\frac{\phi(\tau)\phi'(\tau)}{\phi(-\chi)}\times\frac{1}{1+\phi(\tau)\phi'(\tau)[F(\tau)-F(-\chi)]}\right]. \tag{31}$$

We also notice that the choice of a symmetric initial pulse $I_1(\tau)$ results in an even function $\phi(\tau)$. In this case $F(\chi)$ is an odd function, and the general solution in Eqs. (30), (31) can be written in the form

$$q_1^2(\chi,\tau)=\frac{\phi''(\tau)\phi(\tau)}{2\phi^2(\chi)}\times\left[\frac{1}{1+\phi(\tau)\phi'(\tau)[F(\tau)+F(\chi)]}\right]^2, \tag{32}$$

$$q_2(\chi,\tau)=\frac{1}{2\phi(\chi)}\left[\phi'(\chi)+\frac{\phi(\tau)\phi'(\tau)}{\phi(\chi)}\times\frac{1}{1+\phi(\tau)\phi'(\tau)[F(\tau)+F(\chi)]}\right]. \tag{33}$$

It may be worth noticing that $q_1^2(\chi,\tau)$ and $q_2(\chi,\tau)$ can also be expressed in terms of $\rho(\tau)$ and thus avoiding the wave function ϕ. From (21) and (29), in fact, we get

$$K(\tau)=\int_0^\tau\left[\exp\left(2\int_0^{\tau'}\varrho(\tau'')\,d\tau''\right)\right]d\tau',$$

$$G(\tau)=-K(\tau)+\frac{1}{\rho}\exp\left(2\int_0^\tau\rho(\tau')\,d\tau'\right). \tag{34}$$

The wave amplitudes q_1 and q_2 can be obtained by substitution of $F(\chi) \equiv K(-\chi)$ and $G(\tau)$ into Eqs. (11) and (12).

3.2. About uniqueness

In our procedure the solution of the restricted Cauchy problem is reduced to the solution of a Schrödinger (Hill) equation. The latter is by no means unique, because we did not impose any boundary or asymptotic condition. On the other hand, from physical intuition, we expect that there is a unique solution of the Cauchy problem in our case. How can this apparent discrepancy be resolved?

Given any particular (real) solution $\phi_1(\chi)$ to Eq. (25) the general solution is found as

$$\phi(\chi) = \phi_1(\chi)\left(c_1 + c_2\int_0^\chi \frac{d\chi'}{\phi_1^2(\chi')}\right), \tag{35}$$

c_1 and c_2 being real numbers. By taking the integral in Eq. (28), we obtain

$$F = c_3 - \frac{1}{c_1 + c_2 F_1} = \frac{aF_1 + b}{cF_1 + d}. \tag{36}$$

In Eq. (36) c_3 is an integration constant, and a,b,c,d are real numbers determined, up to an arbitrary common factor, by c_1,c_2,c_3. Eq. (36) tells us that F is determined by ϕ up to an arbitrary linear rational mapping. Indeed, it is known [20] – and could be checked directly – that the Schwarzian derivative is invariant under such a transformation, and thus the potential $I_1(\chi)$ in (22) is invariant. Moreover, starting from Eqs. (28) and (20), it can easily be derived that $-G$ and $-G_1$ are connected by the same linear rational transformation connecting F and F_1. The proof of uniqueness is completed by noting that the right-hand side of (11) is invariant under an arbitrary linear rational transformation simultaneously applied to both F and $-G$. Therefore, any solution of a given Hill equation leads to the same physical solution for the pulse amplitudes q_1,q_2. (Note that q_2 is determined by q_1.)

4. Examples

The formalism discussed in the previous sections can be applied to several pulse shapes, leading to fully analytical solutions for the Cauchy problem. Some of these pulses are presented and analyzed in this section. In subsection 4.4 we also consider a broad class of solutions, having exponential decay as a distinctive feature, which can be generated by a single numerical integration. In all the following examples we impose the condition that the second harmonic wave is zero at the boundary $x = 0$, corresponding to the natural experimental conditions. In the transformed frame this reads $q_2(\chi = -\tau,\tau) = 0$.

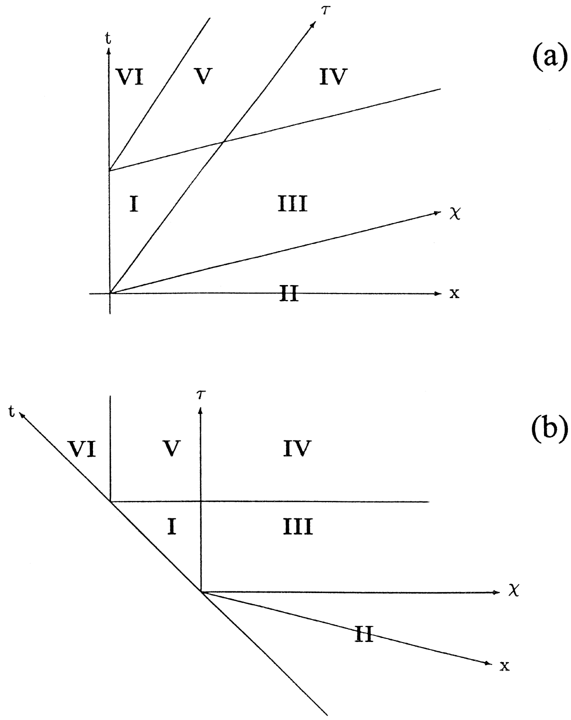

Fig. 1. Space-time regions I to VI of the solution for an initial square pulse at the fundamental frequency. In part (a) the lab coordinates (x,t) are taken as Cartesian ones while in part (b) the same holds for the characteristic coordinates (χ,τ). The interaction occurs in the regions I and II. There is no field in II and VI while in IV and V there is free propagation of the harmonic wave.

4.1. Square pulse

We first consider the initial square pulse given by

$$I_1(\tau) = \begin{cases} A & \text{for } 0 < \tau < \nu t_p \\ 0 & \text{elsewhere}. \end{cases} \tag{37}$$

The solution of this problem requires the definition of several spatio-temporal regions, which are shown in Fig. 1, both in the lab frame (x,t) and in the characteristic frame (χ,τ). The "pieces" of the solution in the respective regions are connected by the conditions that q_1 is continuous and differentiable in χ, whereas q_2 is continuous and differentiable in τ. From Eqs. (8) we get

$$\begin{aligned} &q_1 \equiv 0, \quad q_2 \equiv 0 \quad \text{in regions II and VI,} \\ &q_1 \equiv 0, \quad \partial_\tau q_2 \equiv 0 \quad \text{in regions IV and V.} \end{aligned} \tag{38}$$

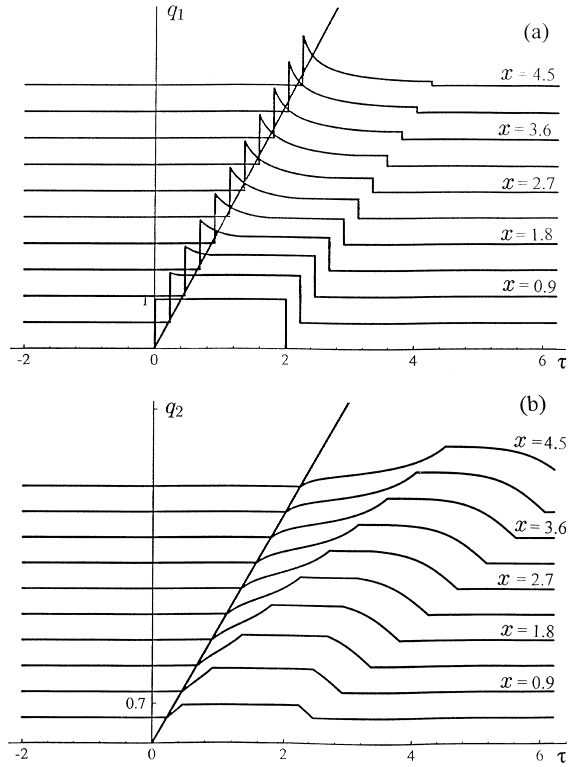

Fig. 2. Solution for an incident fundamental wave being a square pulse. Part (a) shows the fundamental amplitude q_1, part (b) shows the harmonic amplitude q_2. Note that, for $x = 0$, it holds $\tau = \nu t$ with ν being the group velocity mismatch defined by Eq. (4).

The dynamics occurs exclusively in regions I and III and the Cauchy problem of Eq. (37) reduces to the following problems:

(i) the Cauchy problem for the triangle I

$$q_1(-\tau,\tau) = A, \quad q_2(-\tau,\tau) = 0, \tag{39}$$

(ii) and the Goursat problem for the strip III

$$q_1(\chi = 0,\tau) = q_{10}(\tau), \quad 0 < \tau < \nu t_p;$$

$$q_2(\chi,\tau = 0) = 0, \quad 0 < \chi, \tag{40}$$

where q_{10} is known upon (i) has been solved.

The Hill equation for triangle I is easily solved by

$$\phi(\tau) = \cosh(B\tau), \quad B = \sqrt{2A}. \tag{41}$$

Thus, in region I the solution is given by

$$q_1^2(\chi,\tau) = A\,\mathrm{sech}^2[B(\chi+\tau)] = A\,\mathrm{sech}^2(Bx),$$

$$q_2(\chi,\tau) = \frac{B}{2}\tanh[B(\chi+\tau)] = \frac{B}{2}\tanh(Bx). \tag{42}$$

The Goursat problem in the strip III has to be solved with the initial condition

$$q_{10}^2(\tau) = A\,\mathrm{sech}^2(B\tau), \tag{43}$$

thus leading to the solution

$$q_1^2(\chi,\tau) = \frac{A}{[\cosh(B\tau) + B\chi\sinh(B\tau)]^2},$$

$$q_2(\chi,\tau) = \frac{B}{2[B\chi + \coth(B\tau)]}. \tag{44}$$

The complete solution is depicted in Fig. 2.

4.2. Lorentzian pulse

Let us now consider, as a fundamental wave at the boundary $x = 0$, a Lorentzian pulse, that is, a pulse whose intensity is given by

$$I_1(\tau) = \frac{1}{1+\tau^2}. \tag{45}$$

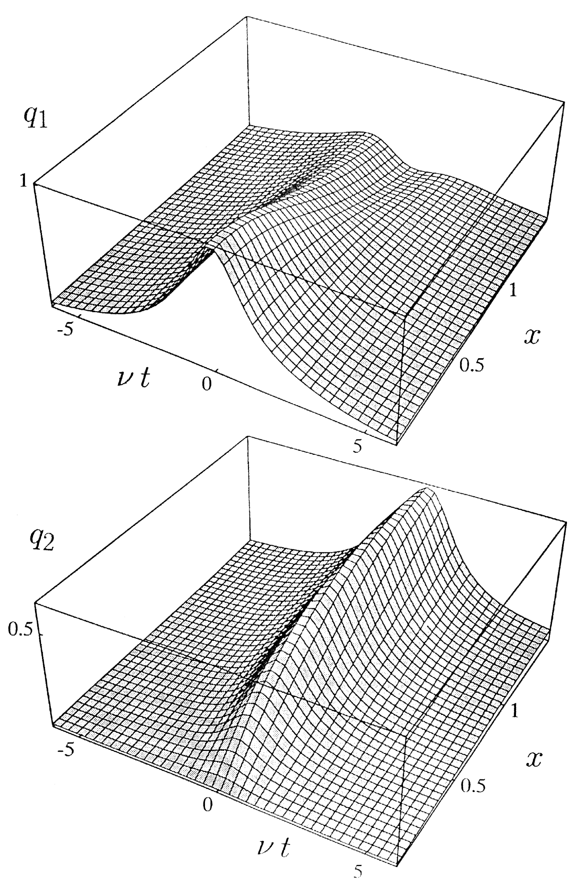

Fig. 3. The amplitudes q_1 and q_2 where the incident intensity $|q_1|^2$ at the fundamental frequency is a Lorentzian shape.

As one can easily verify, such a pulse corresponds to the following ‘‘wave-function’’

$$\phi(\tau) = 1 + \tau^2 . \tag{46}$$

Starting from Eq. (46) one can derive the solutions for both waves in an arbitrary cross section of the nonlinear medium, which are

$$q_1^2(\chi,\tau) = \frac{1}{(1+\chi^2)^2}\frac{1}{\tau^2(1+\tau^2)} \times \frac{1}{[\arctan\chi + \arctan\tau + \chi/(1+\chi^2) + 1/\tau]^2},$$

$$q_2(\chi,\tau) = \frac{\chi}{1+\chi^2} + \frac{1}{(1+\chi^2)^2} \times \frac{1}{\arctan\chi + \arctan\tau + \chi/(1+\chi^2) + 1/\tau}. \tag{47}$$

This solution is depicted in Fig. 3.

4.3. Squared Lorentzian

Here we consider a pulse for which the fundamental wave amplitude, at the boundary, has itself a Lorentzian shape. The intensity is thus a squared Lorentzian given by

$$I_1(\tau) = \frac{b}{2(1+\tau^2)^2} . \tag{48}$$

This pulse corresponds to the following wave function

$$\phi(\tau) = (1+\tau^2)^{1/2} \times \begin{cases} \cos[\alpha \arctan(\tau)], & 1-b=\alpha^2>0, \\ 1, & b=1, \\ \cosh[\beta \arctan(\tau)], & b-1=\beta^2>0, \end{cases} \tag{49}$$

from which we find

$$F(\chi) = \begin{cases} -(1/\alpha)\tan[\alpha \arctan(\chi)], \\ -\arctan(\chi), \\ -(1/\beta)\tanh[\beta \arctan(\chi)]. \end{cases} \tag{50}$$

The outgoing second harmonic wave is strongly dependent on the peak field intensity b. Three regions can be distinguished: low ($b<1$), intermediate ($b=1$) and high ($b>1$) intensities. For the intermediate case the propagated solutions for the two waves in the medium have the particularly simple form

$$q_1^2(\chi,\tau) = \frac{1}{2(1+\chi^2)}\frac{1}{\tau^2(1+\tau^2)} \times \frac{1}{[\arctan(\chi) + \arctan(\tau) + 1/\tau]^2},$$

$$q_2(\chi,\tau) = \frac{\chi}{2(1+\chi^2)} + \frac{1}{2(1+\chi^2)} \times \frac{1}{\arctan(\chi) + \arctan(\tau) + 1/\tau}. \tag{51}$$

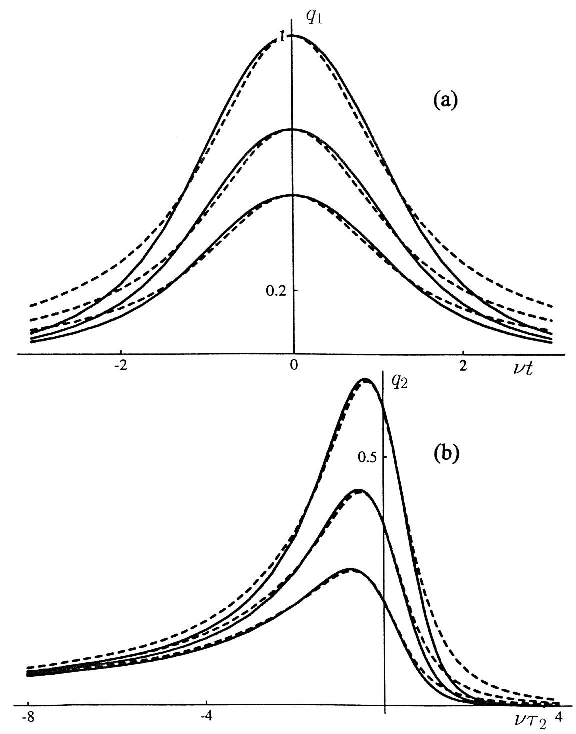

Fig. 4. Solutions where the initial pulse intensity is a squared Lorentzian shape (dashed lines) are compared with solutions where the initial intensity is approximately a sech^2 shape (solid lines). Here, only the initial amplitudes q_1 (part a) and the asymptotic harmonic amplitudes q_2 (part b) are depicted. For the three examples, the initial fundamental pulse shapes differ in their amplitudes, while the half-width is the same for all these pulses.

The solutions for $b>1$ and for $b<1$ can be given as well by substitution of (49), (50) in (32), (33). The explicit formulas are rather messy, however, and therefore will not be presented here. The problem for $b>1$ was already solved by Akhmanov et al. [13]. In Fig. 4 we show three examples of initial q_1- pulses (upper part), together with the corresponding asymptotic q_2- pulses (lower part), for $b=2$ (upper curve), $b=1$ (intermediate curve) and $b=0.5$ (lower curve).

4.4. Exponentially decaying pulses

In examples treated in previous subsections (as well in the next one) the solutions can be explicitly obtained in analytic form. It seems, however, impossible to do the same for an initial pulse with an exponentially decaying shape. Here we start from a particular choice of ‘‘wave functions’’

$$\phi(\tau) = a + \log[b + \cosh^2(\tau)], \tag{52}$$

which corresponds to an initial pulse shape given by

$$I_1(\tau) = \frac{4\left[1+(1+2b)\cosh(2\tau)\right]}{\left[1+2+\cosh(2\tau)\right]^2\left[a+\log\left(b+\cosh^2(\tau)\right)\right]}, \quad (53)$$

the latter exhibiting the exponential decay

$$I_1(t) \underset{|t|\to\infty}{\simeq} \frac{4(1+2b)}{|t|}\exp(-2|t|). \quad (54)$$

For this kind of pulse, we have no closed analytic solutions. However, the numerical solution is obtained by means of the quadrature of Eq. (28). An interesting application relies to the fact that the free parameters a and b may be adjusted to approximate sech^2-shaped pulses of the form

$$\tilde{I}_1(\tau) = A\,\mathrm{sech}^2(\tau). \quad (55)$$

This pulse shape is obviously of experimental interest, since it represents the output of many laser systems. As an example, by choosing $(a,b)=(3.45,0.122)$, $(1.36,0.257)$ and $(0.176,0.579)$ both maxima and half-widths of $I_1(\tau)$ coincide with those of $\tilde{I}_1$ with $A=0.25$, 0.5, 1 respectively. In Fig. 4, the initial pulses and the asymptotic harmonic pulses are depicted, in comparison with the corresponding curves of the squared Lorentzian pulse. It can be seen that the trailing edge of the q_2-pulse is steeper for the sech^2-pulse than for the squared Lorentzian. Apart from that, no striking difference can be seen between these shapes.

4.5. An asymmetric pulse

Here we will give the complete analytic solution for a particular asymmetric initial pulse

$$I_1(\tau) = \frac{1}{2}\frac{1}{1+\tau^2}\left(1-\frac{\epsilon\tau}{\sqrt{1+\tau^2}}\right), \quad \epsilon = \pm 1. \quad (56)$$

For $\epsilon = +1$ the asymptotic behaviour is given by

$$I_1 \to \left(1/4\tau^4\right), \quad \tau\to+\infty; \quad I_1 \to \left(1/\tau^2\right), \quad \tau\to-\infty,$$

and vice versa for $\epsilon=-1$. I.e., the two pulses can be obtained one from the other by the transformation $\tau\to-\tau$. The corresponding ''wave function'' is given by

$$\phi(\tau) = \sqrt{1+\tau^2}+\epsilon\tau, \quad (57)$$

and the function F by

$$F(\chi) = \chi\left(1+\tfrac{2}{3}\chi^2\right)+\frac{2\epsilon}{3}\left[\left(1+\chi^2\right)^{3/2}-1\right]. \quad (58)$$

Through Eqs. (11), (12) we eventually arrive at the complete solution. The asymptotic shape of the harmonic wave is given by the rather simple formulae

$$q_2(\chi,\tau\to\infty) = \frac{1}{2\sqrt{1+\chi^2}}, \quad \epsilon=+1, \quad (59)$$

$$q_2(\chi,\tau\to\infty) = \frac{3}{2}\frac{1}{\sqrt{1+\chi^2}-\chi}-\frac{1}{2\sqrt{1+\chi^2}},$$

$$\epsilon=-1. \quad (60)$$

It is worth noticing that here two initial pulses, one being the time-reversed image of the other, yield quite different results. In particular the asymptotic pulse is symmetric for $\epsilon=+1$, but asymmetric for $\epsilon=-1$, see Fig. 5. For a quantitive comparison of the incident fundamental wave with the asymptotic harmonic wave in one and the same diagram, the respective electric field envelopes E_1, E_2 are more appropriate than our amplitudes q_1, q_2. (We recall that $|q_k|^2 \propto$ photon current densities.) Due to the fact that $E_1/E_2 = q_1/\sqrt{2}\,q_2$ we have introduced the multiplicative factor $\sqrt{2}$ in the figure.

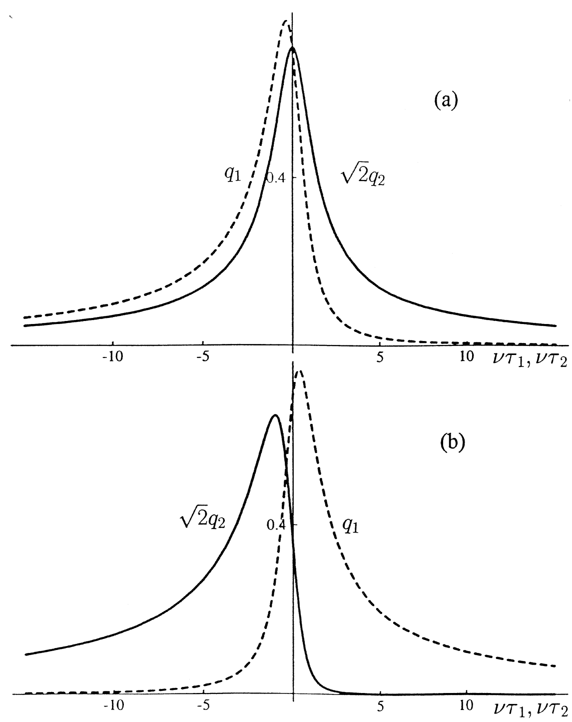

Fig. 5. For two asymmetric initial pulses (dashed lines), one being the mirror image of the other, the asymptotic harmonic waves (solid lines) are given. Parts (a) and (b) correspond respectively to $\epsilon=+1$ and $\epsilon=-1$. The factor $\sqrt{2}$ was introduced for normalization purposes with respect to the energy (cf. the text).

5. Conclusion

We have treated second harmonic generation in one dimension with amplitude-modulated pulses where the fundamental wave was the only incident pulse. We have shown how to reduce this problem to that of solving a zero-eigenvalue Schrödinger equation, the initial pulse shape being formally a repulsive potential.

Instead of starting from a specified pulse shape, we took a properly chosen multi-parametric set of ''wave functions'' $\phi(\tau)$ and easily computed the corresponding set of potentials. The free parameters of these solutions can be used to approximate pulse shapes of interest. To give a complete SHG solution, this method requires at most the single numerical integration of Eq. (28).

Using this approach we were able to obtain solutions for initial pulses approximately of a sech^2-shape with very little numerical effort. We also provide fully analytical solutions for several cases of interest, among these the well-known solutions found by Akhmanov et al. [13], initial square pulses, and asymmetric initial pulses. Using a particular example, we have demonstrated that two asymmetric initial pulses, differing only in the time orientation, give rise to quite different solutions.

We believe that the method introduced here should be useful for optimizing SHG and achieving suitable pulse-shaping in a broad range of working conditions.

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