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Engineering QND measurements for continuous variable quantum information processing

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A novel scheme to realize the whole class of quantum nondemolition (QND) measurements of a field quadrature is suggested. The setup requires linear optical components and squeezers, and allows optimal QND measurements of quadratures, which minimize the information gain versus state disturbance trade-off.

1 Introduction

There is a growing interest for continuous variable (CV) quantum information processing, in particular for implementations based on manipulations of Gaussian states of light in optical circuits. Several quantum protocols [1, 2], including teleportation, error correction, cloning and entanglement purification have been extended to CV systems, which may be easier to manipulate than quantum bits in order to accomplish the desired tasks [3].

In optical implementations, quantum information is encoded in values of a single-mode field quadrature, say $x = 1/2(a^\dagger + a)$, $[a, a^\dagger]$=1 being the mode operators. Therefore, in principle, the most relevant measurement for quantum protocols is provided by homodyne detection. However, the usual implementation of homodyning corresponds to a destructive detection, such that after the measurement we have no longer at disposal a quantum signal for further manipulations and/or measurements. It is thus of interest to devise a scheme for quantum nondemolition (QND) measurements of a field quadrature, in particular for tunable QND measurements, in which the trade-off between information and disturbance may adjusted according to different needs.

In this paper we suggest an all-optical scheme to realize the whole class of QND measurements of a field quadrature [4], from Von Neumann projective measurement to fully non-demolitive, non-informative one. The setup involves only linear optical components (including squeezers) and also allows an optimal QND measurement, which minimizes the information gain versus state disturbance trade-off.

The next section is devoted to the abstract description of a quantum measurement, and to introduce two fidelities in order to quantify the state disturbance and the information gain due to a measurement. In Sect. 3 we analyze with some details the setup for QND measurements of a field quadrature. Sect. 4 closes the paper with some concluding remarks.

2 Quantum measurements

A generic quantum measurement is described by a set of *measurement operators* $\{M_k\}$, with the condition $\sum_k M_k^\dagger M_k = \mathrm{I}$. The POVM of the measurement is given by $\{E_k \equiv M_k^\dagger M_k\}$ whereas its quantum

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operation is expressed as $\varrho \to \sum_k M_k \varrho M_k^\dagger$. This means that, if ϱ is the initial quantum state of the system under investigation, the probability distribution of the outcomes is given by $p_k = \mathrm{Tr}[\varrho\, E_k] = \mathrm{Tr}[\varrho\, M_k^\dagger M_k]$, whereas the conditional output state, after having detected the outcome k, is expressed as $\sigma_k = M_k \varrho M_k^\dagger / p_k$, such that the overall quantum state after the measurement is described by the density matrix $\sigma = \sum_k p_k\, \sigma_k = \sum_k M_k \varrho M_k^\dagger$.

Suppose you have a quantum system prepared in the state ϱ, and that you are interested in measuring the observable K. The so-called Von-Neumann (VN) measurement of K is described by the operators $\{M_k = |k\rangle\langle k|\}$, where the $|k\rangle$'s are the eigenstates of K. Following the above prescription we have for VN measurements $p_k = \langle k|\varrho|k\rangle$, $\sigma_k = |k\rangle\langle k|$ and $\sigma = \sum_k p_k |k\rangle\langle k|$. As a matter of fact, VN measurement (also called *projective* measurement) provides the maximum accessible information about the quantity K, at the price of *erasing* the quantum information of the state being investigated, which is no longer at disposal for further investigations or manipulations.

In opposition to projective measurement one may conceive a *nondemolitive* measurement of K, which preserves the quantum state. This kind of measurement is described by the operators $\{M_k \propto I\}$, proportional to the identity operator. We have uniform p_k and $\sigma = \varrho$, i.e. the quantum state is preserved, however the measurement is completely *non informative*. Overall, such kind of measurement may be viewed as a *blind* quantum repeater, which re-prepares any quantum state received at the input, without giving any information on its characteristics.

Between these two extrema there is a complete class of intermediate cases, i.e. quantum measurements providing only partial information about the distribution $\{p_k\}$ while partially preserving the quantum state of the system. These schemes are sometimes referred to as QND measurements of the quantity K.

Let us now consider a generic quantum measurement $\{Q_k\}$ aimed to provide information about the quantity K. Two questions naturally arise about the characterization of its operation:

i) How much information is provided by the measurement ? Or, in other words, how close are the probability distributions $q_k = \mathrm{Tr}[\varrho\, Q_k^\dagger Q_k]$ and $p_k = \langle k|\varrho|k\rangle$? In order to quantify this resemblance we remind that the space of probability distributions $\{p_k\}_{k=1,\ldots,M}$ is the M-simplex, where a privileged metric (the Fisher metric) exists and induces a distance between probabilities [5] given by $G = \left(\sum_k \sqrt{p_k\, q_k}\right)^2$ which represents a measure of the statistical distinguishability between the two distributions.

ii) How destructive is the measurement ? *I.e.* how far is the output state σ to the input state ϱ ? The *geometric* distance between two density matrices is given by $F = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\varrho}\,\sigma\sqrt{\varrho}}\,\right]\right)^2$. This quantity is the proper generalization to mixed states of the standard quantum overlap used to quantify the (statistical, i.e. by measurements) distinguishability of pure states [6]. F is also characterized by Uhlman theorem, which states that if ϱ and σ are density matrices on a given Hilbert space $\mathcal{H}$, then $F = \max_{\psi,\varphi} |\langle\psi|\varphi\rangle|^2$, where $|\varphi\rangle$ and $|\psi\rangle$ are generic purification of ϱ and σ, i.e. pure states on $\mathcal{H}\otimes\mathcal{H}$ which have ϱ and σ as partial traces. If either ϱ or σ is pure F reduces to the standard overlap. For example, if $\varrho = |\Psi\rangle\langle\Psi|$ then we have $F = \langle\Psi|\sigma|\Psi\rangle$.

3 QND measurements of a field quadrature

Our proposal to realize the whole class of QND measurements of a field quadrature is depicted in Fig. 1. At first, the signal beam is mixed with a probe beam (which will be excited in squeezed vacuum state) in beam splitter of tunable transmissivity τ_1 (BS_1 in the figure). A tunable beam splitter can be easily implemented by a Mach-Zehnder interferometer. In the following we will write the transmittivity in the form $\tau_1 = \cos^2\phi$. For the sake of simplicity, we consider the measurements of the zero-phase quadrature $x = 1/2(a^\dagger + a)$; however, the same analysis is valid for the generic θ-quadrature $x_\theta = 1/2(a^\dagger \mathrm{e}^{\mathrm{i}\theta} + a\mathrm{e}^{-\mathrm{i}\theta})$.

After the beam splitter, the probe beam is revealed by homodyne detection. Taking into account the reflectivity amplitude of BS_1, from an outcome X from the homodyne we infer a value $x = -X/\sin\phi$ for the quadrature of the signal beam. The signal is then displaced by an amount $\alpha^* = -X\tan\phi =$

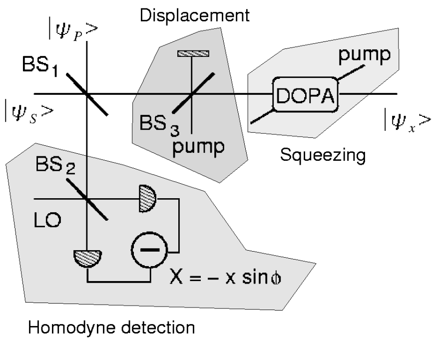

Fig. 1 Schematic diagram of the setup for QND measurements of a field quadrature. The signal is mixed with a squeezed vacuum probe in a beam splitter (BS_1) of tunable transmittivity τ_1. The probe beam is then revealed by homodyne detection, whereas the signal beam is firstly displaced according to the value of the homodyne outcome, and then squeezed according to the transmittivity τ_1 (see text). The local oscillator and the pumps for the displacing and squeezing stages are provided by a common laser source.

$x \tan\phi \sin\phi$, by feedback of the outcome of the homodyne detector, and finally squeezed by an amount r^* such that $\exp\{r^*\} = \sqrt{\tau_1} = \cos\phi$. The displacement transformation $D(\alpha^*) = \exp\{\alpha^* a^\dagger - \bar{\alpha}^* a\}$ can be obtained by mixing the signal with a strong pump beam excited in strong coherent state $|z\rangle$ $|z| \to \infty$ (which may be by the local oscillator of the homodyne detector) in a beam splitter (BS_3 in the figure) with transmittivity approaching unit value $\tau_3 \to 1$ such that $\alpha^* = z\sqrt{1-\tau_3}$. The squeezing transformation is obtained with a degenerate optical parametric amplifier (DOPA), where the pump mode is again provided by the common laser source providing the homodyne LO. Since the cosine is smaller than one, r^* is negative, and this means that we are squeezing the signal in a direction orthogonal to the quadrature we are going to measure.

The probability density for the inferred values of the signal quadrature is given by

$$p(x) = -\sin\phi\, q(X) = \tan\phi \int dy\, |\psi_s(y)|^2\ |\psi_p\left[\tan\phi(y-x_0)\right]|^2\ , \tag{1}$$

where $q(X) = \mathrm{Tr}[|\psi_s,\psi_p\rangle\rangle\langle\langle\psi_p,\psi_s|\, I\otimes|X\rangle\langle X|]$ is the probability density for the homodyne outcomes, and $\psi_j(x),\ j=s,p$ are the signal and probe wave-functions in the quadrature representation, i.e. $|\psi_j\rangle = \int dx\ \psi_j(x)\,|x\rangle$. The conditional output state, after having inferred the value x for the signal quadrature, is given by

$$|\psi_x\rangle = \sqrt{\frac{\sin\phi}{p(x)}}\, S(r^*)\, D(\alpha^*)\, \langle -x\sin\phi|V_\phi|\psi_s,\psi_p\rangle\rangle\ , \tag{2}$$

where $V_\phi = \exp\{i\phi(a^\dagger b + b^\dagger a)\}$ is the evolution operator of the beam splitter BS_1. For a probe mode excited in a squeezed vacuum we have $\psi_p(x) = (2\pi\Sigma^2)^{-1/4}\exp\{-\frac{x^2}{4\sigma_p^2}\}$, where the variance is given by $\sigma_p^2 = 1/4\exp\{\pm 2r\}$ according to the direction of squeezing. We refer to as a *squeezed* probe for the minus sign (squeezing in the direction of the quadrature to be measured) and to as an *antisqueezed* probe for the plus sign (squeezing in the orthogonal direction). The average number of photons carried by the probe is given by $N_p = \sinh^2 r$ in either cases. After minor algebra we get

$$p(x) = |\psi_s(y)|^2 \star G(y,x,\sigma_p^2/\tan^2\phi) \tag{3}$$

$$|\psi_x(y)|^2 = \frac{1}{p(x)}\,|\psi_s(y)|^2\, G(y,x,\sigma_p^2/\tan^2\phi)\ , \tag{4}$$

where $\star$ denotes convolution and $G(y;x,\sigma^2)$ a Gaussian of mean x and variance σ^2.

For $\sigma_p^2/\tan^2\phi \to 0$ i.e. either for strongly squeezed probe or for an almost transparent BS_1 we have $p(x) \to |\psi_s(x)|^2$ and $|\psi_x(y)|^2 \to \delta(y-x)$, which means that we are approaching a projective VN measurement of the quadrature. On the other hand, for $\sigma_p^2/\tan^2\phi \to \infty$ (strongly antisqueezed probe or an almost opaque BS_1) we may write $p(x)$ as a very broad Gaussian, and we have $|\psi_x(y)|^2 \to |\psi_s(y)|^2$, that is we are approaching a non-informative blind quantum repeater. By tuning either the probe squeezing parameter or the transmittivity of BS_1 we may also realize the whole class of intermediate QND measurements of the quadrature.

For Gaussian signals the two fidelities F and G, which measure the state disturbance and the information gain respectively, may be easily evaluated in terms of the single variable $x = \sigma_p/(\sigma_s \tan\phi)$, σ_s^2 being the variance of the signal' wave-function. We have

$$F = \frac{\sqrt{2}x}{\sqrt{1+2x^2}} \quad \text{and} \quad G = 2\,\frac{\sqrt{1+x^2}}{2+x^2}\,.$$

Of course we have $F \to 0$ and $G \to 1$ for $x \to 0$, and vice-versa for $x \to \infty$. However, in general the quantity $F+G$ is not constant, and this means that by varying the squeezing of the probe we obtain different trade-off between information gain and state disturbance. An optimal choice of the probe, corresponding to maximum information and minimum disturbance, maximizes $F+G$. The maximum is achieved for $x \equiv x_m \simeq 1.2$, corresponding to fidelities $F[x_m] \simeq 86\%$ and $G[x_m] \simeq 91\%$. Notice that for a chosen signal, the optimization of the QND measurement can be achieved by tuning the internal phase-shift of the interferometer, without the need of varying the squeezing of the probe. For a nearly balanced interferometer we have $\tan\phi \simeq 1$: in this case the optimal choice for the probe is a state slightly anti-squeezed with respect to the signal, i.e. $\sigma_p \simeq 1.2\,\sigma_s$. Finally, the fidelities are equal for $x \equiv x_e \simeq 1.3$, corresponding to $F[x_e] = G[x_e] \simeq 88\%$. For non Gaussian signals the behavior is similar though no simple analytical form can be obtained for the fidelities. In this case, in order to find the optimal QND measurement, one should resort to numerical means.

4 Conclusions

In conclusions, we have suggested a novel scheme assisted by squeezing and linear feedback to realize an arbitrary QND measurement of a field quadrature. Compared to previous QND proposals [7] the main features of our setup can be summarized as follows: i) it involves only linear coupling between signal and probe, ii) only single mode transformations on the conditional output are needed.

The present setup permits, in principle, to achieve both a projective and a fully non-destructive quantum measurement of a field quadrature. In practice, however, the physical constraints on the maximum amount of energy that can be impinged into the optical channels pose limitations to the precision of the measurements. This agrees with the facts that both an exact repeatable measurement and a perfect state preparation cannot be realized for observables with continuous spectrum [8].

Compared to a vacuum probe, the squeezed/anti-squeezed meters suggested in this paper provide a consistent noise reduction in the desired fidelity figure already for moderate input probe energy. In addition, by varying the squeezing of the probe an optimal QND measure can be achieved, which provides the maximum information about the quadrature distribution of the signal, while keeping the conditional output state as close as possible to the incoming signal.

In order to tune the setup and achieve the whole class of QND measurements we have two independent parameters at disposal: the probe squeezing parameter and the transmittivity of BS_1. This is a another relevant feature of the scheme, since a too large squeezing would increase too much the energy impinged into the apparatus, whereas a tuning based only on the transmittivity would largely affect the detection rate.

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References

[1] A. Furusawa et al., Science **282**, 706 (1998).
[2] L. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 4002 (2000).
[3] B. Julsgaard, A. Kozhekin, and E. S. Polzik, LANL ArXive quant-ph/0106057
[4] M. G. A. Paris, Phys. Rev. A **65**, 012110 (2002).
[5] R. E. Kass, Stat. Sci. **4**, 188 (1989).
[6] S. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[7] M. D. Levenson et al., Phys. Rev. Lett. **57**, 2743 (1986).
A. La Porta et al., Phys. Rev. Lett. **62**, 28 (1989).
S. Pereira et al., Phys. Rev. Lett. **72**, 214 (1994).
J. Poizat and P. Grangier, Phys. Rev. Lett. **70**, 217 (1993).
F. X. Kartner and H. A. Haus, Phys. Rev. A **47**, 4585 (1993).
[8] M. Ozawa, Publ. RIMS Kyoto **21**, 279 (1985).