A quantification of disturbance

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Abstract. I introduce the "entropic disturbance" D that measures the quantity of disturbance induced by an arbitrary evolution of a quantum system. The quantity D satisfies the requirements that a disturbance should possess, as long as one focuses on the irreversibility of any disturbance. Other quantities customarily introduced to gauge disturbance do not weight irreversibility appropriately. An information–disturbance tradeoff is introduced. It states that $I \leqslant D$, where I quantifies the information contained in the state of the system, i.e. the mutual information between the eigenvalues of the initial state and the measurement results.

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INTRODUCTION

The main goal of this paper is to axiomatically introduce a disturbance measure. To this aim, I specify the properties that a measure of disturbance should possess, focusing on the irreversibility aspect of the disturbance. I then introduce a quantity D that satisfies these properties. A natural measure of information I in a quantum system is the mutual information between the eigenvalues of the initial state and the measurement results. This allows to introduce an information–disturbance tradeoff, $I \leqslant D$. Its interpretation is straightforward: Every time an amount I of information is obtained from the measurement apparatus, a disturbance D at least as big is introduced on the system, but, obviously, the system can be disturbed by a process that returns little information. The equality conditions $I = D$ are given showing that the information–disturbance bound is tight: It is saturated by von Neumann projective measurements. This proves that, when considering all the measurements that yield the same information, these types of measurements are among the least disturbing [1].

This bound is an attempt to obtain a quantitative relation that captures the true spirit of Heisenberg's uncertainty principle. The correct interpretation [2, 3] of the Heisenberg–Robertson uncertainty relations [4] relates the uncertainty ΔA in the determination of an observable A with the uncertainty ΔB in the determination of another observable B, where A and B do not commute in general, i.e.

$$\Delta A \Delta B \geqslant |\langle [A,B] \rangle|/2 \ .$$

Thus, such a statement refers to the *preparation* of the system and has nothing to do with the *disturbance* a measurement induces. However, the true spirit of Heisenberg's intuition [5, 6], was that any extraction of information from a quantum system (i.e. a measurement) induces some kind of disturbance on the system. This intuition has yet to be captured in a general quantitative information–disturbance tradeoff relation.

Several of these relations have been put forth and cover many conceivable situations (e.g. see [6, 7, 8, 9, 10]). The main problem in deriving a general version of such a tradeoff lies in the identification of an appropriate definition for the disturbance. Here we focus on the consequences that derive from choosing a disturbance definition that essentially captures the irreversibility aspect. Unfortunately, we have to conclude that, even though a perfectly valid and useful information–disturbance tradeoff can be derived, it cannot be viewed as a direct quantification of the Heisenberg uncertainty relation: Some caveats are in order. In fact, the quantum state is not a physical entity, but it is a purely conceptual construct that contains the information we have on the system. As such, an irreversible evolution of the state can happen also without any interaction with the system itself, but simply through the acquisition of new information. For example, suppose that a preparer is providing me with a system in the state ρ_0. If she does not tell me which is the state, I will be forced to assign it the state of maximal ignorance, $\mathbb{1}/d$. However, as soon as she gives me some information on what the state was, from my point of view the system undergoes the irreversible evolution $\mathbb{1}/d \longrightarrow \rho_0$, even though I may not have interacted with it or measured it in any way. The information–disturbance tradeoff derived in this paper applies also to such situation, even though such situation is not a "measurement" in the Heisenberg sense. Thus, we can conclude that the $I \leqslant D$ relation does not directly capture Heisenberg's view exactly, but it can be seen as a generalization of his intuition. In fact, quantum evolutions are described by completely positive (CP) maps which are a convex set. Thus, it is possible to express the CP-map of the evolution of the system as a convex combination of the "purely informative" part (i.e. the one that entails a state change due only to the acquisition of new information) and of a second part which contains the effect of interactions. This second part is the only one related to Heisenberg's intuition.

This proceedings paper is essentially based on the results derived and analyzed in Ref. [11], where I present a detailed proof of the bound and an analysis of the properties of the disturbance D.

MEASUREMENT

A measurement is by definition an operation that acts on a system and returns some classical information, i.e. a label "l_k" identified by an index k. This operation typically, but not always, changes the state of the measured system (*wave-function collapse* mechanism). When measuring classical objects, the state change is only due to the change in the information we have on the system: The uncertainty in the state is usually reduced. When measuring quantum objects, the state change can have a dynamical nature that perturbs the system. The rigorous description of this mechanism results from the Kraus decomposition of the measurement apparatus [9, 12]. When the system is initially in a state ρ, the kth measurement outcome occurs with a probability $p_k = \mathrm{Tr}[\Pi_k\rho]$, where $\{\Pi_k\}$ is a set of positive operators normalized so that $\sum_k \Pi_k = \mathbb{1}$, the apparatus POVM (Positive Operator-Valued Measure). After the outcome l_k is obtained, the state is changed to

$$\rho'(k) = K_k\,\rho\,K_k^\dagger/p_k\ , \qquad (1)$$

where K_k are the apparatus Kraus operators, in terms of which the POVM is found as $\Pi_k = K_k^\dagger K_k$. On the basis of the above description of the measurement, we can rigorously define the information and the disturbance measures that will be used in the following.

Information: The outcome l_k provides the experimenter with some information I. We are, obviously, interested in the case in which l_k provides some information on the measured system, and it is not just independently generated by the apparatus. Thus, a good measure for I is the mutual information between the measurement results and some property of the system state. A property which is significant and basis-independent is the spectrum of the state's density matrix ρ, i.e. the probability distribution of its eigenvalues λ_j. We then use $I = I(\lambda_j, p_k)$, the mutual information between the eigenvalues of ρ and the measurement results l_k. It is maximal when all eigenvalues are equal (the experimenter has no prior info on the state), and it is null when the state is pure (the experimenter has total knowledge of the state, and can infer all possible properties of the system). [Note that ρ here refers to the state from the experimenter's point of view: It reflects his prior knowledge of the system state. The actual state (from the point of view of the preparer of the system) will be in general purer. The knowledge that can be acquired by the experimenter is weighted by the difference in purity between these two representations of the state.] If the Hilbert space of the system has finite dimension d, we can normalize I with its maximum possible value $\log_2 d$, so to have an adimensional quantity that varies between 0 and 1.

Disturbance: The most general description of an evolution of a quantum system is provided by a CP-map acting on the system state ρ. The reason for a dynamical disturbance during the evolution is that quantum correlations leak out to the environment and are lost. Any quantity D that measures disturbance should then satisfy the following requirements that capture the notion of irreversibility:

i) D should be a function only of the input state ρ and of the apparatus, identified through its Kraus operators $\{K_i\}$, i.e. $D = D(\rho, \{K_i\})$.

ii) D should be null if and only if the transformation $\{K_i\}$ is invertible on ρ. In this case the state change can be undone, and such transformation is not disturbing the system.

iii) Once the state has been disturbed, it should not be possible to decrease D with any successive transformation. This means that D should be monotonically non-decreasing for successive applications of CP-maps [1] (i.e. it should satisfy a sort of pipeline inequality). This requirement captures the notion that a disturbance should be irreversible, and is connected with the concept of "cleanness", see Ref. [13].

iv) D should be continuous: maps and input states which do not differ too much should give similar values of D.

Definitions of disturbance are customarily based on the fidelity or the Bures distance [14] between input and output states. Even though valid information–disturbance relations can be found in this case [7, 15, 16], they do not capture the true spirit of Heisenberg's intuition. In fact, even though a unitary transformation is perfectly reversible, it can rotate a state to an orthogonal configuration, generating the maximum possible fidelity-based disturbance: This quantity does not satisfy the requirements *ii)* and *iii)*. Analogous considerations apply also if we use the entanglement fidelity [17] in place of the fidelity. [One could try enforcing requirements *ii)* and *iii)* by maximizing the fidelity over all possible unitary operators, defining a disturbance of the form $\overline{D} = 1 - \max_{\overline{U}} F(\rho, \overline{U}\rho'\overline{U}^\dagger)$, where F is the fidelity, ρ and ρ' are the input and output states, and the maximization runs over all unitaries $\overline{U}$. Also this definition is inadequate, since $\overline{D}$ is null if ρ and ρ' have the same eigenvalues, which does not necessarily entail that the transformation is invertible, i.e. requirement *ii)* still does not hold.]

A definition of disturbance D that satisfies all the above requirements can be found by recalling that a CP-map $\mathscr{N}$ is invertible even when acting on a system with is entangled with other systems if and only if the map's coherent information $I_c(\rho, \mathscr{N})$ is equal to the von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$ of the input state ρ [18]. The coherent information [18, 19] is defined as

$$I_c \equiv S(\mathscr{N}(\rho)) - S((\mathscr{N} \otimes \mathbb{1})(|\Psi\rangle\langle\Psi|)),$$

where $|\Psi\rangle$ is a purification of ρ and the map $\mathscr{N} \otimes \mathbb{1}$ acts with $\mathscr{N}$ on the system space and with the identity $\mathbb{1}$ on the purification space. The quantity I_c is non-increasing for application of CP-maps (data-processing inequality) [18]. Namely, for any two maps $\mathscr{N}$ and $\mathscr{N}'$, we have $I_c(\rho, \mathscr{N}) \geqslant I_c(\rho, \mathscr{N}' \circ \mathscr{N})$, where $\circ$ denotes composition of maps. Thus, a disturbance measure that satisfies requirements *ii)-iii)* must be a function of $S(\rho) - I_c$ which is non-decreasing and is null when its argument is. I then define

$$D \equiv S(\rho) - I_c = S(\rho) - S(\mathscr{N}(\rho)) + S((\mathscr{N} \otimes \mathbb{1})(|\Psi\rangle\langle\Psi|)),$$

which, in addition to *ii)-iii)*, also satisfies requirements *i)* and *iv)* since it is continuous [11]. Analogously to I, also D can be normalized in d-dimensional Hilbert spaces by dividing it by $\log_2 d$, so that $0 \leqslant D \leqslant 2$.

For the rigorous proof of the information disturbance relation $I \leqslant D$, we refer the reader to Ref. [11]. Here we only give a simple not-very-rigorous, intuition behind the preceding proof. The total quantum information of the initial state ρ can be quantified by $N \simeq S(\rho)$ qubits. The unitary evolution U transfers n of them to the probe space P, where the projective measurement can return a number of bits $b \leqslant n$, due to the Holevo bound [14]. The remaining $N - n$ qubits constitute an upper bound to the quantum capacity to transfer the quantum information in the initial state through the channel Q$\to$Q$'$ consisting of the measurement apparatus. The quantum capacity is measured by the coherent information [19, 20], so that $I_c \leqslant N - n$. Thus,

$$I \simeq b \leqslant n = N - (N - n) \leqslant N - I_c \simeq D\,. \tag{2}$$

Note that in the information–disturbance tradeoff $I \leqslant D$, we have equality $I = D$ if and only if the noise CP-map $\mathscr{N}$ maps different eigenvectors $|j\rangle$ of the initial state ρ into orthogonal subspaces. A typical example is a projective measurement whose Kraus operators are projectors on the basis $|j\rangle$. It is a "classical measurement", where the only uncertainty derives from classical probability. [Notice that, in a d-dimensional Hilbert space, the converse also holds for measurements with d outcomes: If $I = D$ and the measurement POVM has d elements, then the measurement is a von Neumann-type projection, i.e. its Kraus operators are of the form $A_j = \tilde{U}|a_j\rangle\langle a_j|$ where $\tilde{U}$ is a fixed unitary and $|a_j\rangle$ is a basis. The proof of this assertion follows immediately from the fact that the measurement must map a basis $|j\rangle$ of d elements into d orthogonal subspaces, which, in a d-dimensional Hilbert space, must then be one-dimensional.]

CONCLUSIONS

In conclusion, I have introduced a new, legit measure $D(\rho,\{K_i\})$ of the disturbance that a map with Kraus operators $\{K_i\}$ induces on a system in a state ρ. I have derived an information–disturbance tradeoff for such a quantity in the form $I \leqslant D$, where I is the classical info the map $\{K_i\}$ returns on the state ρ. The equality conditions for this bound have been also derived.

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