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Homodyning the exponential phase moments of smoothed Wigner functions

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Abstract
We show that the exponential phase moments associated with the s-ordered generalized Wigner function can be sampled from homodyne data for $s \leqslant 0$. The tomographic sampling functions are evaluated for non-unit homodyne quantum efficiency, and the statistical reliability of the reconstruction is tested by Monte Carlo simulated experiments. The reconstruction of the s-ordered phase distribution from sampled moments is discussed, and the reduction of statistical fluctuations by adaptive techniques is also analysed.

Keywords: Quantum tomography, nonclassical light, operational definition of quantum phase

1. Introduction

One widely used concept of phase in quantum optics relies on examining phase properties of a quantum state via the associated s-ordered quasiprobability distributions $W(x, p, s)$ in phase space [1, 2]. We remind ourselves that $W(x, p, s)$ contains the complete information about the quantum state, as does the density operator and, for $s = 1$, 0 and -1, coincides with the Glauber–Sudershan P function, the Wigner function and the Q function, respectively [3]. Integrating $W(x, p, s)$ expressed in polar coordinates over the radial coordinate yields the s-parametrized 'phase distribution':

$$P(\phi, s) = \int_0^\infty W(r, \phi, s) r \, \mathrm{d}r = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} P^{(k)}(s) \mathrm{e}^{\mathrm{i}k\phi} \quad (1)$$

where $P^{(k)}(s) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\phi} P(\phi, s)$ are the s-parametrized exponential phase moments, i.e. the Fourier coefficients of the s-parametrized phase distribution. In the last decade these quasiprobabilites changed from purely mathematical tools to experimentally determinable quantities using the method of quantum state tomography (see [4–6] for reviews). In particular, s-parametrized phase distributions are intimately connected with specific experimental setups which are designed from a classical point of view for phase measurements. Let us consider an ensemble of classical electric fields $E(t) = E_0 \cos(\omega t + \phi) = x_1 \cos(\omega t) - x_2 \sin(\omega t)$ characterized by the joint probability distribution $P(x_1, x_2) \geqslant 0$ for the quadrature components x_1 and x_2. Once $P(x_1, x_2)$ is determined from measurements the corresponding phase distribution can be calculated via the analogue of equation (1). One possibility to determine $P(x_1, x_2)$ is to measure on each member of the ensemble simultaneously both x_1 and x_2. To this end the signal field is first split into two identical beams by a 50/50 beam splitter and on one beam the quadrature distribution x_1, and on the other one x_2, is measured with the help of two balanced homodyne detectors [7]. Another possibility to obtain $P(x_1, x_2)$ is to use only a single balanced homodyne detector and to measure the rotated quadrature component $x_\Theta = x_1 \cos\Theta + x_2 \sin\Theta$. The desired joint probability $P(x_1, x_2)$ can be reconstructed from the probability distribution $w(x_\Theta, \Theta)$ for the quadrature component x_Θ via the inverse Radon transformation. In the framework of a classical description both experimental schemes, of course, lead to one and the same phase distribution. This is, however, no longer the case in the quantum regime. It is well known that the joint probability distribution determined via the first scheme equals the Q function of the signal field [8, 9]. Compared to it, what is reconstructed via the inverse Radon transformation is the Wigner function $W(x, p)$ of the underlying quantum state [10] or a smoothed Wigner function $W(x, p, s)$ with $s = -(1 - \eta)/\eta$ in the case of a balanced homodyne detector of overall efficiency η [11]. Interestingly, phase detection schemes based on amplification [12, 13], heterodyne detection [14] and six-port detection [15, 16] yield the radius integrated Q function too. Once the Wigner function is reconstructed from the measured homodyne data the expectation value of any physical quantity of interest can in principle be calculated. In this way both the Wigner phase distribution and the canonical phase distribution of a squeezed vacuum state [17] and a weak coherent state [18] have been successfully reconstructed.

Now the question arises if it is possible to avoid the detour of first reconstructing the s-parametrized quasiprobability (which, in turn, introduces classical noise), and to determine the exponential phase moments in equation (1) directly from the measured quadrature distribution, i.e. by sampling over the homodyne data. The analogous reconstruction problem for the canonical phase distribution has been recently discussed in [19]. It turns out that the phase distribution itself cannot be directly sampled from the homodyne data. However, the exponential phase moments can be effectively sampled. The corresponding sampling functions have been derived in terms of integrals containing confluent hypergeometric functions. Using these expressions the canonical phase distribution has been directly determined for various squeezed states from experimental data [20]. Notice that this method leads to much more precise reconstruction than the original procedure, that implied reconstructing first the whole quantum state [17, 18]. Moreover, from these expressions the (quite simple) sampling functions for the Wigner exponential phase moments ($s = 0$) have been derived after some tedious calculations. In this paper we derive this result in a much more straightforward and concise way. Moreover, we do not restrict to the case $s = 0$. Rather we derive simple analytical expressions for the sampling functions needed to determine the exponential phase moments of any s-parametrized phase distribution with $s \leqslant 0$. In a recent paper [21] the same problem has been tackled using a different approach. We notice that, contrary to [21], we obtain a closed expression in terms of known functions not only for the sampling function of the odd exponential phase moments but for the even ones too.

The statistical reliability of the reconstructed exponential phase moments is studied also for non-unit quantum efficiency of the homodyne detector. This allows us to discuss the quality of the reconstruction in realistic situations. Moreover, we analyse the reconstruction of the phase distribution from the sampled exponential moments. In order to improve the reconstruction by the reduction of statistical fluctuations, we adopt an adaptive modification of the sampling functions within an equivalence class [22]. Monte Carlo simulated experiments confirm the reliability of the method.

The paper is organized as follows. In section 2 we express the s-parametrized exponential phase moments in terms of normally ordered moments of the field and briefly review the sampling function for the normally ordered moments. This allows us in section 3 to express the sampling functions for the s-parametrized exponential phase moments in terms of series over Hermite polynomials. Making explicit use of the ambiguity of the desired sampling functions we succeed in deriving for them rather simple expressions in terms of known functions. Moreover, we present an alternative integral representation. The effects of detection losses are considered in section 4, whereas the noise reduction problem is analysed in section 5. Section 6 closes the paper by summarizing the results.

2. Basic relationships

The starting point is the s-ordered quasiprobability distribution of a single mode quantum field in terms of its normally ordered moments $\langle a^{\dagger k} a^l \rangle$ [3,23]. In polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$ the quasiprobability $W(r, \phi, s)$ takes the form of a Fourier series

$$W(r,\phi,s) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} P^{(k)}(r,s)\,\mathrm{e}^{\mathrm{i}k\phi} \tag{2}$$

where the Fourier coefficient $P^{(k)}(r, s)$ is given by

$$P^{(k)}(r,s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(1-s)^{m+k+1}} \frac{\sqrt{2^{2m+k}}}{(m+k)!} r^k \times \exp\left(\frac{-r^2}{1-s}\right) L_m^k\left(\frac{r^2}{1-s}\right) \langle a^{\dagger\, m+k} a^m \rangle \tag{3}$$

with $P^{(-k)}(r, s) = P^{(k)}(r, s)^*$. Here $L_m^k(x)$ denotes the generalized Laguerre polynomial. To obtain the s-parametrized phase distribution we apply the definition (1) to $W(r, \phi, s)$. The integration over r can be performed explicitly using the relation [24]:

$$\int_0^\infty \mathrm{d}x\, x^{\alpha-1} \mathrm{e}^{-x} L_m^k(x) = \frac{(1-\alpha+k)_m}{m!}\Gamma(\alpha) \tag{4}$$

and we get

$$P(\phi,s) = \frac{1}{2\pi}\left\{1 + \mathrm{Re}\sum_{k=1}^{\infty} P^{(k)}(s)\mathrm{e}^{\mathrm{i}k\phi}\right\} \tag{5}$$

where

$$P^{(k)}(s) = \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-1)^m}{(1-s)^{m+(k/2)}} \frac{k\Gamma(m+\frac{k}{2})\sqrt{2^{2m+k}}}{m!(m+k)!} \langle a^{\dagger\, m+k} a^m \rangle. \tag{6}$$

Equation (6) is the sought-after representation of the exponential phase moments in terms of normally ordered moments. In passing we note that expressions for the exponential phase moments in terms of the density matrix elements ϱ_{mn} have been derived several times in the literature (see, e.g., [1]). However, the sampling functions for ϱ_{mn} are rather complicated compared with those for the normally ordered moments and the resulting series are difficult to handle. In contrast the sampling functions for the normally ordered moments are quite simple and the series in question can be more or less directly read off from tables, as we will see in the following.

A realistic homodyne detector of overall efficiency η measures a smoothed ideal quadrature distribution $w(x_\Theta, \Theta, \eta)$ (see, e.g., [11]):

$$w(x_\Theta,\Theta,\eta) = \frac{1}{\sqrt{\pi(1-\eta)}} \int_{-\infty}^{\infty} \mathrm{d}x\, w(x,\Theta) \times \exp\left[-\frac{\eta}{1-\eta}\left(x - \frac{x_\Theta}{\sqrt{\eta}}\right)^2\right] \tag{7}$$

where $w(x_\Theta, \Theta) \equiv w(x_\Theta, \Theta, \eta = 1)$ is the (ideal) quadrature distribution corresponding to the density operator $\hat{\varrho}$ of the signal state. It is well known that $w(x_\Theta, \Theta, \eta)$ can be interpreted as the (ideal) quadrature distribution of the reduced density operator $\varrho(\eta)$ describing the attenuated signal field after passing a (fictitious) beam splitter of intensity transmittivity η [11]. In terms of the normally ordered moments $\langle \hat{a}^{\dagger m}\hat{a}^n\rangle$ the realistic quadrature distribution is [25]

$$w(x_\Theta, \Theta, \eta) = \sum_{k=-\infty}^{\infty} w_k(x_\Theta, \eta)\exp(\mathrm{i}k\Theta) \tag{8}$$

where

$$w_k(x_\Theta, \eta) = \frac{1}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{\sqrt{\eta^{2m+k}}}{\sqrt{2^{2m+k}}m!(m+k)!}\langle \hat{a}^{\dagger m+k}\hat{a}^m\rangle \times H_{2m+k}(x_\Theta)\exp(-x_\Theta^2) \tag{9}$$

for $k \geqslant 0$ and $w_{-k}(x_\Theta, \eta) = w_k^\star(x_\Theta, \eta)$. Here $H_k(x)$ denotes a Hermite polynomial. From equations (8) and (9) follows immediately the reconstruction formula for the normally ordered moments:

$$\langle a^{\dagger n}a^m\rangle = \int_0^{2\pi}\mathrm{d}\Theta\int_{-\infty}^{\infty}\mathrm{d}x_\Theta\, w(x_\Theta, \Theta, \eta) \times \exp[\mathrm{i}(m-n)\Theta]S_{n,m}(x_\Theta, \eta) \tag{10}$$

where the sampling function $S_{n,m}$ is given by

$$S_{n,m}(x_\Theta, \eta) = \left[2\pi\binom{n+m}{m}\sqrt{(2\eta)^{n+m}}\right]^{-1} H_{n+m}(x_\Theta). \tag{11}$$

The sampling function $S_{n,m}$ is not uniquely determined, since there exists a large class of *null functions*, namely functions with zero tomographic average [22]. Adding any number of null functions to a generic sampling function results in a new function which has the same average, and thus it is equivalent in the reconstruction of the given expectation value. This fact will be exploited in section 5 to reduce statistical fluctuations in the reconstruction of phase moments. With respect to the sampling function $S_{n,m}(x_\Theta, \eta)$ we note that the lowest Hermite polynomial occurring in expression (9) for $w_k(x_\Theta, \eta)$ is of the order of $k = |n-m|$. As a result we can use in equation (10) instead of $S_{n,m}$ just as well the function

$$\bar{S}_{n,m}(x_\Theta, \eta) = S_{n,m}(x_\Theta, \eta) + \sum_{s=0}^{|n-m|-1} c_s H_s(x_\Theta) \tag{12}$$

for $|n-m| \geqslant 1$ without changing the reconstructed quantity.

3. Sampling functions for exponential phase moments

In view of equations (6) and (10) we immediately arrive at the following sampling formula for the s-parametrized exponential phase moments ($k \geqslant 1$):

$$P^{(k)}(s, \eta) = \int_{-\infty}^{\infty}\mathrm{d}x_\Theta\int_0^{2\pi}\mathrm{d}\Theta\, w(x_\Theta, \Theta)\mathrm{e}^{-\mathrm{i}k\Theta}K_k(x_\Theta, s, \eta) \tag{13}$$

where the sampling function $K_k(x_\Theta, s, \eta)$ is given by the series

$$K_k(x_\Theta, s, \eta) = \frac{1}{4\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{[\eta(1-s)]^{m+\frac{k}{2}}}\frac{k\,\Gamma(m+\frac{k}{2})}{(2m+k)!}H_{2m+k}(x_\Theta). \tag{14}$$

Obviously the sampling functions do not depend on s and η separately but through the combination $\eta(1-s)$ only. We shall profit from this fact discussing the compensation of detection losses in section 4. For the moment, however, we restrict ourselves to the case of ideal detection, $\eta = 1$. It turns out to be of advantage to consider even and odd exponential phase moments separately. In the case of $k = 2l+1$ equation (14) becomes ($K_k(x_\Theta, s, \eta = 1) \equiv K_k(x_\Theta, s)$)

$$K_{2l+1}(x_\Theta, s) = \frac{(-1)^l}{4\pi}\frac{2l+1}{\sqrt{1-s}}\sum_{n=l}^{\infty}\left(-\frac{1}{1-s}\right)^n\frac{\Gamma(n+\frac{1}{2})}{(2n+1)!} \times H_{2n+1}(x_\Theta). \tag{15}$$

Now we make explicit use of the ambiguity of the sampling function. Repeating the reasoning leading to equation (12), we note that the lower summation limit in equation (15) can be extended to $n = 0$ without changing the reconstructed quantity. For convenience we use for the modified sampling function the same symbol so that we have

$$K_{2l+1}(x_\Theta, s) = \frac{1}{4\pi}(-1)^l\frac{2l+1}{\sqrt{1-s}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1-s)^n}\frac{\Gamma(n+\frac{1}{2})}{(2n+1)!} \times H_{2n+1}(x_\Theta). \tag{16}$$

Analogously we find for the sampling functions belonging to the even exponential phase moments, i.e. for $k = 2l$ with $l \geqslant 1$,

$$K_{2l}(x_\Theta, s) = \frac{(-1)^l}{2\pi}l\sum_{n=1}^{\infty}\left(-\frac{1}{1-s}\right)^n\frac{(n-1)!}{(2n)!}H_{2n}(x_\Theta). \tag{17}$$

We now proceed by considering the special case $s = 0$ separately, then passing to the general case.

3.1. Wigner function, $s = 0$

For $s = 0$ the series on the right-hand side of equation (15) can be directly read off from tables [24]:

$$\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+1/2)}{(2k+1)!}H_{2k+1}(x) = \pi\,\mathrm{sgn}(x) \tag{18}$$

and we immediately obtain ($K_k(x, s=0) \equiv K_k(x)$)

$$K_{2l+1}(x_\Theta) = \frac{(-1)^l}{4}(2l+1)\,\mathrm{sgn}(x_\Theta). \tag{19}$$

In the case of the even moments we use [24]

$$\sum_{n=1}^{\infty}(-1)^n\frac{(n-1)!}{(2n)!}H_{2n}(x_\Theta) = \sum_{n=1}^{\infty}\frac{(n-1)!}{(1/2)_n}L_n^{-1/2}(x^2) = -\gamma - 2\ln 2 + 2 - \ln(x^2) \tag{20}$$

where $(\alpha)_k = \Gamma(k+\alpha)/\Gamma(\alpha)$ is the Pochhammer symbol and γ denotes Euler's constant. Utilizing the ambiguity of the sampling function once again we can neglect the irrelevant constant and get

$$K_{2l}(x_\Theta) = l\frac{(-1)^{l+1}}{\pi}\ln|x_\Theta|. \tag{21}$$

In this way we have rederived in a very concise and straightforward way a result first found in [19]. We emphasize that, as explained in the introduction and noticed in [19], the sampling function $K_k(x_\Theta)$ applies to both the quantum mechanical Wigner function and the classical joint probability distribution $P(x_1, x_2)$. In order to check the statistical

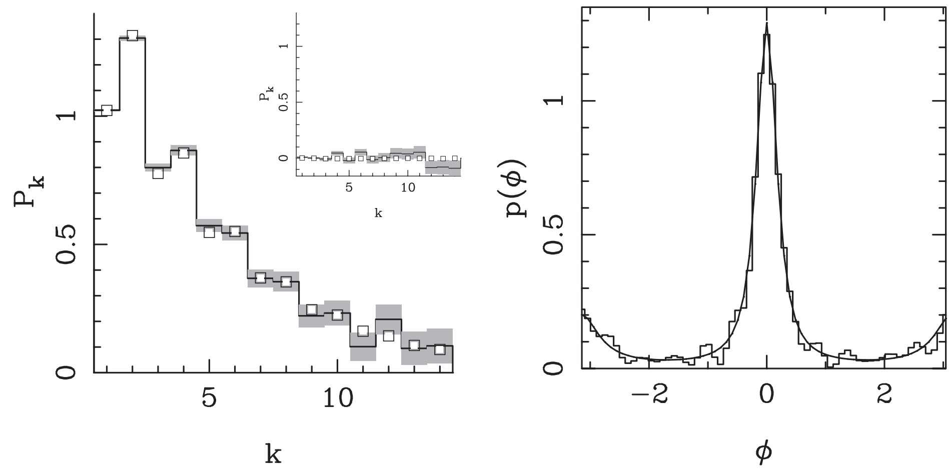

Figure 1. Direct sampling of exponential phase moments ($s = 0$) from homodyne detection. Left: the reconstructed values of the real part of the moments (the imaginary part in the inset). Right: reconstruction of the Wigner phase distribution from moments compared to the theoretical one. Empty squares denote theoretical values, grey regions confidence intervals. The Monte Carlo simulated experiments consist of 10^5 homodyne data at random but known phases on a squeezed coherent state with $\langle a^\dagger a\rangle = 1$ average photons and $\gamma = 0.5$ squeezing fraction. The reconstruction of the phase distribution has been carried out using 29 moments ($k \in [-14, 14]$).

reliability of the reconstruction method we have performed a set of Monte Carlo simulated experiments. In figure 1(a) we report the reconstructed exponential phase moments of a squeezed-coherent state with $\langle a^\dagger a\rangle = 1$ average photons and a squeezing fraction (the fraction of the total energy engaged in squeezing) $\gamma = 0.5$. The homodyne sample consists of 10^5 data (at random but known phases). In figure 1(b) we report the Wigner phase distribution of the same state reconstructed as a Fourier series with the sampled exponential moments as Fourier coefficients (see equation (5)). Of course the series (5) should be truncated, and the quality of the reconstruction depends both on the truncation dimension (the number of moments used in the reconstruction) and on the precision of the reconstructed moments. As a general rule we found that with few moments the resulting distribution is smoother but contains artefacts due to aliasing. On the other hand, using too many moments with a small-size sample produces a noisy distribution. By numerical analysis we found that for a squeezed state with a few photons using a number of moments of the order of ten generally led to a good reconstruction of the phase distribution.

3.2. *Smoothed Wigner function, $s < 0$*

Using the series [24] valid for $s < 0$

$$\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{(2n+1)!}\left(-\frac{1}{1-s}\right)^n H_{2n+1}(x) = 2\sqrt{1+|s|}\,\frac{x}{\sqrt{|s|}} \times {}_1F_1\left(\frac{1}{2};\frac{3}{2};-\frac{x}{|s|^2}\right) = \sqrt{1+|s|}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{x}{\sqrt{|s|}}\right) \quad (22)$$

where ${}_1F_1(a;c;x)$ denotes the confluent hypergeometric function and $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-t^2)\,\mathrm{d}t$ is the error function, we find for the odd sampling function

$$K_{2l+1}(x_\Theta, -|s|) = \frac{(-1)^l}{4}(2l+1)\,\mathrm{erf}(x_\Theta/\sqrt{|s|}). \quad (23)$$

Obviously the sampling function does not depend on x_Θ and s separately, but only through the combination $x_\Theta/\sqrt{|s|}$, i.e. through the rescaled quadrature variable $x'_\Theta = x_\Theta/\sqrt{|s|}$. In the limit $s = 0$, of course, the above expression simplifies to equation (19).

To obtain a closed expression for the sampling function of the even moments we insert the explicit expression of the Hermite polynomials in equation (17). Utilizing the series [24]

$$\sum_{k=0}^{\infty}\frac{(m+k)!}{k!}x^k = \frac{m!}{(1-x)^{m+1}} \quad (24)$$

valid for $|x| < 1$, we obtain after straightforward manipulations

$$K_{2l}(x_\Theta, -|s|) = \frac{(-1)^{l+1}}{\pi}\,l\,\frac{x_\Theta^2}{|s|}\sum_{n=0}^{\infty}\frac{(1)_n\,(1)_n}{(3/2)_n\,(2)_n}\frac{(-x_\Theta^2/|s|)^n}{n!} + \frac{(-1)^l}{2\pi}\frac{l}{1-s}\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{1-s}\right)^n. \quad (25)$$

The first series is just the power series expansion of the generalized hypergeometric function ${}_2F_2(1, 1; 3/2, 2; -x^2/|s|)$ [24], whereas the numerical series yields a constant which can be neglected due to the ambiguity of the sampling function. Thus we obtain

$$K_{2l}(x_\Theta, s) = \frac{(-1)^{l+1}}{\pi}\,l\,\frac{x_\Theta^2}{|s|}\,{}_2F_2\left(1, 1; \frac{3}{2}, 2; -\frac{x_\Theta^2}{|s|}\right). \quad (26)$$

Obviously, the sampling function for the even moments depends on the square of the scaled quadrature variable $x_\Theta/\sqrt{|s|}$. We note that the result (23) for the sampling functions of the odd moments has been found in [21] too using a different approach. The expression (26) for the sampling function of the even phase moments is new. A different handling of the series (17) reveals the close connection between $K_{2l}(x_\Theta)$ and another sampling function well known from optical homodyne tomography. To this end we derive both sides of equation (17) with respect to x_Θ. Making use of $\mathrm{d}H_n(x)/\mathrm{d}x = 2nH_{n-1}(x)$ and of the series [24] ($|t| \leqslant 1$)

$$\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!}t^k H_{2k+1}(x) = \frac{1}{1+t}\,2x\,{}_1F_1\left(1;\frac{3}{2};\frac{tx^2}{1+t}\right) \quad (27)$$

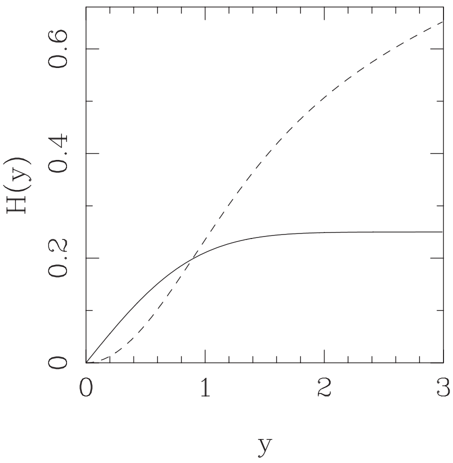

Figure 2. Functions $H_{\mathrm{odd}}(y)$ (full curve) and $H_{\mathrm{even}}(y)$ (broken curve) appearing in the expression of the sampling functions for the exponential phase moments.

we obtain for $s < 0$ and with $x'_\Theta = x_\Theta/\sqrt{|s|}$ and $K_k(x'_\Theta) \equiv K_k(x_\Theta, -|s|)$

$$\frac{\mathrm{d}}{\mathrm{d}x'_\Theta} K_{2l}(x'_\Theta) = \frac{(-1)^{l+1}}{\pi} 2l\, x'_\Theta\, {}_1F_1(1; 3/2; -x'^2_\Theta). \qquad (28)$$

Deriving both sides of equation (28) once more with respect to x'_Θ we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x'^2_\Theta} K_{2l}(x'_\Theta) = (-1)^{l+1}\, l\, f_{00}(x'_\Theta). \qquad (29)$$

Here f_{00} is the well known function for direct sampling of both the vacuum matrix element ϱ_{00} in the Fock basis [3–5] and a smoothed Wigner function [26, 27]. For completeness we note that the expression (27) for the even sampling function becomes the one derived in [21], if we express ${}_1F_1$ by the imaginary error function. We may summarize our results in a compact formula for the desired sampling functions:

$$K_{2l+1}(x_\Theta, -|s|) = (-1)^l (2l+1) H_{\mathrm{odd}}\left(\frac{x_\Theta}{\sqrt{|s|}}\right) \qquad (30)$$

$$K_{2l}(x_\Theta, -|s|) = (-1)^{l+1} l H_{\mathrm{even}}\left(\frac{x_\Theta}{\sqrt{|s|}}\right). \qquad (31)$$

The functions $H_{\mathrm{odd}}(y) = \frac{1}{4}\mathrm{erf}(y)$ and $H_{\mathrm{even}}(y) = \frac{1}{\pi} y^2\, {}_2F_2(1; 1; 3/2; 2; -y^2)$ are plotted in figure 2.

3.3. Alternative integral representation

Once the sampling functions for the exponential phase moments of the Wigner function are known we can easily derive an integral representation of the sampling functions $K_k(x_\Theta, s)$ with $s < 0$. To this end we utilize the following transformation property of the s-ordered quasiprobability distribution [28]:

$$W_{\hat\varrho}(x, p, s) = \eta\, W_{\hat\varrho(\eta)}(\sqrt{\eta}x, \sqrt{\eta}p, 1 - \eta(1-s)). \qquad (32)$$

Here $W_{\hat\varrho}(x, p, s)$ denotes the s-ordered quasiprobability of the signal state described by the density operator $\hat\varrho$. The density operator $\hat\varrho(\eta)$ is the reduced density operator of the attenuated signal field after passing a beam splitter of intensity transmittivity η. Obviously, the Wigner function $W_{\hat\varrho(\eta)}(x, p, s = 0)$ associated with $\varrho(\eta)$ is basically the rescaled s-ordered quasiprobability of the original signal field with $s = -(1-\eta)/\eta$:

$$W_{\hat\varrho(\eta)}(x, p, s = 0) = \frac{1}{\eta} W_{\hat\varrho}\left(\frac{x}{\sqrt{\eta}}, \frac{p}{\sqrt{\eta}}, s = -\frac{1-\eta}{\eta}\right). \qquad (33)$$

From equation (33) follows immediately

$$P_{\hat\varrho(\eta)}(\phi, s = 0) = P_{\hat\varrho}(\phi, s = -(1-\eta)/\eta) \qquad (34)$$

i.e. the phase distribution associated with the Wigner function of the attenuated signal state coincides with the phase distribution associated with the s-ordered quasiprobability distribution of the original signal field with $s = -(1-\eta)/\eta$. Since the quadrature distribution belonging to the state $\varrho(\eta)$ is given by $w(x_\Theta, \Theta, \eta)$ we find in view of equation (13) with $\eta = 1$

$$P^{(k)}_{\hat\varrho(\eta)}(s = 0) \equiv P^{(k)}_{\hat\varrho}(s = 1 - 1/\eta)$$
$$= \int_0^{2\pi} \mathrm{d}\Theta \int_{-\infty}^{\infty} \mathrm{d}x_\Theta\, w(x_\Theta, \Theta, \eta)\, \mathrm{e}^{-\mathrm{i}k\Theta} K_k(x_\Theta). \qquad (35)$$

Substituting (7) for $w(x_\Theta, \Theta, \eta)$ into equation (35) we obtain for the desired function $K_k(x_\Theta, s)$ in equation (13) with $\eta = 1$ after simple manipulations

$$K_k(x_\Theta, -|s|) = \sqrt{\frac{1+|s|}{\pi|s|}} \int_{-\infty}^{\infty} \mathrm{d}x\, K_k(x)$$
$$\times \exp\left[-\frac{1}{|s|}\left(x_\Theta - \sqrt{1+|s|}\, x\right)^2\right]. \qquad (36)$$

Using the expressions for $K_k(x_\Theta, s = 0)$ and tables of integrals one can easily recover the representations (23) and (26), respectively, from equation (36).

4. Effect of detection losses

So far we have considered the sampling functions needed to reconstruct the s-parametrized exponential phase moments from the ideal quadrature distribution $w(x_\theta, \Theta)$. But what is reconstructed if we substitute in equation (13) with $\eta = 1$ for the ideal quadrature distribution $w(x_\Theta, \Theta)$ the realistic distribution $w(x_\Theta, \Theta, \eta)$? Let us denote the reconstructed quantity by $P^{(k)}(s, \eta)$, so that

$$P^{(k)}(s, \eta) = \int_{-\infty}^{\infty} \mathrm{d}x_\Theta \int_0^{2\pi} \mathrm{d}\Theta\, w(x_\Theta, \Theta, \eta)\, \mathrm{e}^{-\mathrm{i}k\Theta} K_k(x_\Theta, s). \qquad (37)$$

For the special case $s = 0$ equation (35) already gives the answer: averaging the sampling function $K_k(x_\Theta)$ with the realistic quadrature distribution yields the exponential phase moment of a smoothed Wigner function, namely the s-parametrized exponential phase moment with $s = -(1-\eta)/\eta$. To answer the question in the general case we substitute in equation (37) both the expression (7) for $w(x_\Theta, \Theta, \eta)$ and the expression (36) for $K_k(x_\Theta, s)$. After a straightforward calculation we find

$$P^{(k)}(s, \eta) = P^{(k)}(s' = -(1 - \eta + |s|)/\eta). \qquad (38)$$

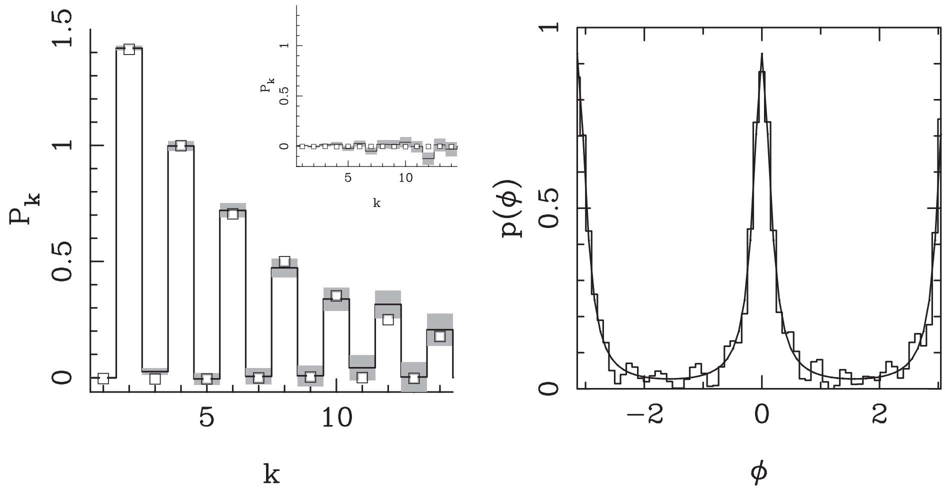

Figure 3. Direct sampling of exponential phase moments ($s = -1$) from inefficient ($\eta = 50\%$) homodyne detection. Left: the reconstructed values of the real part of the moments (the imaginary part in the inset). Right: reconstruction of the Q phase distribution from moments compared to the theoretical one. Empty squares denote theoretical values, grey regions confidence intervals. The Monte Carlo simulated experiments consists of 10^5 homodyne data at random but known phases on a squeezed vacuum with $\langle a^\dagger a\rangle = 1$ average photons. The reconstruction of the phase distribution has been carried out using 29 moments ($k \in [-14, 14]$).

Thus we arrive at the result that applying the sampling function $K_k(x_\Theta, s)$ to the realistic quadrature distribution via equation (37) yields the s'-parametrized exponential phase moment with $s' = -(1-\eta+|s|)/\eta)$. Obviously, s' is less than $-|s|$ for $0 < \eta < 1$ and, as expected, the exponential phase moments of a more smoothed phase distribution are reconstructed.

Now the obvious question arises if it is possible to compensate for the detection loss by using the modified sampling function $K_k(x_\Theta, s, \eta)$. Fortunately equation (14) already gives the answer. This sampling function compensates for the detection loss and can be obtained from the sampling function $K_k(x_\Theta, s)$ simply by replacing s by $s' = 1-\eta(1-s)$ therein. We then find

$$K_k(x_\Theta, -|s|, \eta) = K_k(x_\Theta, s' = 1-\eta(1+|s|)) \qquad (39)$$

provided that $s < 1 - 1/\eta$. Thus we arrive at the result that the modified sampling function is just the sampling function corresponding to perfect detection taken at a larger value of the parameter s. But note that the above condition implies that at best s-parametrized exponential phase moments with $s = 1 - 1/\eta$ can be directly sampled from realistic quadrature distributions. Hence the sampling functions do not allow one to compensate for detection losses in the case of $s > 1 - 1/\eta$. In figure 3 we report the results from Monte Carlo simulated sampling of $s = -1$ ordered exponential phase moments from inefficient ($\eta = 50\%$) homodyne detection. The reconstruction of the 'Q phase' distribution from moments is also reported. The simulated experiments consist of 10^5 homodyne data at random but known phases on a squeezed vacuum with $\langle a^\dagger a\rangle = 1$ average photons. The reconstruction of the phase distribution has been carried out using 29 moments ($k \in [-14, 14]$).

5. Noise reduction

As a matter of fact there exists a large class of functions $F(x_\Theta, \Theta)$ which have zero tomographic average for *any* state of the radiation field [22]:

$$\overline{F} = \int_0^{2\pi} \frac{\mathrm{d}\Theta}{\pi} \int_{-\infty}^{\infty} \mathrm{d}x_\Theta\, w(x,\Theta) F(x_\Theta, \Theta) \equiv 0. \qquad (40)$$

An example is provided by the functions $F_k(x, \phi) = x^k \exp\{\pm \mathrm{i}(k+2)\phi\}$ with $k = 0, 1, \ldots$. This fact can be exploited to reduce statistical fluctuations of the sampled quantities, and ultimately the tomographic noise. The key idea of *adaptive tomography* [22] is that adding null functions to sampling functions does not affect their mean values, but changes statistical errors, which can then be reduced by an optimization method that 'adapts' kernels to homodyne data. Let us consider a generic real sampling function $R(x_\Theta, \Theta)$. By adding M null functions with the constraint of maintaining the function as real, we have the new function

$$H(x_\Theta, \Theta) = R(x_\Theta, \Theta) + \sum_{k=0}^{M-1} \mu_k F_k(x_\Theta, \Theta) + \sum_{k=0}^{M-1} \mu_k^* F_k^*(x_\Theta, \Theta) \qquad (41)$$

where μ_k are complex coefficients. From the definition of a null function, we have $\overline{H} = \overline{R}$, whereas the variance of H is given by[3]

$$\overline{\Delta H^2} = \overline{\Delta R^2} + 2\left\{ \sum_{kl} \mu_k \mu_l^* \overline{F_k F_l^*} + \sum_k \mu_k \overline{R F_k} + \sum_k \mu_k^* \overline{R F_k^*} \right\}. \qquad (42)$$

The variance (42) can be minimized with respect to the coefficients μ_k, leading to the linear set of equations

$$\sum_l \mu_l \overline{F_k F_l^*} = -\overline{R F_k^*}. \qquad (43)$$

The optimization equations (43) can be written in the matrix form [20]

$$\boldsymbol{A}\mu = \boldsymbol{b}. \qquad (44)$$

[3] Notice that the square of a null function is again a null function.

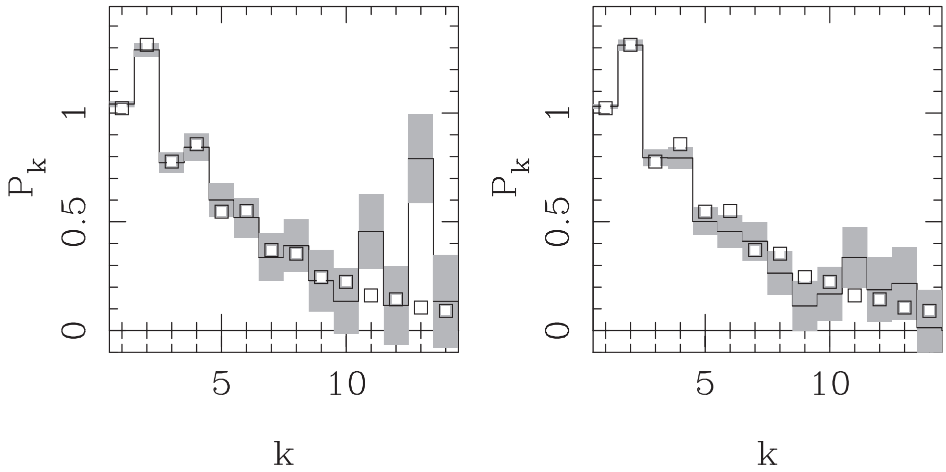

Figure 4. Adaptive noise reduction in the reconstruction of phase moments. Left: the reconstructed values of the real part of the moments using normal sampling functions. Right: the reconstruction with modified sampling functions. Empty squares denote theoretical values, grey regions confidence intervals. The Monte Carlo simulated experiments consist of 20 000 homodyne data at random but known phases on a squeezed coherent state with $\langle a^\dagger a\rangle = 1$ average photons and $\gamma = 0.5$ squeezing fraction. The adaptive noise reduction procedure has been applied using 10 ($k = 0, \ldots, 9$) null functions.

$\boldsymbol{A}$ being a Hermitian $M \times M$ matrix and $\boldsymbol{b}$ a complex vector

$$A_{lk} = \overline{F_k F_l^*} \equiv \overline{x^{k+l} \exp\{\mathrm{i}(k-l)\Theta\}} \qquad b_k = -\overline{R F_k^*}. \tag{45}$$

The vector $\boldsymbol{b}$ depends both on the kernel to be optimized and on the state $\hat{\varrho}$ under examination, whereas the matrix $\boldsymbol{A}$ depends on the state only. By substituting equation (43) in (42) and inverting (44) we obtain

$$\Delta^2 = \overline{\Delta R^2} - \overline{\Delta H^2} = 2\sum_{kl}^{M-1} \mu_k A_{kl}\, \mu_l^* = 2\sum_{kl}^{M-1} b_k (A^{-1})_{kl} b_l^* \tag{46}$$

which expresses the decrease of the variance in terms of $\boldsymbol{A}$ and $\boldsymbol{b}$. The noise reduction procedure proceeds as follows: after collecting a tomographic data sample the quantities $\boldsymbol{A}$ and $\boldsymbol{b}$ are evaluated as experimental averages. Then, by solving the linear systems (44) one obtains the coefficients which are used to build the optimized sampling function. After that the same data sample is used to average the new sampling function and to evaluate the experimental error. Of course, for a finite data sample the mean values of the initial and optimized sampling functions may be different.

We applied this procedure to the reconstruction of the exponential phase moments for $s = 0$ starting from the kernel $R_k(x_\Theta, \Theta) = \exp(-\mathrm{i}k\Theta) K_k(x_\Theta, \Theta)$ (optimization proceeds separately on the real and imaginary parts), and we found that it is effective especially with a reduced number of data, where the direct reconstruction using the sampling functions of the previous sections may be noisy. In figure 4 we plot the reconstructed phase moments (with and without the noise reduction procedure) of a squeezed coherent state with $\langle a^\dagger a\rangle = 1$ average photons and $\gamma = 0.5$ squeezing fraction.

6. Summary

In this paper we have determined the sampling functions needed to determine s-parametrized exponential phase moments from quadrature distributions measured with realistic detectors. These s-parametrized moments are nothing but the Fourier coefficients of the phase distribution obtained by integrating the s-ordered quasiprobability expressed in polar coordinates over the radius. In deriving the sampling functions for the s-parametrized moments we have explicitly taken advantage of their ambiguity. Using this property we could show that the sampling functions needed for the reconstruction of the odd moments are basically given by the error function. For the even moments the sampling functions are proportional to a generalized hypergeometric function ${}_2F_2$. We checked the statistical reliability of the method by means of a set of Monte Carlo simulated experiments, and applied the adaptive tomography technique to minimize fluctuations. Our results confirmed that quantum homodyne tomography is currently the most precise method to determine the intrinsic phase properties of quantum states of radiation.

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