LOCC PROTOCOLS FOR ENTANGLEMENT TRANSFORMATIONS

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We construct the protocols to achieve probabilistic and deterministic entanglement transformations for bipartite pure states by means of local operations and classical communication. A new condition on pure contraction transformations is provided.

The transformation of entangled states by means of local operations and classical communication (LOCC) is a key issue in quantum information processing. In fact, increasing entanglement by means of LOCC with some probability is crucial in practice, since losses and decoherence have detrimental effects in the establishment of entanglement at distance.

In this paper we give a short and simple proof of Lo-Popescu theorem.[1] Then, we provide a new necessary condition for pure-contraction transformations. Finally, we construct explicitly the LOCC protocols to achieve deterministic and probabilistic transformations for bipartite pure states.

We first introduce the main notation. Given a linear operator O we denote by $O^\dagger$, O^*, O^τ, and $O^\ddagger$ the hermitian conjugate, the complex conjugate, the transpose, and the Moore-Penrose inverse of O, respectively. Recall that $O^\ddagger$ is the unique matrix that satisfies $OO^\ddagger O = O$, $O^\ddagger O O^\ddagger = O^\ddagger$, $OO^\ddagger$ and $O^\ddagger O$ hermitian. Notice also that $OO^\ddagger \equiv P_O$ is the orthogonal projector over $\mathrm{Rng}(O)$, whereas $O^\ddagger O \equiv P_{O^\dagger}$ is the orthogonal projector over $\mathrm{Rng}(O^\dagger) \equiv \mathrm{Supp}(O)$. We write the singular value decomposition (SVD) of O as $O = X_O \Sigma_O Y_O$, where Σ_O denotes the diagonal matrix whose entries are the singular values $\sigma_i(O)$ of O taken in decreasing order, and X_O, Y_O are unitary. We write

$$|A\rangle\rangle \equiv \sum_{i,j} a_{ij} |i\rangle_1 \otimes |j\rangle_2, \tag{1}$$

for the bipartite pure states on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\{|i\rangle_1\}$ and $\{|j\rangle_2\}$ are two orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$. One can easily check the relation $A \otimes B|C\rangle\rangle = |ACB^\tau\rangle\rangle$. Finally, we use the notation $A \prec B$ for Hermitian operators A and B to denote the majorization relation[2] $\mathrm{eigv}(A) \prec \mathrm{eigv}(B)$, and in the same fashion we will write $A \prec^w B$ and $A \prec_w B$ for super- and sub-majorization.[2]

We give now a simple proof of Lo-Popescu theorem.[1]

Theorem 1. All LOCC on a pure bipartite entangled state $|\Psi\rangle\rangle$ can be reduced to a contraction by Alice and a unitary transformation by Bob. This is due to the equivalence of any Bob contraction M with the Alice contraction N assisted by Bob's unitary transformation U as follows

$$I \otimes M|\Psi\rangle\rangle = N \otimes U|\Psi\rangle\rangle , \tag{2}$$

where

$$N = K_{M\Psi^\tau} M K_\Psi , \qquad U = K^\dagger_{M\Psi^\tau} K^\dagger_\Psi , \tag{3}$$

and K_O is the unitary operator achieving the transposition of the operator O, namely $O^\tau = K_O O K_O^*$.

Proof. To prove that every LOCC can be reduced to an Alice contraction and a Bob unitary transformation it is sufficient to prove Eq. (2), since: *a)* all possible elementary LOCC in a sequence will be reduced to an Alice contraction and a Bob unitary; *b)* the product of two contractions is a contraction; *c)* unitary transformations are particular cases of contraction.

Given the SVD of any linear operator O one has

$$O^\tau = Y_O^\tau \Sigma_O X_O^\tau = (Y_O^\tau X_O^\dagger) O (Y_O{}^\tau X_O^\dagger)^* \equiv K_O O K_O^* , \tag{4}$$

with $K_O = Y_O^\tau X_O^\dagger$. Hence

$$\Psi M^\tau = (M\Psi^\tau)^\tau = K_{M\Psi^\tau}(M\Psi^\tau)K^*_{M\Psi^\tau} = K_{M\Psi^\tau} M K_\Psi \Psi K_\Psi^* K^*_{M\Psi^\tau} . \tag{5}$$

Then one gets Eq. (2) with N and U given as in Eq. (3).

The main theorem on entanglement transformations is the following.

Theorem 2. The state transformation $|A\rangle\rangle \to |B\rangle\rangle$ is possible by LOCC iff

$$AA^\dagger \prec^w pBB^\dagger , \tag{6}$$

where $p \leq 1$ is the probability of achieving the transformation. A necessary condition to be satisfied is $\mathrm{rnk}(A) \geq \mathrm{rnk}(B)$. In particular, the transformation is deterministic ($p = 1$) iff $AA^\dagger \prec BB^\dagger$.

Theorem 2 unifies the results of Nielsen[3] and Vidal.[4]

In the following we provide a new necessary condition for the case of pure-contraction transformation, namely we prove:

Theorem 3. If there is a *pure* LOCC that achieves the state transformation $|A\rangle\rangle \to |B\rangle\rangle$ with probability p, we must have

$$pBB^\dagger \prec_w AA^\dagger . \tag{7}$$

Proof. According to theorem 1, the pure LOCC transformation $|A\rangle\rangle \to |B\rangle\rangle$ occurring with probability p is given by

$$M \otimes U|A\rangle\rangle = \sqrt{p}|B\rangle\rangle , \tag{8}$$

and we need to have $MAU^\tau = \sqrt{p}B$. Using the SVD of A and B one has $\tilde{M}\Sigma_A\tilde{U} = \sqrt{p}\,\Sigma_B$, with $\tilde{M} = X_B^\dagger M X_A$ and $\tilde{U} = Y_A U^\tau Y_B^\dagger$. Then $\tilde{M}\Sigma_A^2\tilde{M}^\dagger = p\Sigma_B^2$, namely

$$\sum_k S_{kl}\sigma_k^2(A) = p\sigma_l^2(B) , \tag{9}$$

where $S_{kl} \doteq |\langle l|\tilde{M}|k\rangle|^2$ is a sub-stochastic matrix, since

$$\sum_k S_{kl} = \langle l|\tilde{M}\tilde{M}^\dagger|l\rangle \leq ||M^\dagger||^2 \leq 1 ,$$
$$\sum_l S_{kl} = \langle k|\tilde{M}^\dagger \tilde{M}|k\rangle \leq ||M||^2 \leq 1 . \tag{10}$$

This proves that Eq. (7) is a necessary condition for transformation (8).

To construct the explicit protocols that realize entanglement transformations we will use the following lemma:

Lemma 1. $x \prec^w y \iff$ for some v $\quad x \prec v$ and $v \geq y$,

along with Uhlmann theorem:

Theorem 4. For Hermitian operators C and D one has $C \prec D$ if and only if there is a probability distribution p_λ and unitaries W_λ such that

$$C = \sum_\lambda p_\lambda W_\lambda D W_\lambda^\dagger . \tag{11}$$

For Lemma 1 a probabilistic transformation can always be performed through two steps: a deterministic transformation $|A\rangle\rangle \to |Q\rangle\rangle$, followed by a pure-contraction $|Q\rangle\rangle \to |B\rangle\rangle$ that occurs with probability p.

For the deterministic transformation $|A\rangle\rangle \to |Q\rangle\rangle$, one needs to find the contractions M_λ and the unitaries U_λ versus the operators W_λ of Eq. (11), where $C = AA^\dagger$ and $D = QQ^\dagger$, such that

$$M_\lambda \otimes U_\lambda |A\rangle\rangle = \sqrt{q_\lambda}|Q\rangle\rangle . \tag{12}$$

The general solution of Eq. (12) is given by

$$M_\lambda = \sqrt{q_\lambda} Q U_\lambda^* A^\ddagger + N_\lambda (1 - AA^\ddagger) . \tag{13}$$

To guarantee that M_λ is a contraction we can always take

$$U_\lambda^* = Y_Q^\dagger X_Q^\dagger W_\lambda X_A Y_A , \qquad N_\lambda = 0 . \tag{14}$$

In fact from Eqs. (13) and (14) and using Eq. (11) one has

$$\begin{aligned}\sum_\lambda M_\lambda^\dagger M_\lambda &= \sum_\lambda q_\lambda (A^\ddagger)^\dagger Y_A^\dagger X_A^\dagger W_\lambda^\dagger X_Q Y_Q Q^\dagger Q Y_Q^\dagger X_Q^\dagger W_\lambda X_A Y_A A^\ddagger \\ &= \sum_\lambda q_\lambda (A^\ddagger)^\dagger Y_A^\dagger X_A^\dagger W_\lambda^\dagger Q Q^\dagger W_\lambda X_A Y_A A^\ddagger \\ &= (A^\ddagger)^\dagger Y_A^\dagger X_A^\dagger A A^\dagger X_A Y_A A^\ddagger \\ &= (A^\ddagger)^\dagger A^\dagger A A^\ddagger = (AA^\ddagger)^\dagger AA^\ddagger = AA^\ddagger = P_A .\end{aligned} \tag{15}$$

The completeness of the measurement can be guaranteed by the further contraction $M_0 = V(I - AA^\ddagger)$, where V is an arbitrary unitary operator.

Hence, given explicitly Eq. (11) one can perform the contractions M_λ and the unitaries U_λ to achieve the entanglement transformation. The problem of looking for a POVM with minimum number of outcomes (thus minimizing the amount of classical information sent to Bob's side) is reduced to finding the transformation (11) with minimum number of unitaries. One can resort to a constructive algorithm to find a bistochastic matrix D which relates the vectors $\vec{\sigma}_A^2$ and $\vec{\sigma}_Q^2$ of the singular values of A and Q, namely $\vec{\sigma}_A^2 = D\vec{\sigma}_Q^2$. Then Birkhoff theorem allows one to write D as a convex combination of permutation matrices $D = \sum_\lambda q_\lambda \Pi_\lambda$. In terms of Σ_A and Σ_Q one has

$$\Sigma_A^2 = \sum_\lambda q_\lambda \Pi_\lambda^\dagger \Sigma_Q^2 \Pi_\lambda \tag{16}$$

where $\Pi_\lambda = \sum_l |l\rangle\langle\Pi_\lambda(l)|$. In this way one obtains Eq. (11), with $W_\lambda = X_Q \Pi_\lambda X_A^\dagger$. Using Eqs. (13) and (14) for M_λ and U_λ one recovers the result of Ref. 5. Notice that Caratheodory's theorem always allows one to reduce the number of permutations in Eq. (16) to $(d-1)^2+1$, for d-dimensional Alice's Hilbert space.

The second part of the protocol, namely the contraction which provides the state $|B\rangle\rangle$ from $|Q\rangle\rangle$, is present only for probabilistic transformations. It is a pure contraction given by

$$N \otimes V|Q\rangle\rangle = |NQV^\tau\rangle\rangle = \sqrt{p}|B\rangle\rangle\ , \tag{17}$$

where $N = \sqrt{p} X_B \Sigma_B \Sigma_Q^\ddagger X_Q^\dagger$ and $V^\tau = Y_Q^\dagger Y_B$. In fact

$$NQV^\tau = \sqrt{p} X_B \Sigma_B \Sigma_Q^\ddagger \Sigma_Q Y_B = \sqrt{p} X_B \Sigma_B Y_B = \sqrt{p} B\ , \tag{18}$$

where we used the fact that $\Sigma_B \Sigma_Q^\ddagger \Sigma_Q = \Sigma_B$, since for lemma 1 one has $\Sigma_Q^2 \geq p\Sigma_B^2$.

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