Statistical fractional-photon squeezed states

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We construct a large class of squeezed states realized in terms of density matrices. The moments of the canonical variable $\hat{q}$ and the number operator $\hat{n}$ distributions are analytically evaluated. The new states can be simultaneously squeezed in $\hat{q}$ and $\hat{n}$ to any desired amount and exhibit almost Gaussian shape in the former and strongly sub-Poisson distribution in the latter. The probability distributions can be synthetically characterized in terms of the fractional-photon index introduced by Katriel, Rassetti, and Solomon [Phys. Rev. D **35**, 1248 (1987)].

I. INTRODUCTION

Nonclassical properties of special photon states have recently been given a great deal of attention. Among the typical quantum effects which characterize the nonclassical states, two have received most attention, in view of the possible improvement of the signal-to-noise ratio in optical communications and in the interferometric detectors. The first effect is squeezing, in which the uncertainty of one of the two quadrature components of the field is reduced under the coherent-state value (the other quadrature component obviously exhibits increased fluctuations due to Heisenberg's uncertainty principle). The second effect is the sub-Poissonian photon statistics, in which the number distribution is narrower than the Poisson distribution.

The schemes which have been suggested and/or used to generate squeezed states (see, for example, Refs. 1–6) are essentially based on nonlinear optical processes where the field interacts with matter characterized by nonlinear polarizability (kth-order susceptibility in the general case). This corresponds to producing k-photon states[6] (i.e., states in which multiple of k photons are created from the vacuum), such as the usual two-photon squeezed states.[7,8] Schemes have been proposed to also produce strongly sub-Poisson states,[9,10] even if the ordinary squeezed states can exhibit sub-Poisson photon statistics. The improvement in number fluctuations reduction increases the noise in phase, leading to the so-called amplitude squeezed states.[9] Also the schemes to produce such states are based on nonlinear effects and multiphoton processes are involved in the interaction Hamiltonian.

Thus new interest has surfaced in the k-photon processes and in the relationships between squeezing and number fluctuations. On the other hand, the theoretical studies of k-photon squeezing in the general case, i.e., also for $k>2$, run into unexpected difficulties. The first generalizations by the simplest ansatz[11] cannot be handled by analytical techniques as the normal ordering and Taylor expansions methods lead to series which have a zero radius of convergence. The problem can be partly overcome from the computational point of view using Padé approximants.[6] The *non-naïve* way to define k-photon states have been subsequently proposed by D'Ariano and co-workers[12–17] based on the Brandt-Greenberg[18] multiphoton operators and group-theoretical coherent states. These states, allowing analytical calculations of the probability distributions, lead to interesting unexpected relationships between canonical $\hat{q}$ and number $\hat{n}$ moments in form of universal scaling laws, depending only on the photon index k.[16]

In this paper a new larger class of squeezed states is presented, which are mixed states realized in terms of suitable density matrices and which reduce to the previous ones in special cases. The large number of parameters involved allows one to achieve a fine tuning of the probability distributions to match the experimental requirements. Furthermore, all the states can be synthetically characterized in terms of the "fractional photon index" $t^{-1}=h/k$,[19] the matrix elements of h-photon operators between k-photon states depending only on it. Here we perform a thorough analysis of the probability distribution functions for the particular case of $1/r$ coherent states (where r is an integer), showing that they are squeezed in the canonical variable $\hat{q}$ and exhibit sub-Poissonian statistics. Moreover, both of the preceding properties can be attained to any extent by reducing $t=1/r$. In fact both the minimum value of $(\Delta q)^2$ and $(\Delta n)^2/\langle \hat{n} \rangle$ are equal to $1/r$; this result leads to a nice phenomenological interpretation of $\hat{q}$ and $\hat{n}$ squeezing in terms of photon fractioning. On the other hand, the sub-Poisson character should be compared with the one concerning the usual squeezed states—where $(\Delta n)^2$ is restricted by $\langle \hat{n} \rangle^{2/3}$–and the one relative to the amplitude squeezed states defined by Kitagawa and Yamamoto[9] which attain a minimum variance proportional to $\langle \hat{n} \rangle^{1/3}$.

In Sec. II we recall some algebraic background. In Sec. III the fractional boson probability distributions are introduced and the related density matrices are explicitly written. The properties of $1/r$ coherent states are derived in Sec. IV and the probability distributions are analyzed. In Appendix A, B, and C some asymptotic evaluations, involving nonstandard techniques, are reported.

II. ALGEBRAIC BACKGROUND

Brandt and Greenberg introduced in Ref. 18 new Bose operators $b_{(k)}$ and $b^{\dagger}_{(k)}$ satisfying the commutation relations

$$[b_{(k)},b_{(k)}^{\dagger}]=1\ , \tag{2.1}$$

$$[\hat{n},b_{(k)}]=-kb_{(k)}\ , \tag{2.2}$$

where $\hat{n}=a^{\dagger}a$ is the usual number operator. Equations (2.1) and (2.2) show that $b_{(k)}$ and $b_{(k)}^{\dagger}$ are annihilation and creation operators of k photons simultaneously (notice that $b_{(1)}=a$, but $b_{(k)}\neq a^{k}$ for $k\geq 2$).

From (2.1) and (2.2) one can derive the normal-ordered representation

$$b_{(k)}=\sum_{j=0}^{\infty}\alpha_{j}^{(k)}(a^{\dagger})^{j}a^{j+k}\ , \tag{2.3}$$

where

$$\alpha_{j}^{(k)}=\sum_{l=0}^{j}\frac{(-1)^{j-l}}{(j-l)!}\left[\frac{1+\left[\left[\frac{l}{k}\right]\right]}{l!(l+k)!}\right]^{1/2}e^{i\phi_{l}}\ . \tag{2.4}$$

In Eq. (2.4) $[[x]]$ denotes the maximum integer $\leq x$, whereas the phases ϕ_l, $l=0,\ldots,j$ are arbitrary real numbers

In the Fock space $b_{(k)}$ and $b_{(k)}^{\dagger}$ operate as follows:

$$b_{(k)}|sk+\lambda\rangle=\sqrt{s}\,|(s-1)k+\lambda\rangle\ , \tag{2.5a}$$

$$b_{(k)}^{\dagger}|sk+\lambda\rangle=\sqrt{s+1}\,|(s+1)k+\lambda\rangle\ , \tag{2.5b}$$

where $0\leq\lambda\leq k-1$.

From Eqs. (2.5) one can notice that the Fock space splits into k orthogonal subspaces which are invariant under the action of the k-photon operators; the generic Fock state $|sk+\lambda\rangle$ is thus labeled by two quantum numbers s and λ, which are the eigenvalues of the complete set of commuting operators $b_{(k)}^{\dagger}b_{(k)}$ and $\hat{D}_{(k)}=a^{\dagger}a-kb_{(k)}^{\dagger}b_{(k)}$

$$b_{(k)}^{\dagger}b_{(k)}|sk+\lambda\rangle=s|sk+\lambda\rangle\ , \tag{2.6a}$$

$$\hat{D}_{(k)}|sk+\lambda\rangle=\lambda|sk+\lambda\rangle\ . \tag{2.6b}$$

The natural extension of the previous construction is to consider expectation values of the $b_{(h)}$ and $b_{(h)}^{\dagger}$ operators between k-photon states. After a straightforward calculation one obtains ($t=k/h$)

$$\begin{aligned}\langle km|(b_{(h)}^{\dagger})^{u}(b_{(h)})^{v}|kn\rangle &=\frac{([[tn+u-v]]![[tn]]!)^{1/2}}{[[tn-v]]!}\Delta\left[\frac{u-v}{t}\right]\\ &\quad\times\Theta([[tn]]-v)\delta_{m,n+(u-v)t}\ ,\end{aligned} \tag{2.7}$$

where $\Delta(x)=1$ if x is integer and $\Delta(x)=0$ otherwise and $\Theta(x)$ denotes the usual Heaviside function [$\Theta(x)=1$ for $x\geq 0$ and $\Theta(x)=0$ for $x<0$)].

From Eq. (2.7) it appears evident that the matrix elements depend only on $t^{-1}=k/h$ and not explicitly on k or h. This fact suggested to Katriel, Rasetti, and Solomon[19] the introduction of the notion of fractional photon index characterizing probability distributions (indeed, it is impossible to define fractional boson operators $b_{(h/k)}^{\dagger}$ and $b_{(h/k)}$ and consequently fractional boson states as $\hat{N}_{(h/k)}=b_{(h/k)}^{\dagger}b_{(h/k)}$ eigenstates). In Ref. 19 the fractional photon coherent probability are defined selecting a coherent basis in the k-photon Fock space

$$|\omega\rangle_{(s)}=e^{-|\omega|^2/2}\sum_{n=0}^{\infty}\frac{\omega^n}{\sqrt{n!}}|sn\rangle \tag{2.8}$$

or in general using group theoretical coherent states using k-photon Holstein-Primakoff realizations of the group generators. In Sec. III we will show how we can exactly construct density matrices which have the fractional-photon-probability distributions.

III. STATISTICAL FRACTIONAL-PHOTON STATES

In the previous scheme the fractional-boson picture is obtained defining new canonical variables $\hat{Q}_{(h)}$ and $\hat{P}_{(h)}$

$$\hat{Q}_{(h)}=\frac{1}{\sqrt{2}}(b_{(h)}^{\dagger}+b_{(h)})\ , \tag{3.1a}$$

$$\hat{P}_{(h)}=\frac{i}{\sqrt{2}}(b_{(h)}^{\dagger}-b_{(h)})\ , \tag{3.1b}$$

and interpreting the Casimir operator $\hat{D}_{(h)}$ as a *hidden* observable; the number operator corresponds to $\hat{N}_{(h)}=b_{(h)}^{\dagger}b_{(h)}$. For example, when $h=2$ and $k=1$ we obtain the $\frac{1}{2}$-boson picture and $\lambda=0,1$ gives the "boson fraction" quantum number.

The definition of the position $\hat{Q}_{(h)}$-probability distributions for fractional photon states is based on the construction of a complete set of eigenvectors for the two mutually commuting operators $\hat{Q}_{(h)}$ and $\hat{D}_{(h)}$. The diagonalizing procedure is standard and gives the following result:

$$\begin{aligned}|Q,\lambda\rangle_{(h)}&=\sum_{l=0}^{\infty}C_l(Q)|lh+\lambda\rangle\ ,\\ C_l(Q)&=\frac{e^{-Q^2/2}H_l(Q)}{\sqrt{2^l l!\sqrt{\pi}}}\ ,\end{aligned} \tag{3.2}$$

where $H_l(Q)$ are the usual Hermite polynomials of degree l. One can easily check that

$$\begin{aligned}\hat{Q}_{(h)}|Q,\lambda\rangle_{(h)}&=Q|Q,\lambda\rangle_{(h)}\ ,\\ \hat{D}_{(h)}|Q,\lambda\rangle_{(h)}&=\lambda|Q,\lambda\rangle_{(h)}\ .\end{aligned} \tag{3.3}$$

The next step is to consider a k-photon state in the k-photon sector

$$|\omega\rangle_{(k)}=\sum_{m=0}^{\infty}\omega_m|km\rangle \tag{3.4}$$

and then to construct the probability distribution of the canonical variable $\hat{Q}_{(h)}$ for the k-photon state $|\omega\rangle_{(k)}$ ($k\neq h$ and $t=k/h=s/r$, with s and r relatively prime)

$$\mathcal{P}_\omega^{(t)}(Q)=\sum_{\lambda=0}^{h-1}|{}_{(h)}\langle Q,\lambda|\omega\rangle_{(k)}|^2$$
$$=\sum_{\lambda=0}^{r-1}\left|\sum_{m=0}^{\infty}C_{[[tm]]}\omega_m\delta_{\langle\langle tm\rangle\rangle,\lambda/r}\right|^2$$
$$=\sum_{l,m=0}^{\infty}\omega_l^*\omega_m C_{[[tl]]}(Q)C_{[[tm]]}(Q)\times\delta_{\langle\langle tl\rangle\rangle,\langle\langle tm\rangle\rangle}\ , \tag{3.5}$$

where $\langle\langle x\rangle\rangle=x-[[x]]$ denotes the fractional part of x. Equation (3.5) shows clearly that the probability distribution depends only on t and can be thus referred to as *fractional-photon-probability distribution*. In order to characterize completely a physical system in the fractional boson picture it is necessary to assign the probability distributions for every observable $\hat{\Theta}_{(h)}$ (in particular for a complete set of commuting operators).

We can generalize the above procedure, adopted for $\hat{Q}_{(h)}$, to any operator $\hat{\Theta}_{(h)}$ which is an analytic function of $b_{(h)}$ and $b^\dagger_{(h)}$, then commuting with the Casimir operator $\hat{D}_{(h)}$. The probability distribution for $\hat{\Theta}_{(h)}$ is then written as follows:

$$\mathcal{T}_\omega^{(t)}(\Theta)=\sum_{\lambda=0}^{h-1}|{}_{(h)}\langle\Theta,\lambda|\omega\rangle_{(k)}|^2\ , \tag{3.6}$$

where

$$|\Theta,\lambda\rangle_{(h)}=\sum_{l=0}^{\infty}|lh+\lambda\rangle\langle l|\Theta\rangle \tag{3.7}$$

and $N|\Theta\rangle$ are the eigenstates for the operator $\hat{\theta}=\hat{\Theta}_{(1)}$, which is the same function as $\hat{\Theta}_{(h)}$, now in the arguments a and $a^\dagger$. For example, the number operator $\hat{N}_{(h)}$ has the probability distribution

$$\mathcal{N}_\omega^{(t)}(N)=\sum_{\lambda=0}^{h-1}|{}_{(h)}\langle N,\lambda|\omega\rangle_{(k)}|^2$$
$$=\sum_{l,m=0}^{\infty}\omega_m^*\omega_l\delta_{[[tm]],N}\delta_{[[tl]],N}\delta_{\langle\langle tm\rangle\rangle,\langle\langle tl\rangle\rangle}\ . \tag{3.8}$$

Now we require that the observables describing the physical system should be functions of the usual particle operators a and $a^\dagger$ instead of $b_{(h)}$ and $b^\dagger_{(h)}$. More precisely, we look for a quantum state having the probability distributions for the usual observables $\hat{\theta}=\hat{\Theta}_{(1)}$ which are identical to the fractional probabilities $\mathcal{T}_\omega^{(t)}(\Theta)$ for any observable $\hat{\Theta}_{(h)}$.

The attempt to construct such a state leads us to reconsider the distribution $\mathcal{T}_\omega^{(t)}(\Theta)$ in the form

$$\mathcal{T}_\omega^{(t)}(\Theta)=\sum_{\lambda=0}^{r-1}\sum_{l,m=0}^{\infty}{}_{(s)}\langle\omega|lr+\lambda\rangle\langle l|\Theta\rangle\langle\Theta|m\rangle\langle mr+\lambda|\omega\rangle_{(s)}$$
$$=\sum_{l,m=0}^{\infty}\langle l|\Theta\rangle\langle\Theta|\left[\sum_{\lambda=0}^{r-1}(|m\rangle\langle mr+\lambda|\omega\rangle_{(s)})({}_{(s)}\langle\omega|lr+\lambda\rangle\langle l|)\right]|l\rangle\ . \tag{3.9}$$

Equation (3.9) shows clearly that $\mathcal{T}_\omega^{(t)}(\Theta)$ is reproduced by the $\hat{\theta}$-probability distribution of the following mixed state:[20]

$$\mathcal{T}_\omega^{(t)}(\theta)=\mathrm{Tr}(|\theta\rangle\langle\theta|\hat{\rho}_\omega^{(t)})\ , \tag{3.10}$$

with density matrix $\hat{\rho}_\omega^{(t)}$ given by

$$\hat{\rho}_\omega^{(t)}=\sum_{\lambda=0}^{r-1}|\Omega_\lambda^{(t)}\rangle\langle\Omega_\lambda^{(t)}|\ ,$$
$$|\Omega_\lambda^{(t)}\rangle=\sum_{m=0}^{\infty}\Omega_{\lambda,m}^{(t)}|m\rangle\ , \tag{3.11}$$
$$\Omega_{\lambda,m}^{(t)}=e^{i\phi\lambda}\langle mr+\lambda|\omega\rangle_{(s)}\ ,$$

with ϕ_λ arbitrary phases. One can easily check that the density matrix $\hat{\rho}_\omega^{(t)}$ is correctly normalized

$$\mathrm{Tr}\hat{\rho}_\omega^{(t)}=\sum_{m=0}^{\infty}\sum_{\lambda=0}^{r-1}|\Omega_{\lambda,m}^{(t)}|^2=\|\omega\|^2=1\ , \tag{3.12}$$

and that, for example, both $\hat{q}$ and $\hat{n}$ distributions reproduce the distributions (3.5) and (3.8), respectively,

$$\mathcal{P}_\omega^{(t)}(q)=\mathrm{Tr}(|q\rangle\langle q|\hat{\rho}_\omega^{(t)})$$
$$=\sum_{\lambda=0}^{r-1}|\langle q|\Omega_\lambda^{(t)}\rangle|^2$$
$$=\sum_{\lambda=0}^{r-1}\left|\sum_{n=0}^{\infty}\langle q|n\rangle\langle nr+\lambda|\omega\rangle_{(s)}\right|^2$$
$$=\sum_{\lambda=0}^{r-1}|{}_{(r)}\langle q,\lambda|\omega\rangle_{(s)}|^2\ , \tag{3.13}$$

$$\mathcal{N}_\omega^{(t)}(n)=\mathrm{Tr}(|n\rangle\langle n|\hat{\rho}_\omega^{(t)})$$
$$=\sum_{\lambda=0}^{r-1}|\langle n|\Omega_\lambda^{(t)}\rangle|^2$$
$$=\sum_{\lambda=0}^{r-1}|\Omega_{\lambda,n}^{(t)}|^2$$
$$=\sum_{\lambda=0}^{r-1}|\langle nr+\lambda|\omega\rangle_{(s)}|^2=\sum_{\lambda=0}^{r-1}|{}_{(r)}\langle n,\lambda|\omega\rangle_{(s)}|^2\ . \tag{3.14}$$

We can thus conclude that all the fractional probabilities can be realized by means of mixed states with a density matrix $\hat{\rho}_{\omega}^{(t)}$ given by Eq. (3.11): in the particular case of integer t ($r=1$ and $s=t$) the pure multiboson states of Refs. 12–17 are obtained. Thus the physical nature of the fractional photon picture is essentially quantum statistical.

IV. FRACTIONAL COHERENT STATES

In this section we will focus our attention on the particular case of fractional coherent states, i.e., fractional states obtained choosing $|\omega\rangle_{(s)}$ as a coherent state. The most interesting states are the $1/r$ coherent states, in this case the density matrix is given by

$$\hat{\rho}_{\omega}^{(1/r)}=e^{-|\omega|^2}\sum_{\lambda=0}^{r-1}\sum_{l,m=0}^{\infty}|m\rangle\frac{\omega^{mr+\lambda}\omega^{*lr+\lambda}}{\sqrt{(mr+\lambda)!(lr+\lambda)!}}\langle l| \ . \tag{4.1}$$

We observe that $\hat{\rho}_{\omega}^{(1/r)}$ is a periodic function of $\arg\omega$ with period $2\pi/r$. The time evolution under the action of the usual harmonic oscillator Hamiltonian $\hat{H}=\omega_0(a^{\dagger}a+\frac{1}{2})$ maps ω in $\omega\exp[-i(\omega_0/r)t]$.

The probabilities (3.13) and (3.14) are rewritten explicitly

$$\mathcal{P}_{\omega}^{(t)}(q)=e^{-|\omega|^2}\sum_{l,m=0}^{\infty}C_l(q)C_m(q)\omega^{*rl}\omega^{rm}\times\sum_{\lambda=0}^{r-1}\frac{|\omega|^{2\lambda}}{\sqrt{(rl+\lambda)!(rm+\lambda)!}} \ , \tag{4.2}$$

$$\mathcal{N}_{\omega}^{(t)}(n)=e^{-|\omega|^2}\sum_{\lambda=0}^{r-1}\frac{|\omega|^{2(rn+\lambda)}}{(rn+\lambda)!} \ . \tag{4.3}$$

We will show that these states exhibit very intriguing physical features; they can be squeezed in one of the canonical variables $\hat{p}$ or $\hat{q}$ (as already shown in Ref. 19). Furthermore, they have quasi-Gaussian distributions and have strongly sub-Poisson $\hat{n}$ fluctuations. Some analytical evaluations of first two moments for both distributions will better clarify the physical meaning of the "fractional photon index" t.

A. Position probability distributions and squeezing

Figure 1 represents the position probability distribution $\mathcal{P}_{\omega}^{(t)}(q)$ versus q for $\arg\omega=\pi$ (i.e., in the direction of best squeezing) for various values of $\rho=|\omega|$ and t. One notices that the distributions deviate slightly from the Gaussian shape, squeezing occurring mostly as Gaussian narrowing. As one increases ρ, only small local maxima appear on a tail of the distribution, resulting in a weak asymmetry and this last feature becomes more and more apparent as one reduces t. The average value of the distributions increases in the positive (negative) direction for t^{-1} even (odd). For fixed ρ and decreasing t the distribution approaches the vacuum Gaussian, according to a consistent physical interpretation of vacuum state as complete photon fractioning. This last result can also be analytically checked from Eq. (4.1) as follows:

$$\hat{\rho}_{\omega}^{(0)}=\lim_{r\to\infty}e^{-|\omega|^2}\sum_{\lambda=0}^{r-1}\sum_{l,m=0}^{\infty}|m\rangle\frac{\omega^{mr+\lambda}\omega^{*lr+\lambda}}{\sqrt{(mr+\lambda)!(lr+\lambda)!}}\langle l|$$
$$=e^{-|\omega|^2}\sum_{\lambda=0}^{\infty}|0\rangle\frac{|\omega|^{\lambda}}{\lambda!}\langle 0|=|0\rangle\langle 0| \ . \tag{4.4}$$

Focusing our attention on squeezing, we evaluate the second moment of the distributions (divided by the Gaussian one)

$$\frac{\langle(\Delta\hat{q}^2)\rangle}{\langle 0|(\Delta\hat{q}^2)|0\rangle}=1+2\{\langle a^{\dagger}a\rangle-|\langle a^{\dagger}\rangle|^2+\mathrm{Re}[\langle(a^{\dagger})^2\rangle-\langle a^{\dagger}\rangle^2]\} \ , \tag{4.5}$$

where the expectation values are statistical averages with respect to $\hat{\rho}_{\omega}^{(1/r)}$

$$\langle\hat{\mathcal{O}}\rangle=\mathrm{Tr}(\hat{\rho}_{\omega}^{(1/r)}\hat{\mathcal{O}}) \tag{4.6}$$

and are given by

$$\langle a^{\dagger}\rangle=e^{-|\omega|^2}\omega^{*r}\sum_{n=0}^{\infty}\frac{|\omega|^{2n}}{\sqrt{n!(n+r)!}}(1+[[n/r]])^{1/2} \ , \tag{4.7a}$$

$$\langle a^{\dagger}a\rangle=e^{-|\omega|^2}\sum_{n=0}^{\infty}\frac{|\omega|^{2n}}{n!}[[n/r]] \ , \tag{4.7b}$$

$$\langle(a^{\dagger})^2\rangle=e^{-|\omega|^2}\omega^{*2r}\sum_{n=0}^{\infty}\frac{|\omega|^{2n}}{\sqrt{n!(n+2r)!}}[(1+[[n/r]])(2+[[n/r]])]^{1/2} \ . \tag{4.7c}$$

The plots of $\chi_{\omega,(t)}^{(2)}$ vs ρ for some t are shown in Fig. 2 (the low-ρ part of the plot was numerically evaluated in Ref. 19). One can see that in the low-ρ regime squeezing increases as t increases whereas one has the opposite behavior for large ρ. Moreover, in the limit $\rho\to\infty$ the second moment behaves as asymptotically constant (see Appendix A)

$$\chi_{\omega,(t)}^{(2)}=t+O(\rho^{-2}) \ , \tag{4.8}$$

the asymptotic squeezing going to zero like t; this last result gives a suggestive phenomenological interpretation of squeezing as statistical photon fractioning. The apparent contradiction between the last assertion and the previous one—concerning vacuum state as complete photon

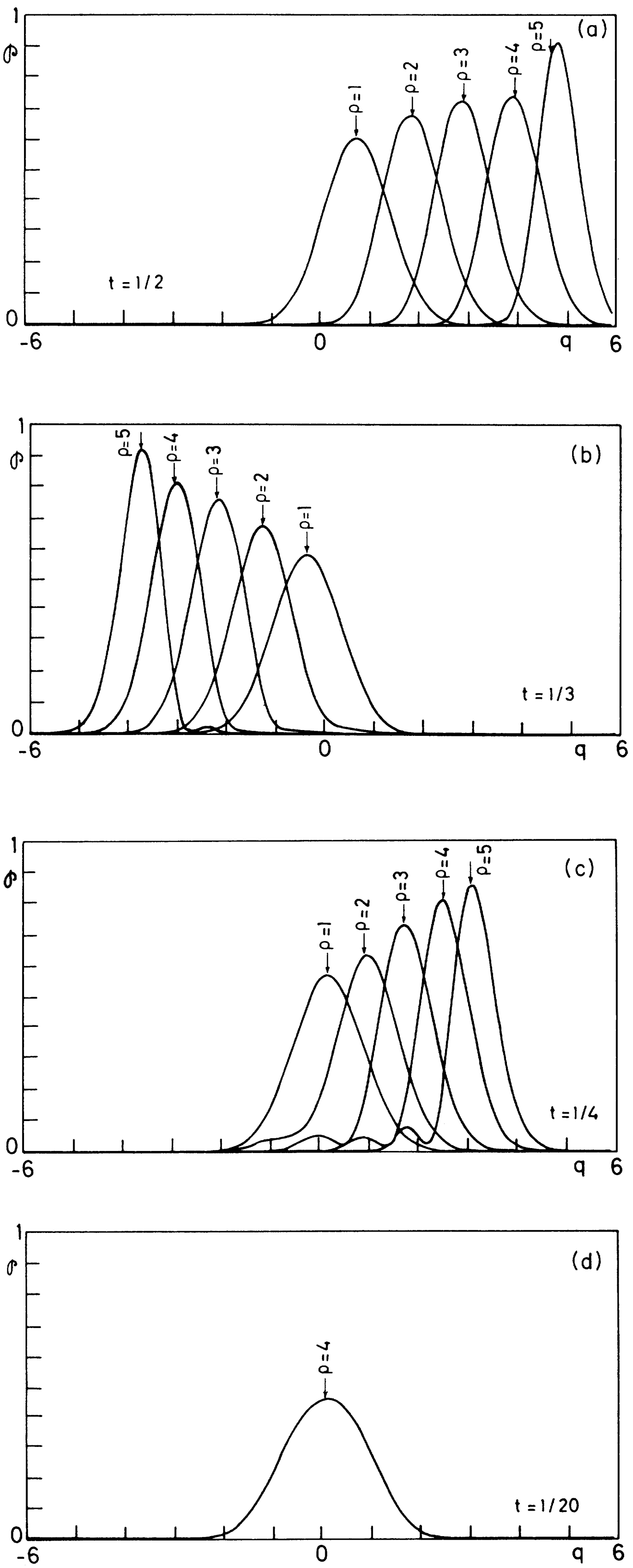

FIG. 1. Fractional coherent state position probability distribution $\mathcal{P}_{\omega}^{(t)}(q)$ vs q for $\arg\omega=\pi$ and for various values of $\rho=|\omega|$ and t (the fractional-photon index): the arrows indicating the values of ρ are directed exactly at the distribution average value.

fractioning—can be understood if one considers that the two limits, $\rho\rightarrow\infty$ and $t\rightarrow 0$, do not commute; performing firstly the $\rho\rightarrow\infty$ limit the totally squeezed states is obtained, whereas exchanging the limits the vacuum state results.

B. Number probability distributions

Figure 3 shows the number probability distribution $\mathcal{N}_{\omega}^{(t)}(n)$ versus n for $\arg\omega=\pi$ and for some values of $\rho=|\omega|$ and t (for the sake of comparison the usual coherent states corresponding to $t=1$ are included as well). One can see that the fractional state has decreasing mean number of photons for decreasing t, according to the intuitive meaning of photon fractioning. Indeed, for large ρ, an asymptotic evaluation gives the result (see Appendix B)

$$\langle\hat{n}\rangle=t\rho^2\left[1+\frac{t-1}{2t}\rho^{-2}+O(\rho^{-4})\right], \tag{4.9}$$

and the average number scales with the fractional index t (at least asymptotically).

As regards the width of the distribution, one can notice that smaller t correspond to narrower distributions; sub-Poisson behavior is evident comparing the fractional distributions with the $t=1$ coherent state distribution. In Appendix B we compute the asymptotic behavior of the variance for large ρ,

$$\Delta n\equiv(\langle\hat{n}^2\rangle-\langle\hat{n}\rangle^2)^{1/2}=t\rho[1+O(\rho^{-2})]\ . \tag{4.10}$$

By means of Eqs. (4.9) and (4.10) we obtain the deviation σ from the Poisson distribution,

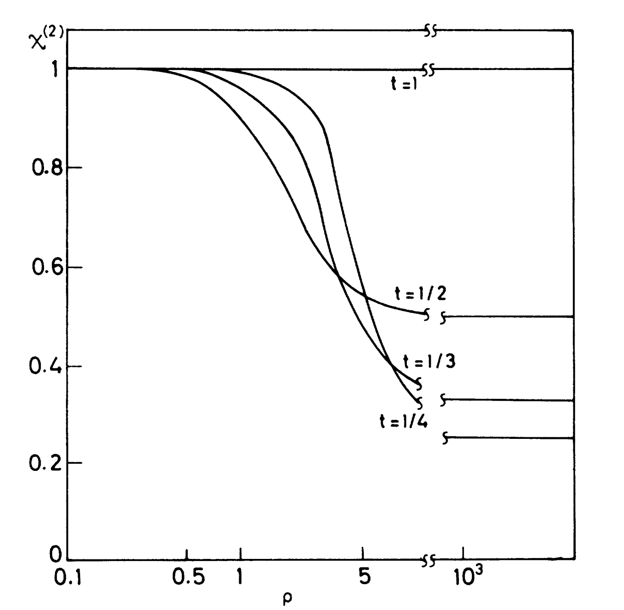

FIG. 2. Dependence of the second moment $\chi^{(2)}$ on the squeezing parameter $\rho=|\omega|$ for $\arg\omega=\pi$ and for various values of t; the asymptotes are those analytically evaluated in Appendix A; the low-ρ part of the plot was numerically evaluated in Ref. 19.

$$\sigma=\frac{(\Delta n)^2}{\langle \hat{n}\rangle}=t[1+O(\rho^{-2})] \;, \tag{4.11}$$

which shows that sharper sub-Poisson behavior can be attained by reducing t.

V. CONCLUSIONS

The class of mixed states constructed in this paper depends on a large (in practice infinite) number of parameter [besides t and ω one can change the Fock states $|\omega\rangle_{(s)}$ in Eq. (3.11) using a general multiphoton state] and this allows the probability distributions to be adapted to the experimental ones. These latter exhibit the fractional photon behavior formally introduced in Ref. 19 by means of expectation values. For integer value of the fractional photon index t^{-1} the usual multiphoton squeezed states are recovered. A thorough analysis of the probability distribution functions corresponding to the fractional coherent states shows that they are almost Gaussian, as regards the canonical variables, and can attain any degree of squeezing. Furthermore, they are sub-Poisson in the $\hat{n}$ fluctuations. As a result we obtain a threefold physical meaning for the "fractional photon index" t (in the squeezing regime): (i) it is the value of the best amount of squeezing; (ii) it is a linear scale parameter for the average number value; (iii) it is the index of the sub-Poisson behavior of the state. The preceding physical interpretation can be summarized in a phenomenological picture in which $\hat{q}$ and $\hat{n}$ squeezing are regarded as photon fractioning.

As the fractional coherent states can be simultaneously squeezed and sub-Poisson to any degree, one may suspect that they should be *amplitude squeezed* states according to the definition of Kitagowa and Yamamoto.[9] Some preliminary numerical results on the quasi-probability-distribution function seem to indicate that indeed this is the case.

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APPENDIX A

In this appendix we study the asymptotic behavior of the second moment $\chi^{(2)}_{(t)}$ for fractional states $|\omega\rangle_{(t)}$ in the limit $|\omega|\to\infty$ with $\arg\omega=\pi$. The complete analytic expression of the second moment is given by

$$\chi^{(2)}_{\omega,(t)}=1+2e^{-x}F_1(x)-4e^{-2x}x^r|F_2(x)|^2 +2e^{-x}x^rF_3(x) \;, \tag{A1}$$

where $x=|\omega|^2$, $r=t^{-1}$, and the auxiliary functions $F_n(x)$ are written as follows:

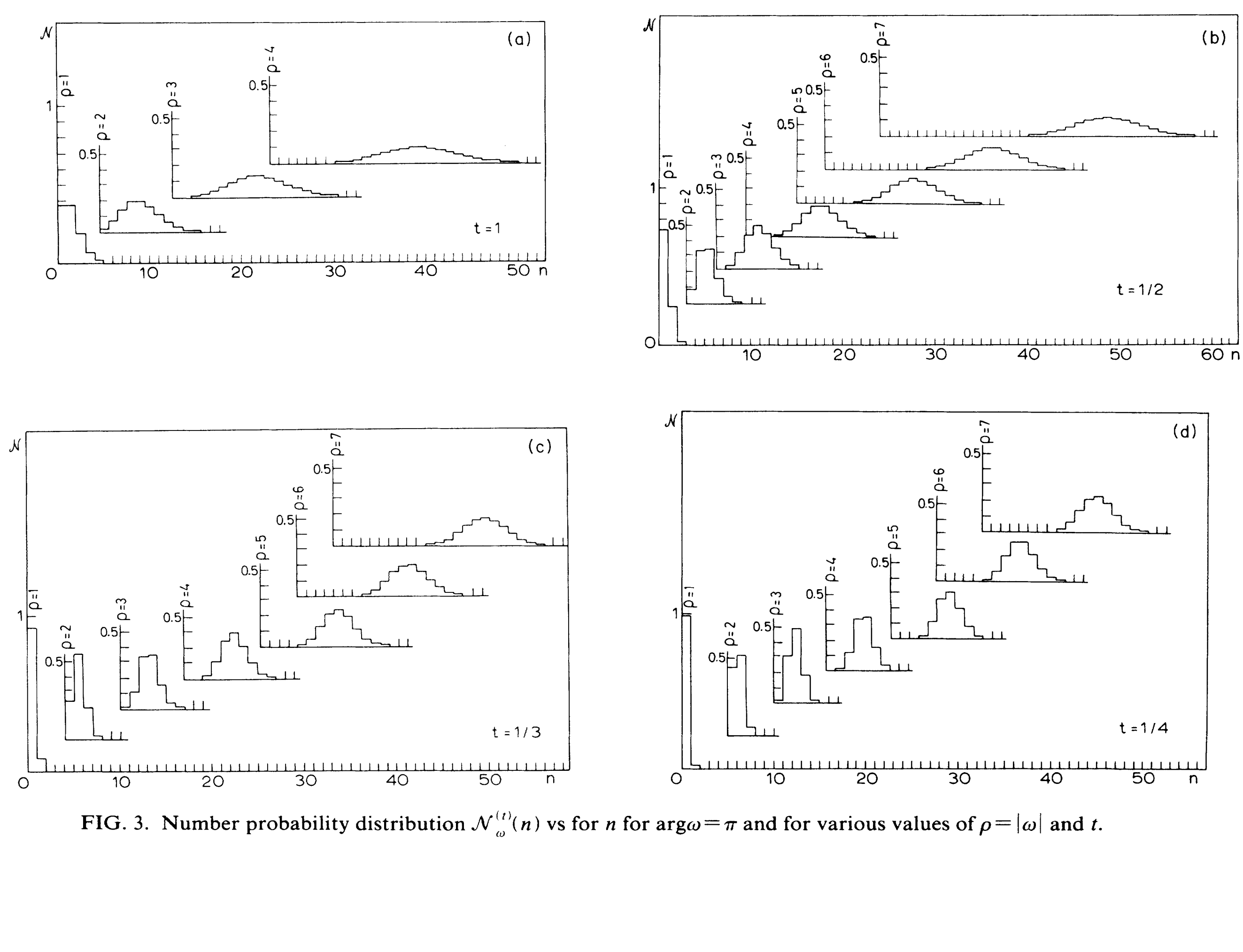

FIG. 3. Number probability distribution $\mathcal{N}^{(t)}_{\omega}(n)$ vs for n for $\arg\omega=\pi$ and for various values of $\rho=|\omega|$ and t.

$$F_1(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}[[n/r]]\ , \tag{A2a}$$

$$F_2(x)=\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n!(n+r)!}}(1+[[n/r]])^{1/2}\ , \tag{A2b}$$

$$F_3(x)=\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n!(n+2r)!}}\{(1+[[n/r]])(2+[[n/r]])\}^{1/2}\ . \tag{A2c}$$

In studying the asymptotic behavior of the functions $F_n(x)$ one needs the following identities for Taylor expansions of the form $\sum_{n=0}^{\infty}c_n h([[n/r]])x^n$, where $h(y)$ is analytic in an integer neighbor of n/r,

$$\sum_{n=0}^{\infty}c_n h([[n/r]])x^n=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}r^{-l-1}\times\sum_{d,s=0}^{r-1}d^l e^{-(2\pi i/r)sd}\times f_l(xe^{2\pi is/r})\ ,$$

$$f_l(x)=\sum_{n=0}^{\infty}c_n h^{(l)}\left[\frac{n}{r}\right]x^n\ , \tag{A3}$$
$$h^{(l)}(y)=\partial_y^l h(y)\ .$$

Disregarding terms of order $\exp\{x[\cos(2\pi/r)-1]\}$ we obtain the intermediate result

$$F_1(x)\sim\sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{n}{r}-\frac{r-1}{2r}\sum_{n=0}^{\infty}\frac{x^n}{n!}=\frac{e^x}{r}\left[x+\frac{1-r}{2}\right]\ , \tag{A4a}$$

$$F_2(x)\sim r^{-1/2}\left[H^{[r]}_{-1/2,\dots,-1/2,0}(x)-\frac{r-1}{4}H^{[r]}_{-1/2,\dots,-1/2,-1}(x)\right]\ , \tag{A4b}$$

$$F_3(x)\sim\frac{1}{2r}H^{[2r]}_{-1/2,\dots,1/2}(x)+\frac{1}{r}H^{[2r]}_{-1/2,\dots,-1/2,1/2,\ -1/2,\dots,-1/2}(x)\ , \tag{A4c}$$

where the functions $H^{[r]}_{\alpha_0,\dots,\alpha_{r-1}}(x)$ are defined as follows (α_k real):

$$H^{[r]}_{\alpha_0,\dots,\alpha_{r-1}}(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\prod_{k=0}^{r-1}(n+1+k)^{\alpha_k}\ . \tag{A5}$$

In Appendix C an asymptotic expansion for the functions $H^{[r]}_{\alpha_0,\dots,\alpha_{r-1}}(x)$ is derived, which gives

$$H^{[r]}_{\alpha_0,\dots,\alpha_{r-1}}(x)\sim x^{\|\alpha\|}e^x\left\{1+\left[\left[\frac{\|\alpha\|+1}{2}\right]+\sum_{k=0}^{r-1}k\alpha_k\right]\frac{1}{x}+O(x^{-2})\right\}\ , \tag{A6}$$

where $\|\alpha\|=\sum_{k=0}^{r-1}\alpha_k$.

With the preceding formula we can write the desired asymptotic expansions for the functions $F_n(x)$,

$$F_1(x)\sim\frac{e^x}{r}\left[x+\frac{1-r}{2}\right]\ , \tag{A7a}$$

$$F_2(x)\sim r^{-1/2}x^{-(r-1)/2}e^x\left[1+\frac{1}{x}\left[\frac{1-r^2}{8}\right]+O(x^{-2})\right]\ , \tag{A7b}$$

$$F_3(x)\sim\frac{e^x}{r}x^{-r+1}\left[1+\frac{1}{x}\left[\frac{1-r^2}{2}\right]+O(x^{-2})\right]\ . \tag{A7c}$$

Collecting all terms in (A1) we obtain the final result

$$\chi^{(2)}_{\omega,(t)}\sim1+\frac{2}{r}\left[x+\frac{1-r}{2}\right]-4\frac{x}{r}\left[1+\frac{1}{x}\left[\frac{1-r^2}{4}\right]\right]+2\frac{x}{r}\left[1+\frac{1}{x}\left[\frac{1-r^2}{2}\right]\right]+O(x^{-2})=\frac{1}{r}+O(x^{-2})\ . \tag{A8}$$

APPENDIX B

We are interested in the analytic asymptotic evaluation of the average number and variance of the fractional states. Using the probability distribution (3.6) we can write the complete analytic expression of $\langle\hat{n}^q\rangle$ $(t=1/r)$,

$$\langle\hat{n}^q\rangle=e^{-x}\sum_{n=0}^{\infty}[[tn]]^q\frac{x^n}{n!}\ . \tag{B1}$$

Using again the identities (A3) we can evaluate the sum in Eq. (B1) disregarding terms of order $\exp\{x[\cos(2\pi t)-1]\}$,

$$\langle\hat{n}^q\rangle=e^{-x}\sum_{l=0}^{q}(-1)^l\begin{bmatrix}q\\l\end{bmatrix}t^{q+1}\left[\sum_{d=0}^{r-1}d^l\right]\sum_{n=0}^{\infty}\frac{x^n}{n!}n^{q-l}=e^{-x}\sum_{l=0}^{q}(-1)^l\begin{bmatrix}q\\l\end{bmatrix}t^{q+1}\left[\sum_{d=0}^{r-1}d^l\right]x\Lambda_{q-l-1}(x)\ , \tag{B2}$$

where the function $\Lambda_\alpha(x)$ is defined as follows:

$$\Lambda_\alpha(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}(n+1)^\alpha\ . \tag{B3}$$

In Appendix C the function $\Lambda_\alpha(x)$ is asymptotically evaluated. Inserting the result (C9) in the right-hand side of (B2) we obtain

$$\langle \hat{n}^q \rangle = (tx)^q \left[1 + \frac{q}{2} \left[q - \frac{1}{t} \right] x^{-1} + O(x^{-2}) \right] . \tag{B4}$$

The asymptotic expansions for average number and variance are thus given by

$$\langle \hat{n} \rangle = tx \left[1 + \frac{t-1}{2t} x^{-1} + O(x^{-2}) \right] , \tag{B5}$$

$$\Delta n = (\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2)^{1/2} = t\sqrt{x}\,[1 + O(x^{-1})] . \tag{B6}$$

APPENDIX C

The asymptotic behavior of the functions

$$H^{[r]}_{\alpha_0,\ldots,\alpha_{r-1}}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \prod_{k=0}^{r-1} (n+1+k)^{\alpha_k} \tag{C1}$$

is obtained expanding the product in powers of $(n+1)^{-1}$,

$$\prod_{k=0}^{r-1} (n+1+k)^{\alpha_k} = (n+1)^{\|\alpha\|} \times \left[1 + (n+1)^{-1} \left[\sum_{k=1}^{r-1} k\alpha_k \right] + O((n+1)^{-2}) \right] , \tag{C2}$$

where $O((n+1)^{-2})$ denotes terms of order $(n+1)^{-2}$. Putting (C2) into (C1) we have

$$H^{[r]}_{\alpha_0,\ldots,\alpha_{r-1}}(x) = \Lambda_{\|\alpha\|}(x) + \left[\sum_{k=1}^{r-1} k\alpha_k \right] \Lambda_{-1+\|\alpha\|}(x) , \tag{C3}$$

$$\Lambda_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (n+1)^\alpha .$$

The asymptotic behavior of the functions $\Lambda_\alpha(x)$ can be evaluated for $\alpha < 0$ by means of the following identity:

$$r^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-rt} t^{-\alpha-1} dt, \quad r > 0, \quad \alpha < 0 . \tag{C4}$$

The functions $\Lambda_\alpha(x)$ are thus written in integral form

$$\Lambda_\alpha(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-t} t^{-\alpha-1} \exp(xe^{-t}) dt = \frac{e^x}{\Gamma(-\alpha)} \int_0^1 [-\ln(1-z)]^{-\alpha-1} e^{-xz} dz . \tag{C5}$$

Expanding the logarithm in powers of z, one has

$$\Lambda_\alpha(x) = \frac{e^x}{\Gamma(-\alpha)} \int_0^1 \left[\sum_{n=1}^{\infty} \frac{z^n}{n} \right]^{-\alpha-1} e^{-xz} dz = \frac{e^x}{\Gamma(-\alpha)} \int_0^1 z^{-\alpha-1} \left[1 - \frac{\alpha+1}{2} z + O(z^2) \right] e^{-xz} dz . \tag{C6}$$

All the integrals in the (C6) expansion can be generated deriving only the integral with respect to x,

$$\Lambda_\alpha(x) = \frac{e^x}{\Gamma(-\alpha)} \left[1 + \frac{\alpha+1}{2} \partial_x + O(\partial^2) \right] \times \int_0^1 z^{-\alpha-1} e^{-xz} dz . \tag{C7}$$

The integral in (C7) can be asymptotically evaluated,

$$\int_0^1 z^{-\alpha-1} e^{-xz} dz = x^\alpha \int_0^x t^{-\alpha-1} e^{-t} dt = x^\alpha \Gamma(-\alpha)[1 + O(e^{-x})] . \tag{C8}$$

The asymptotic behavior of $\Lambda_\alpha(x)$ is thus obtained putting (C8) into (C7)

$$\Lambda_\alpha(x) = x^\alpha e^x \left[1 + \frac{\alpha(\alpha+1)}{2} x^{-1} + O(x^{-2}) \right] , \tag{C9}$$

and by means of the preceding formula and Eq. (C3) one finally writes the desired asymptotic expansion

$$H^{[r]}_{\alpha_0,\ldots,\alpha_{r-1}}(x) \sim x^{\|\alpha\|} e^x \left\{ 1 + \left[\left[\frac{\|\alpha\|+1}{2} \right] + \sum_{k=0}^{r-1} k\alpha_k \right] \frac{1}{x} + O(x^{-2}) \right\} . \tag{C10}$$

We point out that expansions (C9) and (C10) can be analytically prolonged to positive values of α ($\|\alpha\|$, respectively) using the following trick

$$\Lambda_\beta(x) = (x\partial_x + 1)^{[[\beta]]+1} \Lambda_{-1+\langle\langle \beta \rangle\rangle}(x) , \tag{C11}$$

which enables us to use both (C9) and (C10) for any real value of the parameters.

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