

# Informationally complete measurements on bipartite quantum systems: Comparing local with global measurements

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Informationally complete measurements allow the estimation of expectation values of any operator on a quantum system, by changing only the data processing of the measurement outcomes. In particular, an informationally complete measurement can be used to perform quantum tomography, namely to estimate the density matrix of the quantum state. The data processing is generally nonunique, and can be optimized according to a given criterion. In this paper we provide the solution of the optimization problem which minimizes the variance in the estimation. We then consider informationally complete measurements performed over bipartite quantum systems focusing attention on universally covariant measurements, and compare their statistical efficiency when performed either locally or globally on the two systems. Among global measurements we consider the special case of Bell measurements, which allow us to estimate the expectation of a restricted class of operators. We compare the variance in the three cases: local, Bell, and unrestricted global—and derive conditions for the operators to be estimated such that one type of measurement is more efficient than the other. In particular, we find that for factorized operators and Bell projectors the Bell measurement always performs better than the unrestricted global measurement, which in turn outperforms the local one. For estimation of the matrix elements of the density operator, the relative performances depend on the basis on which the state is represented, and on the matrix element being diagonal or off-diagonal, however, with the global unrestricted measurement generally performing better than the local one.

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## I. INTRODUCTION

One of the most distinctive features of quantum mechanics is that the information on the state of the quantum system cannot be retrieved in a single measurement [1]. Among the different methods for retrieving information on the state—including quantum tomography, state discrimination, and quantum state estimation [2]—the informationally complete measurement [3–5] (shortly named *infocomplete* in the following) is the most versatile, allowing one to obtain the expectation  $\langle O \rangle$  of any operator  $O$  of the system, and with the additional possibility of adapting the processing of outcomes, depending on the quantity to be estimated and on the set of states on which the measurement is performed. A particularly interesting case is when the measurement is *covariant* [6,7] with respect to a group of physical transformations. This means that there is an action of a group on the probability space which maps events into events in a covariant fashion, namely that when the quantum system is group transformed, the probability of the given event becomes the probability of the transformed event. This situation is very natural, and occurs in many applications (e.g., heterodyne detection [8], measurement of the spin direction [9], transmission of reference frames [10], estimation of unitary transformations [11], etc.).

In this paper we study covariant infocomplete measurements for bipartite quantum systems. The problem of optimization of data processing will be solved using the method of *frame theory* [12,13], which recently has been shown [14,15] to be perfectly suited to optimize covariant infocomplete measurements. Dealing with a bipartite (or multipartite) quantum system, the study of the possibility to perform

information-theoretic tasks locally on nonlocal states has become a new paradigm in the field of quantum information. In this respect consider for example the study of local implementation of nonlocal quantum gates [16], telecloning [17], remote manipulation and preparation of quantum states [18], and local discrimination of nonlocal states [19]. The problem of comparing the performances of local and entangled estimation schemes has not been addressed yet, and this is precisely the purpose of the present paper, where we compare the statistical efficiency of local infocomplete measurements with that of nonlocal ones. In particular, the possibility of having a maximally entangled infocomplete measurement—or *Bell quorum*—has never been considered. Here we will show that, although an infocomplete Bell measurement strictly does not exist, however, Bell measurements can be used to estimate the expectation of any operator  $O$  having both partial traces  $\text{Tr}_1[O]$  and  $\text{Tr}_2[O]$  proportional to the identity, e.g., maximally entangled operators and traceless factorized observables. This possibility is interesting also in view of the relevant problem of the classification of measurements according to separability criteria, which still remains unsolved.

The statistical efficiency of an infocomplete measurement is quantified by the noise in the estimation of the operator expectation  $\langle O \rangle$ . This generally depends both on the state and on the operator  $O$  to be estimated. However, for unbiased estimations the squared expectation is independent of the measurement, whereas if the measurement is also fully covariant (i.e., covariant under the full group of unitary transformations) the square of the estimation averaged over the ensemble of possible input states depends only on the measurement: this greatly simplifies the evaluation of the

variances, and allows a simple comparison of statistical efficiencies of different infocomplete measurements.

The paper is organized as follows. After summarizing some useful notation and elements of the *theory of frames* in Sec. II, in Sec. III we give a general method for optimizing the data processing. In Sec. IV we consider covariant infocomplete measurements for bipartite quantum systems, in which case the optimal data processing is provided by the so-called *canonical dual frame*. In Secs. V–VII we derive the minimal noise (averaged over an ensemble of covariant states) for factorized, global, and Bell infocomplete covariant measurements, and finally we conclude the paper in Sec. VIII by comparing the resulting statistical efficiencies of the different types of measurements.

## II. NOTATION AND ELEMENTS OF THE THEORY OF FRAMES

An infocomplete measurement is characterized by a positive operator-valued measurement (POVM) that spans the whole space of linear operators. The *theory of frames* [12,13] provides simple and powerful tools to establish whether a set of operators is complete in the whole operator space, and in addition gives a simple algebraic rule for constructing all possible expansion coefficients—representing all possible data processing of measurement outcomes—in terms of the so-called *dual frames*. Upon denoting with  $\{P_i\}$  the elements of an infocomplete POVM [14], dual frames correspond to sets of operators  $\{D_i\}$  in terms of which we can write the operator expansion as follows:

$$\sum_{i=1}^N \text{Tr}[D_i^\dagger O] P_i = O, \quad (1)$$

where  $N$  denotes the number of outcomes of the POVM (in the following we will not specify it anymore in the sum limits). When considering expansions of operators over a Hilbert space  $\mathcal{H}$  it is convenient to exploit the natural isomorphism between operators  $O$  on  $\mathcal{H}$  and vectors  $|O\rangle\rangle$  in  $\mathcal{H}^{\otimes 2}$ , defined through the equation

$$|O\rangle\rangle \doteq \sum_{m,n} \langle m|O|n\rangle |m\rangle|n\rangle. \quad (2)$$

We will make repeated use of the following identities [20]:

$$A \otimes B |C\rangle\rangle = |ACB^\tau\rangle\rangle, \quad (3)$$

$$\text{Tr}_1[|A\rangle\rangle\langle\langle B|] = A^\tau B^*, \quad (4)$$

$$\text{Tr}_2[|A\rangle\rangle\langle\langle B|] = AB^\dagger, \quad (5)$$

where  $\tau$  and  $*$  denote transposition and complex conjugation with respect to the given fixed basis in Eq. (2). The main results from frame theory are the following [14]. First, a the POVM  $\{P_i\}$  is infocomplete if and only if the *frame operator*,

$$F = \sum_i |P_i\rangle\rangle\langle\langle P_i|, \quad (6)$$

is invertible. A dual frame  $\{D_i\}$  is given by the following one

(usually referred to as *canonical*):

$$|D_i\rangle\rangle = F^{-1}|P_i\rangle\rangle. \quad (7)$$

In general there exist infinitely many alternate duals  $\{D'_i\}$ , which can be obtained from the canonical one as follows:

$$D'_i = D_i + Y_i - \sum_j \text{Tr}[P_j^\dagger D_i] Y_j, \quad (8)$$

for arbitrary set  $\{Y_j\}$ . Finally, whenever a POVM is not complete, the span of its elements coincides with the *support* (the orthogonal complement of the kernel) of the operator  $F$ . Then the operators in such a subspace can be reconstructed through Eq. (1), where the canonical dual is now defined through the Moore-Penrose generalized inverse  $F^\ddagger$  of  $F$ , which corresponds to invert  $F$  on its support.

## III. OPTIMIZATION OF THE DATA PROCESSING

In the following we will restrict our attention only to finite dimension  $d < \infty$  of the system Hilbert space. Apart from the case of POVM  $\{P_i\}$  with exactly  $d^2$  linearly independent elements  $P_i$  (whence with discrete sample space), for infocomplete POVM's there always exist infinitely many alternate duals. This feature provides a wide freedom in choosing the dual frame, which can be exploited in order to optimize the data processing, e.g., to minimize the variance in the estimation. Therefore suppose that an experimenter is collecting statistics using an infocomplete measuring apparatus—e.g., in a quantum tomographic setup—with the aim of estimating the expectations of any desired operator. The noise in the estimation of the operator  $O$  can be evaluated by [21]

$$\delta O^2(\rho; P) \doteq \sum_i |\text{Tr}[D_i^\dagger O]|^2 \text{Tr}[P_i \rho] - |\text{Tr}[O \rho]|^2, \quad (9)$$

which clearly depends on the state  $\rho$ . Notice that we consider generally complex operators  $O$ , i.e., we can include the case of external products  $O = |i\rangle\langle j|$  of orthonormal basis, which are needed to estimate the matrix elements  $\rho_{ij}$  of the density operator  $\rho$ . In a Bayesian scheme the noise (9) can be minimized for a given prior probability distribution  $d\mu(\rho)$  of the input states  $\rho$ , corresponding to the prior state  $\bar{\rho} = \int d\mu(\rho) \rho$ . The dual frame  $\{D_i\}$  can be chosen in order to minimize such a noise, which in turn corresponds to minimize the following norm:

$$\frac{1}{d^2} \|c\|_\pi^2 = \frac{1}{d^2} \sum_i |\text{Tr}[D_i^\dagger O]|^2 \text{Tr}[P_i \rho], \quad (10)$$

where  $|c\rangle = \sum_{i=1}^N \text{Tr}[D_i^\dagger O] |e_i\rangle$  is a vector in  $\mathbb{C}^N$ , and  $\{e_i\}$  denotes the canonical basis of  $\mathbb{C}^N$ . We will now state a useful condition for optimality. Let us consider the linear mapping  $\Lambda: \mathbb{C}^N \rightarrow \mathcal{H}^{\otimes 2}$

$$\Lambda|c\rangle = \sum_i c_i |P_i\rangle\rangle, \quad (11)$$

where  $c_i = \langle e_i | c \rangle$ . The matrix elements representing  $\Lambda$  in the bases  $|e_i\rangle \in \mathbb{C}^N$  and  $|m\rangle|n\rangle \in \mathcal{H}^{\otimes 2}$  are given by

$$\Lambda_{mn,i} = (P_i)_{mn}. \quad (12)$$

We can define  $\Gamma$  to be a *generalized inverse* of  $\Lambda$ , shortly  $g$  inverse, if  $\Lambda = \Lambda\Gamma\Lambda$  holds (in the following, for all properties of the various kinds of  $g$  inverse see Ref. [22]). We now show that any  $g$  inverse  $\Gamma$  must have matrix elements of the form

$$\Gamma_{i,mn} = (D_i^*)_{mn}, \quad (13)$$

where  $\{D_i\}$  is a dual frame for  $\{P_j\}$ . In fact,  $(\Gamma\Lambda)_{ij} = \text{Tr}[D_i^\dagger P_j]$ , and consequently the condition for  $\Gamma$  to be a  $g$  inverse can be written as  $(\Lambda\Gamma\Lambda)_{mn,j} = \sum_i (P_i)_{mn} \text{Tr}[D_i^\dagger P_j]$ . Then, by informational completeness of the POVM  $\{P_j\}$ , the reconstruction formula  $\sum_i \text{Tr}[D_i^\dagger O] P_i = O$  must hold, and necessarily  $\{D_i\}$  is a dual. Moreover, since  $\{D_i\}$  is a dual, than the  $g$  inverse is also *reflexive*, namely  $\Gamma\Lambda\Gamma = \Gamma$ , as can be easily checked from the expansion  $(\Gamma\Lambda\Gamma)_{i,mn} \equiv \sum_j \text{Tr}[D_i P_j] (D_j^\dagger)_{nm} = (D_i^*)_{mn} \equiv (\Gamma)_{i,mn}$ . In addition, the  $g$  inverse  $\Gamma$  is also a *least-square* inverse, since  $\|\Lambda\Gamma O - O\| = 0$ , as a consequence of the expansion  $(\Lambda\Gamma O)_{mn} \equiv \sum_i \text{Tr}[D_i^\dagger O] \times (P_i)_{mn} = (O)_{mn}$ . Therefore, summarizing, any dual frame  $\{D_i\}$  corresponds to a reflexive and least-square  $g$  inverse  $\Gamma$  of  $\Lambda$  through Eqs. (12) and (13).

Now, the quantity we want to minimize is  $\|\Gamma O\|_\pi$ , where the norm  $\|\bullet\|_\pi$  is defined through

$$\|c\|_\pi^2 = \langle c | \pi | c \rangle, \quad (14)$$

$\pi$  being the diagonal matrix (in the canonical basis  $e_i$ ) with entries  $\pi_{ii} = \text{Tr}[P_i \rho]$  (for finite dimension the POVM is trace class). Notice that when  $\text{Tr}[P_i \rho]$  does not depend on  $i$  (as in the case where the average over the prior distribution of states gives  $\bar{\rho} \propto I$  and the POVM is covariant so that  $\text{Tr}[P_i \bar{\rho}] \propto \text{Tr}[U_i \nu U_i^\dagger] = \text{Tr}[\nu]$ ) then  $\pi \propto I$ . The quantity to be minimized is then simply  $\|c\|^2 = \langle c | c \rangle$ , and the solution to the minimum-norm problem is provided by any matrix  $\Gamma$  such that  $\Gamma\Lambda = \Lambda^\dagger \Gamma^\dagger = \Lambda^\dagger \Gamma^\dagger \Gamma\Lambda$ . Along with the reflexivity and least-square properties, the minimum-norm condition uniquely determines the  $g$  inverse in terms of the Moore-Penrose pseudoinverse  $\Gamma \equiv \Lambda^\ddagger$ . Since  $(\Gamma\Lambda)_{ij} = \text{Tr}[D_i^\dagger P_j]$ , the previous identity is equivalent to

$$\langle\langle D_i | P_j \rangle\rangle = \langle\langle P_i | D_j \rangle\rangle = \sum_k \langle\langle P_i | D_k \rangle\rangle \langle\langle D_k | P_j \rangle\rangle. \quad (15)$$

Now it is easy to check that the canonical dual  $D_i$ , defined as  $|D_i\rangle = F^{-1} |P_i\rangle$ , satisfies the previous condition, as  $\sum_k |D_k\rangle \langle\langle D_k| = F^{-1}$ , and

$$\langle\langle D_i | P_j \rangle\rangle = \langle\langle P_i | D_j \rangle\rangle = \sum_k \langle\langle P_i | D_k \rangle\rangle \langle\langle D_k | P_j \rangle\rangle = \langle\langle P_i | F^{-1} | P_j \rangle\rangle. \quad (16)$$

In the general case in which  $\pi \neq I$ , analogous proof as in the previous case [22] leads to the following condition for minimization of  $\|\Gamma O\|_\pi$ :

$$\pi\Gamma\Lambda = \Lambda^\dagger \Gamma^\dagger \pi = \Lambda^\dagger \Gamma^\dagger \pi\Gamma\Lambda. \quad (17)$$

The results of the present and the previous sections also apply in the case of continuous POVM by suitably replacing the discrete index  $i$  with a continuous one, and sums with

integrals with some care about summability of integrals in the case of noncompact groups.

#### IV. INFOCOMPLETE COVARIANT MEASUREMENTS FOR BIPARTITE SYSTEMS

In the following we will consider quantum measurements that are covariant under the action of a group  $\mathbf{G}$  of unitary operators  $U_g, g \in \mathbf{G}$ , on the Hilbert space  $\mathcal{H}^{\otimes 2}$  of a bipartite system,  $\mathcal{H}$  denoting the Hilbert space of the two identical systems, for finite dimension  $d = \dim(\mathcal{H})$ . As is well known [6], the POVM of the measurement has the form

$$dg P_g = dg U_g \xi U_g^\dagger, \quad (18)$$

where  $\xi$  is a suitable positive operator such that the POVM is normalized, and  $dg$  denotes a (suitably normalized) Haar invariant measure on the group (for what we need in the following, the group is unimodular, which guarantees that an invariant measure always exists). For later convenience, we will normalize the Haar measure over the group as  $\int_{\mathbf{G}} dg = d$  ( $d$  is the dimension of the Hilbert space on which the group is represented). For infocomplete measurements every operator can be expanded over the POVM density, namely  $P_g$  spans the whole linear space of linear operators. Clearly, all expansions for bounded operators must be square summable over the group  $\mathbf{G}$ . However, since we are considering only finite dimensions, all operators are bounded, and since we consider only compact groups (which then admit normalizable invariant Haar measure), all group integrals are bounded too. In the present case, the noise in Eq. (9) can be rewritten as

$$\delta O^2(\rho; P) \doteq \int_{\mathbf{G}} dg |\text{Tr}[D_g^\dagger O]|^2 \text{Tr}[P_g \rho] - |\text{Tr}[O \rho]|^2. \quad (19)$$

We will consider the following classes of pure states with uniform prior probability distribution: (a) all pure input states; (f) all factorized pure input states; (e) all maximally entangled input states (also called EPR states). The averaged noise in all three cases can be evaluated with the following integrals:

$$\delta_a O^2[P] = \frac{1}{d^2} \int_{\text{SU}(d^2)} dh \delta O^2(U_h |0\rangle) \langle\langle 0 | U_h^\dagger; P \rangle\rangle, \quad (20)$$

$$\delta_f O^2[P] = \frac{1}{d^2} \int_{\text{SU}(d)} dh \int_{\text{SU}(d)} dh' \delta O^2((V_h \otimes V_{h'}) |0\rangle \langle 0|^{\otimes 2} \times (V_h^\dagger \otimes V_{h'}^\dagger); P), \quad (21)$$

$$\delta_e O^2[P] = \frac{1}{d} \int_{\text{SU}(d)} dh \delta O^2((V_h \otimes I) \mathcal{I}(V_h^\dagger \otimes I); P), \quad (22)$$

where  $\mathcal{I} = (1/d) |I\rangle \langle\langle I|$ , while  $|0\rangle$  and  $\langle 0|$  are arbitrary reference pure states in  $\mathcal{H}^{\otimes 2}$  and  $\mathcal{H}$ , respectively. Notice that the average of  $|\langle O \rangle|^2 \equiv |\text{Tr}[O \rho]|^2$  over input states does not depend on the POVM, but only on the set of states over which

the average is estimated. Upon denoting by overline the average over input states, one has

$$\overline{|\langle O \rangle|_a^2} \doteq \frac{1}{d^2} \int_{\text{SU}(d^2)} dh \text{Tr}[U_h^{\otimes 2}|0\rangle\langle 0|^{\otimes 2}U_h^{\dagger \otimes 2}(O \otimes O^\dagger)], \quad (23)$$

$$\overline{|\langle O \rangle|_f^2} \doteq \frac{1}{d^2} \int_{\text{SU}(d)} dh \int_{\text{SU}(d)} dh' \text{Tr}[(V_h \otimes V_{h'})^{\otimes 2}|0\rangle\langle 0|^{\otimes 4} \times (V_h^\dagger \otimes V_{h'}^\dagger)^{\otimes 2}(O \otimes O^\dagger)], \quad (24)$$

$$\overline{|\langle O \rangle|_e^2} \doteq \frac{1}{d} \int_{\text{SU}(d)} dh \text{Tr}[(V_h \otimes I)^{\otimes 2} \mathcal{I}^{\otimes 2} (V_h^\dagger \otimes I)^{\otimes 2} (O \otimes O^\dagger)]. \quad (25)$$

These integrals can be evaluated by exploiting the following identities (which are corollaries of Schur's lemmas):

$$\int_{\text{SU}(d)} dg U_g X U_g^\dagger = \text{Tr}[X] I, \quad (26)$$

$$\int_{\text{SU}(d)} dg U_g^{\otimes 2} X U_g^{\dagger \otimes 2} = \frac{2}{d+1} \text{Tr}[P_S^{\mathcal{H}} X] P_S^{\mathcal{H}} + \frac{2}{d-1} \text{Tr}[P_A^{\mathcal{H}} X] P_A^{\mathcal{H}}, \quad (27)$$

where  $P_S^{\mathcal{K}}$  and  $P_A^{\mathcal{K}}$  denote the projection on the symmetric and antisymmetric subspaces of  $\mathcal{K}^{\otimes 2}$ , respectively. The result is

$$\overline{|\langle O \rangle|_a^2} = \frac{2}{d^2(d^2+1)} \text{Tr}[P_S^{\mathcal{H}^{\otimes 2}}(O \otimes O^\dagger)],$$

$$\overline{|\langle O \rangle|_f^2} = \frac{4}{d^2(d+1)^2} \text{Tr}[(P_S^{\mathcal{H}}{}_{13} \otimes P_S^{\mathcal{H}}{}_{24})(O \otimes O^\dagger)],$$

$$\overline{|\langle O \rangle|_e^2} = \frac{2}{d^3(d+1)} \text{Tr}[(P_S^{\mathcal{H}}{}_{13} \otimes P_S^{\mathcal{H}}{}_{24})(O \otimes O^\dagger)], \quad (28)$$

where  $X_{ij}$  denotes an operator acting on  $\mathcal{H}_i \otimes \mathcal{H}_j$ . Finally, recalling that  $P_S = \frac{1}{2}(I+E)$  where  $E$  is the swap operator  $E|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle$ , and that  $\text{Tr}[E(A \otimes B)] = \text{Tr}[AB]$ , we get

$$\overline{|\langle O \rangle|_a^2} = \frac{1}{d^2(d^2+1)} (\text{Tr}[|O|^2] + |\text{Tr}[O]|^2), \quad (29)$$

$$\overline{|\langle O \rangle|_f^2} = \frac{1}{d^2(d+1)^2} (\text{Tr}[|O|^2] + |\text{Tr}[O]|^2 + \text{Tr}[|\text{Tr}_1[O]|^2] + |\text{Tr}_2[O]|^2), \quad (30)$$

$$\overline{|\langle O \rangle|_e^2} = \frac{1}{2d^3(d+1)} (\text{Tr}[|O|^2] + |\text{Tr}[O]|^2 + \text{Tr}[|\text{Tr}_1[O]|^2] + |\text{Tr}_2[O]|^2). \quad (31)$$

Notice that

$$\overline{|\langle O \rangle|_e^2} = \frac{d+1}{2d} \overline{|\langle O \rangle|_f^2}.$$

One can easily show that the first integral in Eq. (19) is independent of the input state ensemble, so that the first term of the noise depends only on the POVM, whereas the second term depends only on the input ensemble. This can be verified using identities (26) and (27). One has

$$\delta O_x^2[P] = \frac{1}{d^2} \int_{\mathbf{G}} dg |\text{Tr}[D_g^\dagger O]|^2 \text{Tr}[P_g] - \overline{|\langle O \rangle|_x^2}, \quad (32)$$

where  $x=a, f, e$ , and a dual that optimizes one of these noise parameters optimizes all of them. According to Sec. III, we are in the situation where the canonical dual is optimal, namely it provides the optimal processing function minimizing the noise (19). In the following section we will then consider processing functions obtained from the canonical dual.

## V. PRODUCT OF LOCAL INFOCOMPLETE MEASUREMENTS

As the first example of infocomplete POVM we consider

$$P_{g,h}^{\text{loc}} = U_g \nu U_g^\dagger \otimes U_h \nu' U_h^\dagger, \quad (33)$$

where the elements  $g, h$  belong to  $\text{SU}(d)$ , and  $\nu, \nu'$  are pure states in  $\mathbb{C}^d$ . Such a POVM describes a measurement that can be performed locally by two separate parties, with classical communication needed in order to evaluate the processing function.

It can be easily shown by Schur's lemma that  $P_g \otimes P_h$  is actually a POVM. The canonical dual can be written as [14]

$$D_{g,h} = [(d+1)U_g \nu U_g^\dagger - I] \otimes [(d+1)U_h \nu' U_h^\dagger - I]. \quad (34)$$

The noise  $\delta_x O^2[P^{\text{loc}}]$  in the evaluation of the expectation value of an operator  $O$  is given by

$$\delta_x O^2[P^{\text{loc}}] = \frac{1}{d^2} \int_{\text{SU}(d)} dg \int_{\text{SU}(d)} dh |\text{Tr}[D_{g,h}^\dagger O]|^2 - \overline{|\langle O \rangle|_x^2}. \quad (35)$$

Substituting the expression for the canonical dual and exploiting the identities in Eqs. (26) and (27), we obtain

$$\delta_x O^2[P^{\text{loc}}] = \frac{1}{d^2} \{ (d+1)^2 \text{Tr}[|O|^2] + |\text{Tr}[O]|^2 - (d+1) \times \text{Tr}[|\text{Tr}_1[O]|^2 + |\text{Tr}_2[O]|^2] \} - \overline{|\langle O \rangle|_x^2}. \quad (36)$$

## VI. GLOBAL INFOCOMPLETE MEASUREMENT

We will now consider the POVM

$$P_g^{\text{glob}} = U_g \nu U_g^\dagger, \quad (37)$$

where  $g$  now belongs to  $\text{SU}(d^2)$ ,  $\nu$  is a pure state in  $\mathbb{C}^d \otimes \mathbb{C}^d$ , and  $\int_{\text{SU}(d^2)} dg = d^2$ . The canonical dual set is given by [14]

$$D_g = (d^2 + 1)U_g \nu U_g^\dagger - I. \quad (38)$$

The average noise over all pure states can be evaluated as in the previous section, and one obtains

$$\delta O_x^2[P^{\text{glob}}] = \frac{1}{d^2} \{ (d^2 + 1) \text{Tr}[|O|^2] - |\text{Tr}[O]|^2 \} - \overline{|\langle O \rangle|_x^2}. \quad (39)$$

### VII. BELL MEASUREMENT

Finally, we will consider the Bell POVM

$$P_g^{\text{Bell}} = U_g \otimes I |I\rangle \langle I| U_g^\dagger \otimes I = d(U_g \otimes I) \mathcal{I}(U_g^\dagger \otimes I) \quad (40)$$

with  $g$  belonging to  $\text{SU}(d)$ . Using Eq. (26) it is easy to verify that Eq. (40) actually defines a POVM. On the other hand, such a POVM is not infocomplete since operators whose partial trace is not proportional to the identity cannot be spanned. This can be seen directly from the reconstruction formula (1). We can evaluate explicitly the frame operator as follows:

$$\begin{aligned} F &= \int_{\text{SU}(d)} dg |P_g\rangle \langle P_g| \\ &= d^2 \int_{\text{SU}(d)} dg U_g \otimes I \otimes U_g^* \otimes I (\mathcal{I}_{12} \otimes \mathcal{I}_{34}) U_g^\dagger \otimes I \otimes U_g^T \otimes I \\ &= d \left\{ \mathcal{I}_{13} \otimes \mathcal{I}_{24} + \frac{1}{d^2 - 1} (\mathcal{I}_{13} - \mathcal{I}_{13}) \otimes (\mathcal{I}_{24} - \mathcal{I}_{24}) \right\}, \quad (41) \end{aligned}$$

where  $\mathcal{I}_{ij}$  denotes the projector  $\mathcal{I}$  acting on  $\mathcal{H}_i \otimes \mathcal{H}_j$ . Indeed,  $F$  has a nontrivial kernel. The Bell POVM in Eq. (40) can be used, however, to obtain the expectation values of any operator of the form

$$O = \alpha_{00} I + \sum_{i,j \neq 0} \alpha_{ij} V_i \otimes W_j, \quad (42)$$

where  $V_i$  and  $W_j$  are orthogonal bases for the operators with  $V_0 = W_0 = I$  (and hence with  $V_i$  and  $W_j$  traceless for  $i, j \neq 0$ ). These are exactly the operators in the support of  $F$ . Notice also that any maximally entangled state is of the form (42).

Since one has

$$F^\ddagger |U_g\rangle \langle U_g|_{12} |U_g^*\rangle \langle U_g^*|_{34} = \frac{d^2 - 1}{d} |U_g\rangle \langle U_g|_{12} |U_g^*\rangle \langle U_g^*|_{34} - \frac{d^2 - 2}{d^2} |I\rangle \langle I|_{13} |I\rangle \langle I|_{24}, \quad (43)$$

the canonical dual is given by

$$D_g = \frac{d^2 - 1}{d} |U_g\rangle \langle U_g| - \frac{d^2 - 2}{d^2} I. \quad (44)$$

As long as operators  $O$  of the form (42) are considered, the average noise can still be evaluated through Eq. (32), and is given by

$$\delta O_x^2[P^{\text{Bell}}] = \frac{1}{d} \int_{\text{SU}(d)} dg |\text{Tr}[D_g O]|^2 - \overline{|\langle O \rangle|_x^2}. \quad (45)$$

Evaluation of the integral gives the following result:

$$\begin{aligned} \delta O_x^2[P^{\text{Bell}}] &= \frac{1}{d^2} \{ (d^2 - 1) \text{Tr}[|O|^2] + |\text{Tr}[O]|^2 \} \\ &\quad - \left( \frac{d^2 - 1}{d^3} \right) \text{Tr}[|\text{Tr}_1[O]|^2 + |\text{Tr}_2[O]|^2] - \overline{|\langle O \rangle|_x^2}. \quad (46) \end{aligned}$$

### VIII. COMPARISON AND CONCLUSIONS

We now compare the three infocomplete POVM's in terms of their statistical efficiency, namely we compare their variance (19). We have

$$\begin{aligned} \delta O_x^2[P^{\text{glob}}] - \delta O_x^2[P^{\text{Bell}}] &= \frac{2}{d^2} (\text{Tr}[|O|^2] - |\text{Tr}[O]|^2) \\ &\quad + \frac{d^2 - 1}{d^3} \text{Tr}[|\text{Tr}_1[O]|^2 + |\text{Tr}_2[O]|^2], \\ \delta O_x^2[P^{\text{loc}}] - \delta O_x^2[P^{\text{glob}}] &= \frac{2}{d} \left( \text{Tr}[|O|^2] + \frac{1}{d} |\text{Tr}[O]|^2 \right) \\ &\quad - \frac{d + 1}{d^2} \text{Tr}[|\text{Tr}_1[O]|^2 + |\text{Tr}_2[O]|^2], \\ \delta O_x^2[P^{\text{loc}}] - \delta O_x^2[P^{\text{Bell}}] &= \frac{2(d + 1)}{d^2} \text{Tr}[|O|^2] \\ &\quad - \frac{d + 1}{d^3} \text{Tr}[|\text{Tr}_1[O]|^2 + |\text{Tr}_2[O]|^2]. \quad (47) \end{aligned}$$

Clearly, for operators  $O$  having  $\text{Tr}_1[O] = \text{Tr}_2[O] = 0$  (e.g., for operators of the form  $O = A \otimes B$  with both  $A$  and  $B$  traceless) one has

$$\delta^2 O_x[P^{\text{Bell}}] \leq \delta^2 O_x[P^{\text{glob}}] \leq \delta^2 O_x[P^{\text{loc}}]. \quad (48)$$

For off-diagonal matrix elements in a factorized basis, namely with  $O = |i\rangle \langle j| \otimes |n\rangle \langle m|$  and  $i \neq j$  and/or  $n \neq m$ , the inequalities (48) also hold true, and, specifically, for diagonal matrix elements one has  $\delta^2 O_x[P^{\text{glob}}] = \delta^2 O_x[P^{\text{loc}}]$ . Also for maximally entangled projectors  $O = (1/d) |V\rangle \langle V|$  the variances are bounded as in Eq. (48), whereas for  $O = (1/d) |V\rangle \langle W|$  and  $\text{Tr}[W^\dagger V] = 0$  one has

$$\delta^2 O_x[P^{\text{glob}}] \leq \delta^2 O_x[P^{\text{loc}}], \quad (49)$$

and generally the expectation of  $O$  cannot be estimated using the Bell POVM, unless  $W^\dagger V \propto I$ . This means that for matrix elements in an orthogonal Bell basis  $|V_n\rangle$  where  $V_n$  are traceless (apart from the identity) and form a (projective) representation of a group (such as the usual orthonormal basis with Pauli matrices for  $d=2$ , or shift-and-multiply operators in dimension  $d>2$ ), the Bell POVM is optimal for diagonal matrix elements, whereas for off-diagonal matrix elements it cannot be used, and for these it is better to use the global measurement than the local one.

We conclude that for factorized traceless operators, for Bell projectors, and for the density matrix estimation in fac-

torized basis the Bell measurement always performs better than the unrestricted global measurement, which in turn outperforms the local one. For the diagonal matrix elements in the factorized basis the Bell measurement cannot be used, and global and local POVM's perform equally well. Finally, for matrix elements in an orthogonal Bell basis the Bell POVM is optimal for diagonal matrix elements, whereas for off-diagonal matrix elements it cannot be used, and for these elements the global measurement performs better than the local one.

A further interesting study continuing the present analysis is the evaluation of the noise for infocomplete measurements

with minimal number of outcomes, comparing local versus global measurements.

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