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RAPID COMMUNICATIONS

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Homodyne detection of the density matrix of the radiation field

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We provide a simple analytic relation that connects the density operator $\hat{\rho}$ of the electromagnetic field with the tomographic homodyne probabilities for generic quantum efficiency η of detectors. The problem of experimentally "sampling" a general matrix element $\langle\psi|\hat{\rho}|\varphi\rangle$ is addressed in the statistically rigorous sense of the central-limit theorem. We show that experimental sampling is possible also for nonunit efficiency, provided that η satisfies a lower bound related to the "resolutions" of vectors $|\psi\rangle$ and $|\varphi\rangle$ in the quadrature representations. For coherent and number states the bound is $\eta>\frac{1}{2}$. On the basis of computer-simulated experiments we show the feasibility of detecting delicate quantum probability oscillations, which otherwise would be smeared out by inefficient detection.

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The possibility of "measuring" the density matrix of a quantum state entered the realm of experiments only recently [1], in the domain of quantum optics. Here, it was recognized [2] that a complete characterization of the state of a field mode a ($[a,a^\dagger]=1$) is achieved by measuring the quadratures $\hat{x}_\phi=\frac{1}{2}(\hat{a}^\dagger e^{i\phi}+\hat{a}e^{-i\phi})$ of the field at all phases ϕ, with the help of balanced homodyne detection.

The experimental determination of density matrix elements of the radiation state opens new interesting perspectives. In principle, for example, it becomes possible to measure the probability distribution of any physical quantity. Here, it should be emphasized that, even in the case in which a method for directly observing a field observable is available, the reconstruction of the probability distribution from its moments is a delicate matter, mostly due to low quantum efficiency of the detectors. This is the case, for example, of the photon number, where nonclassical effects, such as the probability oscillations due to squeezing [3], are difficult to detect in practice and have not, to our knowledge, been reported yet. For the same reason, other interesting quantum features, such as oscillation of the Wigner function of a Schrödinger cat [2], remain experimentally precluded [4].

Hence, the problem is twofold: on the one hand, a direct method to measure density matrix elements is needed [5]; on the other hand, the same method should allow us to systematically overcome the smearing effect due to nonunit quantum efficiency by properly averaging over a larger set of experimental data. In this Rapid Communication we provide a way of achieving both goals: we give analytic relations for "sampling" a general matrix element, which hold even for quantum efficiency $\eta<1$, depending on the chosen vector set of the matrix representation. Here, by "sampling" we simply mean retrieving the experimental datum just as the result of averaging a given function over a finite set of instrumental results; the analytic form of the averaged function will depend on which quantum matrix element of the density matrix is desired.

The homodyne-tomography technique originated from the simple idea that the probability distributions $p(x,\phi)$ of the outcomes x of $\hat{x}_\phi$ (for $0\leq\phi\leq\pi$) is just the Radon transform (or "back-projection") of the Wigner function $W(\alpha,\bar{\alpha})$; in other words, every function $p(x,\phi)$ at different values of ϕ is the marginal probability—or "projection"—of

$W(\alpha,\bar{\alpha})$ along the line $\alpha=re^{i\phi}$, $r\in(-\infty,+\infty)$ in the complex plane. Hence, $W(\alpha,\bar{\alpha})$ can be evaluated as the inverse Radon transform (or "back projection") of $p(x,\phi)$, namely [2]

$$W(\alpha,\bar{\alpha})=\int_{-\infty}^{+\infty}\frac{dr|r|}{4}\int_{0}^{\pi}\frac{d\phi}{\pi}\int_{-\infty}^{+\infty}dx\ p(x,\phi)\times\exp[ir(x-\alpha_{\phi})]\ , \tag{1}$$

where $\alpha_{\phi}=\mathrm{Re}(\alpha e^{-i\phi})$. Equation (1) can be written in a form suited to statistical sampling upon exchanging the averaging integrals over ϕ and x with respect to the outer integral over r. One thus obtains

$$W(\alpha,\bar{\alpha})=\int_{0}^{\pi}\frac{d\phi}{\pi}\int_{-\infty}^{+\infty}dx\ p(x,\phi)\ K(x-\alpha_{\phi})\ , \tag{2}$$

where the kernel $K(x)$ is given by

$$K(x)=-\frac{1}{2}\,\mathrm{P}\,\frac{1}{x^{2}}\equiv-\lim_{\varepsilon\to0^{+}}\frac{1}{2}\ \mathrm{Re}\,\frac{1}{(x+i\varepsilon)^{2}}\ . \tag{3}$$

In Eq. (3) the symbol P denotes the Cauchy principal value. From Eqs. (2) and (3) it is clear that $W(\alpha,\bar{\alpha})$ cannot be sampled statistically, because the kernel $K(x-\alpha_{\phi})$ is an unbounded function of x and ϕ. In the currently adopted experimental techniques [1] the back projection (2) is obtained by "filtering" data—as in the usual x-ray medical tomography—namely upon introducing a cutoff that sets the resolution for $W(\alpha,\bar{\alpha})$. The density matrix in the number representation is then obtained through further Fourier transforms and integrations with Hermite polynomials.

As noticed in Ref. [5], a filtering of Eq. (2) amounts to tampering with the quantum state (it makes the state "more classical"). In the same Ref. [5] a technique is presented, which provides the matrix elements in the number representation in terms of averages on data, avoiding the evaluation of $W(\alpha,\bar{\alpha})$ as an intermediate step (see also Ref. [6]). In the following the problem of experimentally sampling a density matrix element will be addressed in the rigorous sense of the central-limit theorem.

For a general matrix element $\langle\psi|\hat{\rho}|\varphi\rangle$—between arbitrary vectors ψ and ϕ in the Hilbert space—a sampling formula should have a form similar to Eq. (2), with the matrix element in place of $W(\alpha,\bar{\alpha})$ and a suited integral kernel pertaining to the matrix element. Then, if (and only if) the kernel is bounded, every moment of the kernel is bounded for all possible distributions $p(x,\phi)$, and according to the central-limit theorem [7], the matrix element can be sampled on sufficiently large sets of data. Moreover, the average values for different experiments are normal-Gaussian distributed, allowing estimation of confidence intervals, or "errors."

In order to obtain the integral kernel for the matrix element we start from the operator identity

$$\hat{\rho}=\int\frac{d^{2}\alpha}{\pi}\,\mathrm{Tr}(\hat{\rho}e^{-\bar{\alpha}a}e^{\alpha a^{\dagger}})\ e^{-\alpha a^{\dagger}}e^{\bar{\alpha}a}, \tag{4}$$

which, by the change of variables $\alpha=(i/2)re^{i\phi}$, becomes

$$\hat{\rho}=\frac{1}{4}\int_{-\infty}^{+\infty}dr|r|\int_{0}^{\pi}\frac{d\phi}{\pi}\,\mathrm{Tr}(\hat{\rho},e^{ir\hat{x}_{\phi}})e^{-ir\hat{x}_{\phi}}\ . \tag{5}$$

Equation (4) is just the operator form of the Fourier-transform relation between Wigner functions and characteristic functions for a general density matrix. Upon evaluating the trace average in Eq. (5) in terms of $p(x,\phi)$—using the complete set $\{|x_{\phi}\rangle\}$ of eigenvectors of $\hat{x}_{\phi}$—and then exchanging the integrals over x and ϕ with respect to the outer integral over r, one obtains the operator equivalent of (2), namely

$$\hat{\rho}=\int_{0}^{\pi}\frac{d\phi}{\pi}\int_{-\infty}^{+\infty}dx\ p(x,\phi)K(x-\hat{x}_{\phi})\ , \tag{6}$$

where K is still given by Eq. (3). From Eq. (6) one immediately recognizes that a matrix element $\langle\psi|\hat{\rho}|\varphi\rangle$ can be experimentally sampled if the corresponding matrix element of the operator kernel $K(x-\hat{x}_{\phi})$ is bounded. The case of nonunit quantum efficiency η requires only a slight generalization. In fact, low efficiency detectors in a homodyne scheme simply produce a probability $p_{\eta}(x,\phi)$ that is a Gaussian convolution of the ideal probability $p(x,\phi)$. In terms of the generating functions for the $\hat{x}_{\phi}$ moments, one has [8]

$$\int_{-\infty}^{+\infty}dx\ p_{\eta}(x,\phi)e^{irx}=e^{-\frac{1-\eta}{8\eta}r^{2}}\int_{-\infty}^{+\infty}dx\ p(x,\phi)e^{irx}\ . \tag{7}$$

Upon substituting Eq. (7) into Eq. (5) [along the same lines that lead us to Eq. (6)] one has the sampling formula for low-efficiency detection

$$\hat{\rho}=\int_{0}^{\pi}\frac{d\phi}{\pi}\int_{-\infty}^{+\infty}dx\ p_{\eta}(x,\phi)K_{\eta}(x-\hat{x}_{\phi})\ , \tag{8}$$

where the kernel is given by

$$K_{\eta}(x)=\frac{1}{2}\ \mathrm{Re}\int_{0}^{+\infty}dr\ r\ \exp\left(\frac{1-\eta}{8\eta}\,r^{2}+irx\right)\ . \tag{9}$$

One should stress that, although the kernel in Eq. (9) is an unbounded function of x for $y\leqslant1$, when considered as a function of $x-\hat{x}_{\phi}$ it has matrix elements $\langle\psi|K_{\eta}(x-\hat{x}_{\phi})|\varphi\rangle$ that are bounded if $\langle\psi|e^{-ir\hat{x}_{\phi}}|\varphi\rangle$ decays faster than $\exp[-(1-\eta)r^{2}/8\eta]$. Hence, taking into account that the Fourier transform is just equivalent to a unitary rotation by $\pi/2$ (i.e., $\hat{x}_{\phi}\to\hat{x}_{\phi+\frac{\pi}{2}}$), one can readily assert that the matrix element $\langle\psi|K_{\eta}(x-\hat{x}_{\phi})|\varphi\rangle$ is bounded if the following inequality is satisfied for all phases $0\leqslant\phi\leqslant\pi$:

$$\eta>\frac{1}{1+4\varepsilon^{2}(\phi)}\ , \tag{10}$$

where

$$\frac{2}{\varepsilon^{2}(\phi)}=\frac{1}{\varepsilon_{\psi}^{2}(\phi)}+\frac{1}{\varepsilon_{\varphi}^{2}(\phi)} \tag{11}$$

and $\varepsilon^2_\psi(\phi)$ is the "resolution" of the vector $|\psi\rangle$ in the $\hat{x}_\phi$ representation, namely,

$$|_\phi\langle x|\psi\rangle|^2 \simeq \exp\left[-\frac{x^2}{2\varepsilon^2_\psi(\phi)}\right]. \quad (12)$$

In Eq. (12) the symbol $\simeq$ stands for the leading term as a function of x, and $|x\rangle_\phi \equiv e^{ia^\dagger a\phi}|x\rangle$ denotes eigenkets of the quadrature $\hat{x}_\phi$. Upon maximizing Eq. (10) with respect to ϕ, one obtains the overall bound

$$\eta > \frac{1}{1+4\varepsilon^2}, \quad \varepsilon^2 = \min_{0\le\phi\le\pi}\{\varepsilon^2(\phi)\}. \quad (13)$$

We now consider some examples of interest for applications.

Quadrature representation

Here the resolution is $\varepsilon=0$, and hence it is not possible to "sample" experimentally the density matrix in this representation, even for $\eta=1$. Here, for the reader's convenience, we give the representation of the integral kernel for $\eta=1$ [9],

$$_\phi\langle x_1|K(x-\hat{x}_\phi)|x_2\rangle$$
$$= \frac{|x_1-x_2|}{\sin^2\phi}\exp\left\{-2i\,\frac{x_1-x_2}{\sin\phi}\,[x-\tfrac{1}{2}(x_1+x_2)\cos\phi]\right\}. \quad (14)$$

Coherent-state representation

The resolution is $\varepsilon=1/2$, and sampling is possible for $\eta>1/2$ [see Eq. (13)]. Explicitly one has

$$\langle\alpha|K_\eta(x-\hat{x}_\phi)|\beta\rangle = 2\kappa^2\langle\alpha e^{i\phi}|\beta e^{i\phi}\rangle e^{-2(\kappa x - w_\phi)^2}$$
$$\times\Phi[-\tfrac{1}{2},\tfrac{1}{2};2(\kappa x-w_\phi)^2], \quad (15)$$

with $\kappa=\sqrt{\eta/(2\eta-1)}$, $w_\phi=\frac{1}{2}(\beta e^{i\phi}+\bar{\alpha}e^{-i\phi})$, and $\Phi(\alpha,\beta;z)$ denoting the degenerate hypergeometric function.

Number-state representation

Again the resolution is $\varepsilon=1/2$, and sampling is possible for $\eta>1/2$. One has

$$\langle n|K_\eta(x-\hat{x}_\phi)|n+\lambda\rangle$$
$$= e^{-i\lambda\phi}2\kappa^{\lambda+2}\sqrt{\frac{n!}{(n+\lambda)!}}e^{-\kappa^2x^2}$$
$$\times\sum_{\nu=0}^{n}\frac{(-)^\nu}{\nu!}\binom{n+\lambda}{n-\nu}(2\nu+\lambda+1)!\,\kappa^{2\nu}$$
$$\times\mathrm{Re}\,\{(-i)^\lambda D_{-(2\nu+\lambda+2)}(-2i\kappa x)\}. \quad (16)$$

In Eq. (16) $D_\sigma(z)$ denotes the parabolic cylinder function.

Squeezed-state representation

The leading term in Eq. (12) comes solely from the squeezed vacuum (the coherent part just shifts the quadrature). If one considers a squeezed vacuum with squeezing parameter s, the probability of the quadrature $\hat{x}_\phi$ is the Gaussian $|_\phi\langle x|\psi\rangle|^2=\sqrt{2s_\phi/\pi}\,e^{-2s_\phi^2x^2}$, where $s_\phi=|s^{1/2}\sin\phi-is^{-1/2}\cos\phi|^{-2}$. Hence, if we fix $s<1$, for simplicity, the smallest resolution is $\varepsilon=s/2$ and the matrix element can be experimentally sampled for $\eta>(1+s^2)^{-1}$. From this and the above examples one is led to conjecture that $\eta=1/2$ is actually an absolute bound.

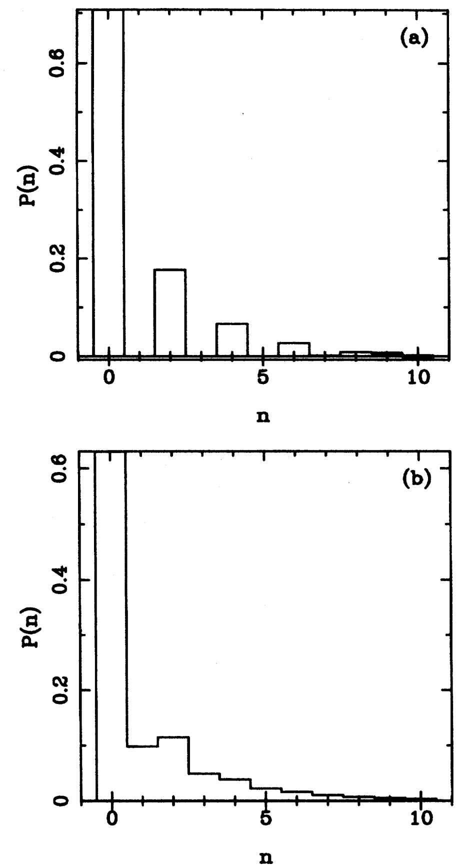

FIG. 1. Tomographic reconstruction of the photon-number probability of a squeezed vacuum ($\langle\hat{n}\rangle=1$) with detection efficiency $\eta=0.8$. Homodyne data are computer simulated. (Here we averaged over 27 phases using 200 blocks of 5×10^5 data for each phase.) Experimental errors (confidence intervals) are represented by the gray-shaded thickness of horizontal lines. (a) Exact reconstruction based on Eq. (16). (b) Reconstruction from the same data without taking into account quantum efficiency [namely using Eq. (16) for $\eta=1$]. The last would be the best experimentally achievable result using the current tomographic techniques of Refs. [1,5,6].

Particularly interesting is the possibility of recovering the density matrix in the number-state representation even for quantum efficiency $0.5<\eta<1$. From Eq. (16) a numerical algorithm for reconstructing the matrix elements is immediately devised. Here, on the basis of computer simulated homodyne data, we show how some interesting nonclassical effects can be experimentally detected using the new tomographic reconstruction.

In Fig. 1(a) the oscillations of the photon number probability distribution of a squeezed vacuum with $\langle\hat{n}\rangle=1$ are reconstructed from data for $\eta=0.8$; the agreement with the theoretical distribution is striking. Figure 1(b) shows the result that would be obtained without properly accounting for nonunit efficiency η (as in Ref. [1]). It is evident how the new method allows us to recover very delicate quantum oscillations, which would be almost completely smeared out by low efficiency at detectors.

In conclusion, we have presented a simple analytic relation that connects the density operator of the field with the homodyne tomographic probability densities. A statistically rigorous study based on the central-limit theorem shows that it is possible to sample experimentally a general matrix element $\langle\psi|\hat{\rho}|\varphi\rangle$ even for nonunit quantum efficiency, provided η satisfies the lower bound (13) determined by the resolution of the vectors $|\psi\rangle$ and $|\varphi\rangle$ in the quadrature representations. For coherent and number states the lower bound is $\eta>1/2$. On the basis of computer-simulated experiments we have seen how it is possible to detect experimentally delicate interference oscillations in the probabilities of nonclassical states, overcoming the destroying effect of nonunit detection efficiency.

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