INFORMATION GAIN IN QUANTUM COMMUNICATION CHANNELS

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1 INTRODUCTION

In quantum communications the performances of an amplifier depend on the scheme of the channel in which the device is inserted. For example, both gain and noise figure of the amplifier depend on the kind of coding at the transmitter, and detection at the receiver. [1] In an optimized ideal channel, detection and coding are ideal, and both are "matched" on the same observable; the alphabet probability is optimized in order to satisfy the physical constraints on the line. In this case, the channel capacity is already achieved, and there is no need of amplification. However, when the channel is nonideal—either because of quantum mismatch between transmitter and receiver, because of losses along the line, or as a result of nonunit quantum efficiency at detectors—an appropriate amplification can improve the transmitted information, ideally achieving the channel capacity for infinite gain.

In this paper we analyze the effect of an amplifier insertion in a quantum nonideal channel, for coherent and quadrature/squeezed channels. Nonefficient heterodyne and homodyne detections are described by means of the Wigner and marginal-Wigner POM's [2–4] respectively, whereas the quantum evolution along the communication line is given in terms of CP-maps, [5,6] whose infinitesimal versions are "master" and "Langevin" equations. We show that for a large class of linear amplifiers (attenuators) these equations have differential representations on the POM's in form of Fokker-Planck or Ornstein-Uhlenbeck equations, with Gaussian Green-function solutions. This allows a simple description of the device in terms of drift and diffusion coefficients.

Within the above approach we show that an ideal amplifier matched with an

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ideal POM has a pure-drift differential representation, corresponding to a POM rescaling which leaves mutual information invariant. For nonunit quantum efficiency, an amplifier matched with ideal detection can improve the transmitted information, eventually recovering the channel capacity for infinite gains: this can be signalled by the fact that the diffusion coefficient becomes negative and, correspondingly, the noise figure becomes lower than unit. When mismatched, an amplifier can be re-matched with the detector either by "squeezing" the idler, or by means of measurement-and-feedback controls, with both measurement and feedback matched with detection.

2 BASICS OF QUANTUM COMMUNICATIONS

In a quantum communication channel information transmission between source and user is described as follows. [7] The source consists of a coder which maps letters θ from an alphabet Θ into a set of density operators ρ_θ on the Hilbert space $\mathcal{H}$ of the dynamical system supporting communication. The alphabet Θ (either a discrete set, a subset of R^d, or a mixed discrete-continuous set) is distributed according to an "a priori" probability $dP(\theta)$, which also determines the mixture of parametrized states

$$\bar{\rho} \doteq \int dP(\theta)\rho_\theta \,. \tag{1}$$

The set Θ, the map ρ_θ and the probability $dP(\theta)$ will be globally refereed to as "coding", and denoted by $\mathcal{C} = \{dP(\theta), \rho_\theta, \theta \in \Theta\}$. We will consider two main types of coding: i) the quadrature/squeezed coding $\mathcal{C}_x$, where the parametrized states ρ_λ ($\lambda \in R$) are squeezed states for the quadrature $X = \frac{1}{2}(a+a^\dagger)$ with average $\langle X \rangle = \lambda$; ii) the coherent coding $\mathcal{C}_\alpha$, which uses coherent states $\rho_\alpha = |\alpha\rangle\langle\alpha|$, $\alpha \in C$. Any device inserted in the communication line produces a Schrödinger evolution $\rho \to A_S(\rho)$ of the density matrix ρ carrying information. Thus the series "coder+device" is equivalent to a new coder with coding $\mathcal{C} \circ \mathcal{A} = \{dP(\theta), A_S(\rho_\theta), \theta \in \Theta\}$, where $\mathcal{A} \doteq \{\rho \to A_S(\rho)\}$ denotes the device map.

At the end of the communication line the user determines the transmitted letter θ as the result of a quantum measurement on the system. This is conveniently described by means of a probability-operator-valued measure [2–4] (POM) $d\mu(\zeta)$ on $\mathcal{H}$ for the outcome $\zeta \in R^{d'}$, namely a resolution of identity $\int d\mu(\zeta) = 1$ with $d\mu(\zeta) \geq 0$ [we use notation $d\mu(\zeta) \equiv \mu(d\zeta)$]. The POM's generalize the customary orthogonal-projector-valued measures associated to quantum observables. When the measurement is performed on the system state ρ the output probability distribution is given by

$$dP[\rho](\zeta) \doteq \mathrm{Tr}[\rho d\mu(\zeta)] \,. \tag{2}$$

Each detection apparatus is described by a POM. [8] In the following we will denote the detection by $\mathcal{D} = \{d\mu(\zeta), \zeta \in R^{d'}\}$. We will only consider homodyne detection $\mathcal{D}_x = \{d\mu(x), x \in R\}$ and heterodyne detection $\mathcal{D}_\alpha = \{d^2\mu(\alpha, \bar{\alpha}), \alpha \in C\}$: the explicit form of the POM for such detectors will be given in the next section. If the detector is preceded by a device described by the Heisenberg evolution $d\mu \to A_H(d\mu)$, the series "device+detector" is equivalent to a new detector with $\mathcal{A} \circ \mathcal{D} = \{A_H(d\mu(\zeta)), \zeta \in R^d\}$. This equivalence should be compared with that "coder+device$\simeq$coder", and reflects duality $A_H \equiv A_S^\vee$ under trace (2).

In terms of the detector POM the conditional probability of receiving ζ is given by Eq. (2) with $\rho \equiv \mathcal{A}_S(\rho_\theta)$. The amount of information transfer between the coder and the detector is described by the "mutual information" $I(\theta;\zeta)$ between the random variables θ and ζ. Here, also in consideration of the symmetry property $I(\theta;\zeta) = I(\zeta;\theta)$ we prefer to label information with the maps of coding, detection, and other devices along the line. Then the mutual information is given by

$$I[\mathcal{C} \circ \mathcal{A} \circ \mathcal{D}] = \int dP(\theta) \int dP[\rho_\theta](\zeta) \log \frac{dP[\rho_\theta]}{dP[\bar{\rho}]}(\zeta) \,, \tag{3}$$

where $dP[\rho_\theta]/dP[\bar{\rho}]$ is the Radon-Nikodym derivative of $P[\rho_\theta]$ with respect to $P[\bar{\rho}]$. The maximum of I over $dP(\theta)$ (with $\mathcal{A}$ the identical map) is the "capacity" C of the information channel; the global maximum obtained by varying also $\mathcal{C}$ and $\mathcal{D}$ is the "ultimate quantum capacity" of the system. Using the Holevo-Ozawa-Yuen information bound, [7,3] it can be proved that the ultimate quantum capacity is achieved by number coding/detection, with $\bar{\rho}$ as the thermal state for photon number $\bar{n}$ corresponding to the average power physically allowed along the line. In the following we are interested only in coherent and quadrature/squeezed channels. For such cases the channel capacities are given by

$$C[\mathcal{C}_x \circ \mathcal{D}_x] = \log(1 + 2\bar{n}) \,, \qquad C[\mathcal{C}_\alpha \circ \mathcal{D}_\alpha] = \log(1 + \bar{n}) \,. \tag{4}$$

For the quadrature channel the capacity is achieved by squeezed states $|\lambda; r\rangle$, with squeezing parameter $e^{2r} = 1 + 2\bar{n}$ and zero-average Gaussian superposition $dP(\lambda)$ with variance $\sigma^2 = \bar{n} - \sinh^2 r$. For the coherent channel again the probability $dP(\alpha, \bar{\alpha})$ is Gaussian, and corresponds to a thermal $\bar{\rho}$ with $\bar{n}$ average photons.

3 HOMODYNE AND HETERODYNE DETECTION WITH NON-UNIT QUANTUM EFFICIENCY

The photon-count distribution $P_\eta(n)$ for a phototube cathode small with respect to radiation coherence length is given by the Bernoulli convolution

$$P_\eta(n) = \sum_{m=n}^{\infty} \binom{m}{n} \eta^n (1-\eta)^{m-n} \langle m|\rho|m\rangle,$$

with η denoting the quantum efficiency of the detector. In other words, the detector is equivalent to an ideal detector ($\eta = 1$) preceded by a beam splitter with transmissivity η. Using this simple rule it is easy to evaluate the POM's of homodyne and heterodyne detectors equipped with nonideal photocounters. [9] For a homodyne detector (with strong local oscillator and two equal photodetectors of efficiency η) the POM is simply the Gaussian convolution

$$d\mu_\eta(x) = \int \frac{dx'}{\sqrt{\pi(1-\eta)/2\eta}} \exp\left[-\frac{2\eta}{1-\eta}(x - x')^2\right] |x'\rangle\langle x'| \,, \tag{5}$$

where $|x\rangle$ are eigenstates of the quadrature $a_\phi = \frac{1}{2}(a^\dagger e^{i\phi} + a e^{-i\phi})$ at phase ϕ with respect to the local oscillator (in the following we will consider either $\phi = 0$, or $\phi = \pi/2$, for simplicity). For $\eta = 1$ one recovers the POM $d\mu(x) = dx|x\rangle\langle x|$ of the ideal homodyne detector. For an heterodyne detector with photodetector efficiency

η (independent on frequency in the range between signal and image frequencies), the POM is again a Gaussian convolution of the ideal POM, namely

$$d^2\mu_\eta(\alpha,\bar{\alpha}) = \frac{d^2\alpha}{\pi}\int \frac{d^2\alpha'}{\pi(1-\eta)/\eta}\exp\left[-\frac{\eta}{1-\eta}|\alpha-\alpha'|^2\right]|\alpha'\rangle\langle\alpha'| \,. \quad (6)$$

It is convenient to write both POM's (5) and (6) in terms of the Wigner POM

$$d^2w_s(\alpha,\bar{\alpha}) \doteq \frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi}e^{\alpha\bar{\beta}-\bar{\alpha}\beta+\frac{1}{2}s|\beta|^2}D(\beta)\,, \quad (7)$$

where $D(\beta) = \exp(\beta a^\dagger - \bar{\beta}a)$ is the displacement operator. The probability associated to the POM (7) is the Wigner function $W_s(\alpha,\bar{\alpha})$ for ordering parameter s. It is nonnegative for $s \leq -1$. Comparing Eqs. (6) and (7) it is easy to show that $d^2\mu_\eta(\alpha,\bar{\alpha}) \equiv d^2w_{1-2\eta^{-1}}(\alpha,\bar{\alpha})$, whereas the homodyne POM can be written as marginal POM of d^2w_s, namely

$$d\mu_s(x) = \int\frac{dy}{\pi}\frac{d^2w_s(x,y)}{dxdy}\,, \qquad s = 1-\eta^{-1}\,. \quad (8)$$

In Eq. (8) we used the change of variables $\alpha = x + iy$. Notice that all instrumental POM are nonnegative for quantum efficiency $0 \leq \eta \leq 1$.

4 AMPLIFICATION AND LOSS

4.1 CP-maps

An amplifier is an open system [5] where the amplified mode of radiation—the "signal" mode—resonantly interacts with other modes (parametric amplifier) or with matter degrees of freedom (active medium amplifier). "Pump" modes provide the necessary energy for amplification, whereas "idler" modes account for resonance condition and phase-matching. Also a beam splitter or a lossy cavity/fiber are open systems, with the signal mode gradually lost into external modes. The dynamical evolution of the density matrix ρ of an open system is described by a map of the form

$$\rho_t = A_S(\rho) \doteq \mathrm{Tr}_P\left[U_t\,\rho_P\otimes\rho\,U_t^\dagger\right]\,, \quad (9)$$

where U_t is a unitary operator, and ρ_P is the density matrix for the "probe" (=idler+pump). Corresponding to the Schrödinger-picture (9) one has the Heisenberg evolution of POM's $d\mu$, which is defined through duality $A_H = A_S^\vee$ under trace (2). The Heisenberg map A_H is a unit-preserving normal completely positive (CP) map, [6] and transforms POM's into POM's. The Schrödinger map evolves density matrices, preserving positivity, trace and convex linear combinations. In terms of the unitary evolution U_t the CP-map A_H is written as follows

$$d\mu_t = A_H(d\mu) = \mathrm{Tr}_P\left[\rho_P\otimes\hat{1}\,U_t^\dagger\,\hat{1}\otimes d\mu\,U_t\right]\,. \quad (10)$$

The normal unit preserving CP-maps (and their dual) admit the Stinespring representation [5]

$$A_H(d\mu) = \sum_k V_k^\dagger d\mu V_k\,, \qquad A_S(\rho) = \sum_k V_k\rho V_k^\dagger\,, \qquad \sum_k V_k^\dagger V_k = 1\,. \quad (11)$$

4.2 Master and Langevin equations

The infinitesimal versions of Eqs. (9) and (10) are referred to as "master" and "Langevin" equations, respectively. Lindblad proved that the most general form of master equation is [11]

$$d\rho = dA_S(\rho) = \sum_k D[A_k]\rho\ , \qquad D[A]\rho \doteq A\rho A^\dagger - \frac{1}{2}\left\{A^\dagger A, \rho\right\}\ , \tag{12}$$

$\{,\}$ denoting the anticommutator, and A_k being (complex) operators. The Langevin equation corresponding to Eq. (12) is [12]

$$d\, d\mu = dA_H(d\mu) = \sum_k D^\vee[A_k]d\mu\ , \qquad D^\vee[A]d\mu \doteq A^\dagger d\mu A - \frac{1}{2}\left\{A^\dagger A, d\mu\right\}\ . \tag{13}$$

4.3 Fokker-Planck equations

When a differential representation of the Langevin equation is available, one obtains a Fokker-Planck equation (FPE). The explicit form of the FPE generally depends on the analytic form of the POM under consideration, and coincides with the differential equation for the probability distribution. For the Wigner functions $W_s(\alpha,\bar\alpha)$ a differential representation is available for all superoperators that are polynomial functions of a and $a^\dagger$. In Table 1 the representation of right and left multiplication of ρ by a and $a^\dagger$ are reported along with the composition rules for obtaining the representation of any monomial.

Superoperator	Wigner representation	Superoperator	Wigner representation
$L[a] \doteq a\cdot$	$\alpha - \frac{1}{2}(s-1)\partial_{\bar\alpha}$	$R[a] \doteq \cdot a$	$\alpha - \frac{1}{2}(s+1)\partial_{\bar\alpha}$
$L[a^\dagger] \doteq a^\dagger\cdot$	$\bar\alpha - \frac{1}{2}(s+1)\partial_\alpha$	$R[a^\dagger] \doteq \cdot a^\dagger$	$\bar\alpha - \frac{1}{2}(s-1)\partial_\alpha$
$L[O_1O_2] = L[O_1]L[O_2]$		$R[O_1O_2] = R[O_2]R[O_1]$	

Table 1: Left and right superoperators and corresponding Wigner differential representations.

For the homodyne POM the FPE [in this case referred to as Ornstein-Uhlenbeck equation (OUE)] can be obtained from the Wigner representation using marginal integration (8) and dropping out boundary terms for the exponential decay of W_s at infinity. As an example, the OU representation of $D[a]$ can be obtained as follows

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dvdu}{4\pi^2} e^{\frac{s}{8}(u^2+v^2)} \mathrm{Tr}\left[D[a]e^{iu(X-x)+iv(Y-y)}\right] =$$
$$\int_{-\infty}^{\infty}\frac{dy}{\pi}\left[\frac{1}{2}(\partial_x x + \partial_y y) + \frac{1-s}{8}(\partial^2_{xx} + \partial^2_{yy})\right] W_s(x,y) =$$
$$\left(\frac{1}{2}\partial_x x + \frac{1-s}{8}\partial^2_{xx}\right)\int_{-\infty}^{\infty}\frac{dy}{\pi} W_s(x,y)\ , \tag{14}$$

where $a = X + iY$ is the quadrature decomposition of a. Notice that there are Wigner differential operators which admit no marginal form [for example, the operator $\alpha\partial_\alpha = \frac{1}{2}(x+iy)(\partial_x - i\partial_y)$ has terms linear in y having no marginal representation]. On the other hand, the OU differential operators are marginal representations of many different Wigner operators: hence, the OU representation is neither injective, nor defined on the whole superoperator algebra, whereas the Wigner representation is one-to-one. In Table 2 the representation of the superoperators used within this paper are reported.

Superoperator	Wigner	Marginal-X
$D[a]$	$\frac{1}{2}(\partial_\alpha \alpha + \partial_{\bar\alpha}\bar\alpha) + \frac{1-s}{2}\partial^2_{\alpha\bar\alpha}$	$\frac{1}{2}\partial_x x + \frac{1-s}{8}\partial^2_{xx}$
$D[a^\dagger]$	$-\frac{1}{2}(\partial_\alpha \alpha + \partial_{\bar\alpha}\bar\alpha) + \frac{1+s}{2}\partial^2_{\alpha\bar\alpha}$	$-\frac{1}{2}\partial_x x + \frac{1+s}{8}\partial^2_{xx}$
$[a^2,\cdot]$	$2\alpha\partial_{\bar\alpha} - s\partial^2_{\bar\alpha\bar\alpha}$	—
$[a^{\dagger 2},\cdot]$	$-2\bar\alpha\partial_\alpha + s\partial^2_{\alpha\alpha}$	—
$[a,[a,\cdot]]$	$\partial^2_{\bar\alpha\bar\alpha}$	$\frac{1}{4}\partial^2_{xx}$
$[a^\dagger,[a^\dagger,\cdot]]$	$\partial^2_{\alpha\alpha}$	$\frac{1}{4}\partial^2_{xx}$
$D[a_\phi]$	$\frac{1}{4}\partial^2_{\alpha\bar\alpha} - \frac{1}{8}(e^{2i\phi}\partial^2_{\alpha\alpha} + e^{-2i\phi}\partial^2_{\bar\alpha\bar\alpha})$	$\frac{1}{8}\sin^2\phi\partial^2_{xx}$
$[a^2 - a^{\dagger 2},\cdot]$	$2(\alpha\partial_{\bar\alpha} + \bar\alpha\partial_\alpha) - s(\partial^2_{\alpha\alpha} + \partial^2_{\bar\alpha\bar\alpha})$	$2\partial_x x - \frac{1}{2}s\partial^2_{xx}$
$i[Y, a\cdot + \cdot a^\dagger]$	$\frac{1-s}{4}\partial^2_{\bar\alpha\bar\alpha} + \frac{1-s}{4}\partial^2_{\alpha\alpha} + \frac{1-s}{2}\partial^2_{\alpha\bar\alpha} + \frac{1}{2}(\alpha\partial_{\bar\alpha} + \bar\alpha\partial_\alpha + \partial_\alpha\alpha + \partial_{\bar\alpha}\bar\alpha)$	$\partial_x x + \frac{1-s}{4}\partial^2_{xx}$
$i[X, -ia\cdot + \cdot ia^\dagger]$	$\frac{1-s}{4}\partial^2_{\bar\alpha\bar\alpha} + \frac{1-s}{4}\partial^2_{\alpha\alpha} + \frac{s-1}{2}\partial^2_{\alpha\bar\alpha} + \frac{1}{2}(\alpha\partial_{\bar\alpha} + \bar\alpha\partial_\alpha - \partial_\alpha\alpha - \partial_{\bar\alpha}\bar\alpha)$	0
$i[Y, a^\dagger\cdot + \cdot a]$	$-\frac{1+s}{4}\partial^2_{\bar\alpha\bar\alpha} - \frac{1+s}{4}\partial^2_{\alpha\alpha} - \frac{1+s}{2}\partial^2_{\alpha\bar\alpha} + \frac{1}{2}(\alpha\partial_{\bar\alpha} + \bar\alpha\partial_\alpha + \partial_\alpha\alpha + \partial_{\bar\alpha}\bar\alpha)$	$\partial_x x - \frac{1+s}{4}\partial^2_{xx}$
$i[X, ia^\dagger\cdot - \cdot ia]$	$-\frac{1+s}{4}\partial^2_{\bar\alpha\bar\alpha} - \frac{1+s}{4}\partial^2_{\alpha\alpha} + \frac{1+s}{2}\partial^2_{\alpha\bar\alpha} + \frac{1}{2}(\alpha\partial_{\bar\alpha} + \bar\alpha\partial_\alpha - \partial_\alpha\alpha - \partial_{\bar\alpha}\bar\alpha)$	0
$i[Y,[a^\dagger,\cdot]]$	$-\frac{1}{2}(\partial^2_{\alpha\alpha} + \partial^2_{\alpha\bar\alpha})$	$-\frac{1}{4}\partial^2_{xx}$
$i[X,[ia^\dagger,\cdot]]$	$\frac{1}{2}(-\partial^2_{\alpha\alpha} + \partial^2_{\alpha\bar\alpha})$	0
$i[Y,[a,\cdot]]$	$\frac{1}{2}(\partial^2_{\bar\alpha\bar\alpha} + \partial^2_{\alpha\bar\alpha})$	$\frac{1}{4}\partial^2_{xx}$
$i[X,[-ia,\cdot]]$	$\frac{1}{2}(\partial^2_{\bar\alpha\bar\alpha} - \partial^2_{\alpha\bar\alpha})$	0

Table 2: Master equation superoperators and corresponding Wigner and marginal-X differential representations

In the following we will consider FPE's of the form

$$\partial_t W_s(\alpha,\bar\alpha;t) = \left[Q(\partial_\alpha\alpha + \partial_{\bar\alpha}\bar\alpha) + 2D_s\partial^2_{\alpha\bar\alpha}\right] W_s(\alpha,\bar\alpha;t)\,, \tag{15}$$

having Gaussian solution

$$\begin{aligned} W_s(\alpha,\bar\alpha;t) &= \frac{1}{\pi\Delta_s^2(t)}\exp\left[-\frac{|\alpha-\alpha_0 e^{-Qt}|^2}{\Delta_s^2(t)}\right], \\ \Delta_s^2(t) &= \frac{D_s}{Q}(1-e^{-2Qt}) + \Delta_s^2(0)e^{-2Qt}\,. \end{aligned} \tag{16}$$

The corresponding marginal OUE is

$$\partial_t P_s(x;t) = \left[Q\partial_x x + \frac{1}{2}D_s\partial^2_{xx}\right] P_s(x;t)\,, \tag{17}$$

again with Gaussian solution

$$\begin{aligned} P_s(x;t) &= \frac{1}{\sqrt{2\pi d_s^2(t)}}\exp\left[-\frac{(x-x_0e^{-Qt})^2}{2d_s^2(t)}\right] \\ d_s^2(t) &= \frac{D_s}{2Q}(1-e^{-2Qt}) + d_s^2(0)e^{-2Qt}\,. \end{aligned} \tag{18}$$

It is clear that Eq. (17) could also be obtained as marginal OUE of FPE's different from (15). Moreover, notice that the diffusion terms in (15) and (17) generally depend on the quantum efficiency of detection through the parameter s. For negative drift $Q < 0$ (amplification) negative diffusion coefficients are admissible, without violating positivity of variances in (16) and (18) during the time evolution. In the following we will see that Eqs. (15) and (17) describe many different linear devices, with Gaussian solutions (16) and (18) modeling both coherent and squeezed coding (in Table 3 the parameters of the probability distributions for these coding are reported).

	Coherent state Wigner repr.	Thermal state Wigner repr.	$dP(\alpha,\bar{\alpha})$ coherent channel
Δ_s^2	$\frac{1}{2}(1-s)$	$\frac{1}{2}(1-s)+\bar{n}$	$\bar{n}$
α_0	α	0	0
	Squeezed state Marginal-X repr.	Thermal super. Marginal-X repr.	$dP(\lambda)$ quadrature channel
d_s^2	$\frac{1}{4}(\frac{1}{1+2\bar{n}}-s)$	$\frac{1}{4}(1-s)+\frac{\bar{n}}{2}$	$\bar{n}(\bar{n}+1)/2\bar{n}+1$
x_0	λ	0	0

Table 3: Variance and average of the Gaussian distributions in Eqs. (16) and (18), corresponding to optimal coding achieving capacities (4).

4.4 Gain, noise figure, and mutual information

Both gain and noise figure of any device inserted in a quantum communication channel generally depend on the whole chain of devices in the line, namely they depend on the overall equivalent detection and coding referred at the output and input of the device, respectively. The gain is defined as the ratio between output and input signals, the "signal" being the amplitude of the modulation $S \doteq \mathrm{Tr}\{\Delta\rho O\}$ of the detected operator O with respect to the reference state ρ_0 ($\Delta\rho = \rho - \rho_0$) [at the beginning of the line the state ρ_0 is usually the vacuum state, but evolution along the line generally leads to nonvacuum ρ_0]. The device is named "linear" when the gain does not depend on the input signal (i.e. on the input state ρ). Notice that, in general, a device can be linear for a particular detection scheme, being nonlinear for other schemes. For homodyne and heterodyne detection the gain is given by

$$g[\Delta\rho, \mathcal{A}\circ\mathcal{D}_x^{(\eta)}] = \frac{\mathrm{Tr}[\Delta\rho A_H(X_\eta)]}{\mathrm{Tr}[\Delta\rho X_\eta]}\ , \quad g[\Delta\rho, \mathcal{A}\circ\mathcal{D}_\alpha^{(\eta)}] = \frac{\mathrm{Tr}[\Delta\rho A_H(Z_\eta)]}{\mathrm{Tr}[\Delta\rho Z_\eta]}\ , \tag{19}$$

where $X_\eta^n = \int_{-\infty}^{\infty} d\mu_\eta(x) x^n$, and similarly $Z_\eta^n = \int d^2\mu_\eta(z,\bar{z}) z^n$. Notice that for heterodyne detection the gain g is complex and carries information on field phase.

Besides the gain, the most significant characteristic of a device is its noise figure R. This represents the degradation of the signal-to-noise ratio (SNR) from the input to the output, and thus it is defined as $R = SNR_{in}/SNR_{out}$. For homodyne detection the noise is just the customary root mean square (r.m.s.) of the detected observable X. For on-off modulation and binary alphabet the noise is mediated on the (equally likely) states ρ and ρ_0. More generally, for an alphabet corresponding to an *a priori* matrix $\bar{\rho}$ as in Eq. (1) one evaluates the average noise N over the alphabet probability, namely

$$N[\mathcal{C}_x \circ \mathcal{D}_x^{(\eta)}] = \int dP(\theta) \langle \Delta X_\eta^2 \rangle_{\rho_\theta} = \langle \Delta X_\eta^2 \rangle_{\bar\rho} - \overline{\Delta \langle X_\eta \rangle_{\rho_\theta}^2} \,, \tag{20}$$

where $\langle \Delta O^2 \rangle_\rho = \mathrm{Tr}(\rho O^2) - [\mathrm{Tr}(\rho O)]^2$ and $\overline{\Delta f^2(\theta)} = \int dP(\theta) f^2(\theta) - [\int dP(\theta) f(\theta)]^2$. For linear devices the gain does not depend on θ, and thus the noise figure is simply given by

$$R[\mathcal{C}_x \circ \mathcal{A} \circ \mathcal{D}_x^{(\eta)}] = \frac{N[\mathcal{C}_x \circ \mathcal{A} \circ \mathcal{D}_x^{(\eta)}]}{N[\mathcal{C}_x \circ \mathcal{D}_x^{(\eta)}]} \frac{1}{g^2} \,. \tag{21}$$

For heterodyne detection one has to define the noise of a complex variable, namely the noise of joint detection of two real variables. Hence, generally there are two noises $\frac{1}{2}\left[\langle |Z|_\eta^2 \rangle_{\rho_\theta} - |\langle Z_\eta \rangle_{\rho_\theta}|^2 \pm |\langle \Delta Z_\eta^2 \rangle|\right]$, where $|Z|_\eta^2 = \int d^2 \mu_\eta(z, \bar z) |z|^2$, corresponding to the eigenvalues of the covariance matrix. For "isotropic" devices $\langle \Delta Z_\eta^2 \rangle = 0$ and the noise is simply

$$N[\mathcal{C}_\alpha \circ \mathcal{D}_\alpha^{(\eta)}] = \frac{1}{2} \left[\langle \Delta |Z|_\eta^2 \rangle_{\bar\rho} - \overline{\Delta |\langle Z_\eta \rangle_{\rho_\theta}|^2} \right] \,, \tag{22}$$

where $\langle \Delta |O|^2 \rangle_\rho = \mathrm{Tr}(\rho |O|^2) - |\mathrm{Tr}(\rho O)|^2$, $\overline{\Delta |f(\theta)|^2} = \int dP(\theta) |f(\theta)|^2 - |\int dP(\theta) f(\theta)|^2$ and the "modulus operator" has already been defined. For linear devices the figure of noise is then defined similarly to Eq. (21). For devices described by equations (15) and (17) the gain is $g = \exp(-Qt)$, independently on the signal, and for both coherent and homodyne detection. The device is a linear amplifier for negative drift $Q < 0$, and a linear attenuator for $Q > 0$. From Eqs. (20-22) and Table 2 one obtains the noise figures

$$\begin{aligned} R[\mathcal{C}_x \circ \mathcal{A} \circ \mathcal{D}_x^{(\eta)}] &= 1 + \frac{\eta(1+2\bar n)}{1 + 2\bar n(1-\eta)} \frac{2D_{1-\eta^{-1}}}{Q} (g^{-2} - 1) && (23) \\ R[\mathcal{C}_\alpha \circ \mathcal{A} \circ \mathcal{D}_\alpha^{(\eta)}] &= 1 + \eta \frac{D_{1-2\eta^{-1}}}{Q} (g^{-2} - 1) \,. && (24) \end{aligned}$$

For optimized coding and ideal detection the noise figure is bounded as $R \geq 1$. [13] On the other hand, for $\eta < 1$ the noise figure may become lower than unit, signaling that the device (preamplifier) is improving detection performance. Noise figures $R < 1$ are equivalent to negative diffusion coefficients, whereas $R > 1$ corresponds to positive diffusion. An ideal amplifier for ideal detection has $R = 1$, corresponding to pure (negative) drift $-\nabla_\zeta \cdot \zeta$ which gives the POM rescaling $A_H(d\mu(\zeta)) = d\mu(g^{-1}\zeta)$ ($\zeta \in R^{d'}$ is used as in Sect. 2). The POM rescaling leaves the mutual information invariant: therefore, for ideal detection an amplifier can only degrade the mutual information, and improvements are possible only for $\eta < 1$. In the following we will see that in order to improve the mutual information for $\eta < 1$ the amplifier needs to be matched with the detector POM, namely one one should use a quadrature (or phase sensitive) amplifier for homodyne, and a coherent (or phase insensitive) amplifier for heterodyne detection. Here we only give the mutual information after the insertion of a device modeled by Eqs. (15) and (17). For squeezed/quadrature and coherent channels one has

$$\begin{aligned} I[\mathcal{C}_x \circ \mathcal{A} \circ \mathcal{D}_x^{(\eta)}] &= \frac{1}{2} \log \left[1 + \frac{\bar n(\bar n + 1)/2\bar n + 1}{\frac{D_{1-\eta^{-1}}}{2Q}(g^{-2} - 1) + \frac{1}{4}\left(\eta^{-1} - \frac{2\bar n}{1+2\bar n}\right)} \right] \,, && (25) \\ I[\mathcal{C}_\alpha \circ \mathcal{A} \circ \mathcal{D}_\alpha^{(\eta)}] &= \log \left[1 + \frac{\bar n}{\frac{D_{1-2\eta^{-1}}}{Q}(g^{-2} - 1) + \eta^{-1}} \right] \,. && (26) \end{aligned}$$

4.5 Modeling master equations for linear amplifiers

The following master equation models a linear phase insensitive device

$$\partial_t \rho_t = 2\left[AD[a^\dagger] + BD[a]\right]\rho_t\,, \tag{27}$$

with superoperator D defined in Eq. (12). The device is phase insensitive as a consequence of invariance $D[ae^{-i\phi}] = D[a]$. $B > A$ leads to attenuation, whereas $A > B$ produces amplification. For example, with A and B proportional to atomic populations on the upper and lower lasing levels respectively, Eq. (27) describes an active medium amplifier in the linear regime (far from saturation). On the other hand, for $A = \frac{\Gamma}{2}\bar{m}$ and $B = \frac{\Gamma}{2}(\bar{m}+1)$ the same equation describes a field mode damped with photon lifetime Γ^{-1} toward the thermal distribution with $\bar{m}$ average photons. Eq. (27) has the following general CP-map solution [14]

$$\begin{aligned}\rho_t &= \mathrm{Tr}_P[U_t\,\rho\otimes\nu\,U_t^\dagger]\,,\\ U_t &= \begin{cases}\exp[-\arctan\sqrt{e^{\Gamma t}-1}(ab^\dagger - a^\dagger b)] & (B > A)\,,\\ \exp[-\mathrm{arctanh}\sqrt{1-e^{-\Gamma t}}(a^\dagger b^\dagger - ab)] & (A > B)\,,\end{cases}\end{aligned} \tag{28}$$

with $\Gamma/2 = |A-B|$ and ν the thermal state of the idler mode b with average photons $\bar{m} = \min\{A,B\}/|A-B|$ [A and B are non negative, otherwise one would have negative idler photons]. Thus, Eq. (27) also models either parametric amplification with thermal idler, or loss due to frequency conversion. [15] From Table 2 one obtains the Wigner representation of Eq. (27) in form of the FPE (15) with $Q = B - A$ and $2D_s = A + B + s(A-B)$. For ideal heterodyne detection ($s = -1$) the device is ideal for $B = 0$ (ideal phase insensitive amplifier). On the contrary, linear attenuation is always nonideal. The diffusion coefficient can be negative for $s = 1 - 2\eta^{-1} < -1$, and the amplifier can improve both SNR and mutual information when $A - B > \eta A$. The mutual information for the coherent channel is

$$I[\mathcal{C}_\alpha \circ \mathcal{A} \circ \mathcal{D}_\alpha^{(\eta)}] = \log\left[1 + \frac{\bar{n}}{\left(\frac{A}{B-A} + \eta^{-1}\right)(g^{-2}-1) + \eta^{-1}}\right] \tag{29}$$

with $g = \exp[(A-B)t]$: for ideal amplification ($B = 0$) the channel capacity is recovered in the limit of infinite gain g.

The phase insensitive amplifier modeled by Eq. (27) is "matched" to heterodyne detection (the field is detected with no preferred phase). The amplifier can be made ideal for the matched detection, and the information loss for $\eta < 1$ is completely recovered in the limit $g \to \infty$. Such information gain for nonefficient detection is described by negative diffusion coefficient D and, correspondingly, by noise figures $R < 1$. On the other hand, when there is a mismatch between amplifying CP-map and detecting POM, it is no longer possible to recover the whole lost information, even for infinite gain. This happens, for example, when amplifier (27) is inserted in a quadrature channel using homodyne detection. In this case the amplifier is described by the OUE (17), and the diffusion coefficient $\frac{1}{2}D_s$ now is evaluated for $s = 1 - \eta^{-1}$, as a result of the POM marginal projection (8). One has $D_{1-\eta^{-1}} = A + \frac{1}{2}\eta^{-1}(B-A)$: for $\eta = 1$ $D_0 = \frac{1}{2}(A+B) > 0$, and the amplifier is no longer ideal. For $\eta < 1$ the diffusion coefficient becomes negative when $A - B > 2\eta A$, namely the amplifier is able to improve the SNR only for quantum efficiencies half than the corresponding ones for heterodyne detection. Despite for $\eta < \frac{1}{2}$ the amplifier can reduce the SNR, it can no longer completely recover the lost information. In fact, from Eq. (25) one has

$$I[\mathcal{C}_x \circ \mathcal{A} \circ \mathcal{D}_x^{(\eta)}] = \frac{1}{2}\ln\left[1 + \frac{4\bar{n}(\bar{n}+1)}{(2\bar{n}+1)\frac{2A}{B-A}(g^{-2}-1) + (2\bar{n}+1)\eta^{-1}g^{-2} - 2\bar{n}}\right] \tag{30}$$

which is always lower than the channel capacity in Eq. (4), and leads to $I = \frac{1}{2}\ln(1+2\bar{n})$ for $B = 0$ and $g \to \infty$.

In order to improve the performance of the amplifier for homodyne detection one can "squeeze" the master equation (27) as follows

$$\partial_t \rho_t = 2\left\{AD[a^\dagger] + BD[a]\right\}\rho_t + C^*[a,[a,\rho_t]] + C[a^\dagger,[a^\dagger,\rho_t]]\,, \tag{31}$$

with $|C|^2 \leq AB$. The master equation (31) could be obtained either by "squeezing" the atomic lasing bath, or using transformations (31) with b in a squeezed-thermal state. For a parametric amplifier with the idler in a squeezed-thermal state with N photons in total and $\bar{m}_s$ squeezing photons, one has $A = \frac{\Gamma}{2}(N+1)$, $B = \frac{\Gamma}{2}N$ and $|C|^2 = \frac{\Gamma}{4}[N(N+1) - \bar{m}(2\bar{m}+1)]$, where now $\bar{m}$ denotes the pure thermal photons. The limiting case $|C|^2 = AB$ corresponds to the idler in a squeezed vacuum. From Table 2 one can see that the OUE of the device has the form (17) with the same drift $Q = B - A$ of the unsqueezed amplifier, but with diffusion $D_{1-\eta^{-1}} = A + \frac{1}{2}\eta^{-1}(B-A) + \mathrm{Re}(C)$. For $\mathrm{Re}(C) < 0$ the overall effect of squeezing is a reduction of the diffusion: for $\eta = 1$ and $\bar{m} = 0$ one has $D_0 = \frac{\Gamma}{16}N^{-1} + O(N^{-2})$, namely at infinite squeezing the amplifier becomes ideal for homodyne detection. Also, in this limit for $\eta < 1$ the diffusion coefficient becomes negative $D_{1-\eta^{-1}} = \frac{Q}{2}(\eta^{-1} - 1)$ and from Eq. (25) one obtains

$$\begin{aligned} I[\mathcal{C}_x \circ \mathcal{A} \circ \mathcal{D}_x^{(\eta)}] &= \frac{1}{2}\ln\left[1 + \frac{4\bar{n}(\bar{n}+1)}{(2\bar{n}+1)(\eta^{-1}-1)g^{-2}+1}\right] \\ &= \log(1+2\bar{n}) + O(g^{-2})\,, \end{aligned} \tag{32}$$

recovering the channel capacity.

As a consequence of noninjectivity of marginal integration (8) there are many different ways to achieve ideal quadrature amplification. For example, the popular phase sensitive amplifier is obtained by means of a degenerate parametric amplifier (with strong coherent pump) having effective Hamiltonian of the form

$$H = i\frac{\lambda}{2}(a^2 - a^{\dagger 2})\,. \tag{33}$$

Again, from Table 2 one can see that Hamiltonian (33) corresponds to an OUE of the form (17) with $Q = \lambda$ and $\frac{1}{2}D_{1-\eta^{-1}} = \frac{Q}{2}(\eta^{-1} - 1)$, leading to ideal amplification for $\lambda < 0$, and ideal attenuation for $\lambda > 0$.

Before concluding this subsection, we want to remark that it is not possible to recover lost information when the loss occurs before amplification. For example, let us consider the maps $\mathcal{L}$ and $\mathcal{A}$ both corresponding to Eq. (27), but the former describing a loss ($A = 0$, $B = \Gamma/2$), and the latter an ideal phase insensitive amplifier ($A = t^{-1}\log g$, $B = 0$). Then, the mutual informations for the coherent channel when the amplifier is inserted before and after loss are: $I[\mathcal{C}_\alpha \circ \mathcal{A} \circ \mathcal{L} \circ \mathcal{D}_\alpha^{(\eta)}] = \log\left[1 + \frac{\bar{n}}{1+(1-\eta\gamma)/\eta\gamma g^2}\right]$, $I[\mathcal{C}_\alpha \circ \mathcal{L} \circ \mathcal{A} \circ \mathcal{D}_\alpha^{(\eta)}] = \log\left[1 + \frac{\gamma\bar{n}}{1+(1-\eta)/\eta g^2}\right]$. It is clear that only in the first case the information is recovered, however with an increased power impinged into the line.

4.6 Ideal amplification via feedback

In Ref. [16] it is shown that it is possible to produce squeezing using feedback mediated by intracavity quantum non demolition (QND) measurements. The cavity supports a second mode b, which is coupled to the signal mode a via the interaction $H_I = \chi X_a Y_b$, and which is simultaneously homodyne detected (measurement of X_b) with efficiency ϵ. The evolution due to feedback has the form $\partial_t \rho_t|_{fb} = i\frac{\Lambda}{\epsilon\chi} I(t - 0^+)[Y_a, \rho_t]$, where $I(t - 0^+)$ is the detected photocurrent immediately before the feedback action. When fastly decaying to the vacuum ($\chi^2/\gamma \ll 1$) the mode b can be adiabatically eliminated, and the following master equation for the reduced density matrix of the signal mode is obtained

$$\partial_t \rho_t = L\rho_t + \frac{\Lambda^2}{2h} D[Y]\rho_t + \Gamma D[X]\rho_t + i\Lambda[Y, [X\rho_t + \rho_t X]] \; . \qquad (34)$$

In Eq. (34) L is the original Liouvillian for a, $h = \epsilon\Gamma$, $\Gamma = \chi^2/\gamma$, and γ is the damping rate of mode b. From Table 2 one can see that the master equation (34) admits the OU representation (17). The term with $D[Y]$ adds a contribution $\frac{\Lambda^2}{16h}$ to diffusion D_s and could be made arbitrarily small for $h \propto \chi^2 \to \infty$. The term with $D[X]$—describing the QND measurement of X—has no effect on the POM $d\mu_\eta(x)$. Finally, the last term in Eq. (34) adds Λ to the drift, and $\frac{Q}{2}(\eta^{-1} - 1)$ to the diffusion $D_{1-\eta^{-1}}$. One thus concludes that the present intracavity feedback for $\Lambda < 0$ (within the validity limits of the adiabatic approximation) can either convert a loss L into amplification, or improve the original amplifier, adding an ideal term to the OU representation of the Liouvillian L.

The above measurement-and-feedback control has both steps and feedback matched with detection (measurement of X and commutator with Y which drives X). Again, if there is no matching with detection, the device is no longer improved. This happens, for example, if the roles of X and Y are exchanged in the above example. In Ref. [17] other master equations are considered, also from complex-amplitude and heterodyne feedback: using Table 2 one could immediately obtain the corresponding FPE or OUE, checking if feedback is effective in improving the device. The effectiveness of feedback loop is the result of competition between driving terms and positive-diffusion terms due to unavoidable introduction of noise by the measurement step in the quantum-limited feedback loop: as noticed in Ref. [17] the positive diffusion is doubled when the measurement step is achieved by heterodyne detection (the latter having effective quantum efficiency 1/2, due to joint detection of both quadratures. [3])

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