Repeatable Two-Mode Phase Measurement

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Abstract. A measurement scheme to perform a repeatable phase detection on a two-mode field is presented. The interaction with the probe state (the output state of a phase-insensitive high-gain amplifier) is described by a Hamiltonian which is physically realizable in the rotating wave approximation. Information on the system is obtained through unconventional heterodyne measurements performed on the probe field after the interaction with the system. The expressions for the probability distribution and the state reduction are given.

1 Introduction

The estimation of a phase shift on a two-mode field is not subjected to the limitations due to the complementarity principle that are suffered by a single mode field [1, 2, 3]. A two-mode field corresponds to a complex photocurrent $\hat{Z}$, and a proper self-adjoint operator $\hat{\phi} = \arg(\hat{Z})$ is well-defined as long as $[\hat{Z}, \hat{Z}^\dagger] = 0$ [4]. Indeed, the output photocurrent $\hat{Z}$ of an ideal heterodyne detector has just this property [5]. For this reason, unconventional field heterodyning (i.e. with the signal and image-band modes both nonvacuum) has revealed promising possibilities [4, 6] for ax exact phase measurement in terms of two-mode fields. In particular, after finding the physical (realizable) states that approach heterodyne eigenstates, we have shown [6] that the ideal sensitivity limit $\delta\phi = 1/\overline{n}$ can be achieved for large mean number of photons $\overline{n}$.

In this paper, we provide a feasible scheme to perform a repeatable phase measurement on a generic two-mode field. The repeatability of the measurement is based on the fact that a heterodyne detection is carried out on the probe two-mode field only, after a suitable interaction with the system signal. The property of repeatability allows to check the evolution of the signal under successive measurements, and possibly also to drive the evolution itself by state reduction (i.e. selecting the state after measurement). Finally—more interesting for foundations—a repeatable phase measurement is a good candidate for detecting Schrödinger-cat states (see, for example, Ref. [7]).

2 Approaching the heterodyne eigenstates

In Ref. [6] we presented a feasible scheme that achieves ideal phase detection on a two-mode field. The main idea is to perform an unconventional heterodyne measurement with non vacuum image-band mode on a two-mode field, in order to get a probability distribution of the output photocurrent as sharp as possible. The signal photons are tuned to optimize the r.m.s. phase sensitivity to the ideal limit $\delta\phi = 1/\overline{n}$.

As proved in Ref. [5], the output photocurrent of an ideal heterodyne detector corresponds to the complex operator $\hat{Z} = a + b^\dagger$, where a and b denote the signal and the image-band mode, respectively. Indeed the heterodyne detector jointly measures the real and the imaginary part of $\hat{Z}$, which are expressed as a function of the quadratures $\hat{c}_\phi = \frac{1}{2}(c^\dagger e^{i\phi} + \text{h.c.})$ of the single modes $c = a, b$ as follows

$$\hat{Z}_1 = \text{Re}\hat{Z} = \hat{a}_0 + \hat{b}_0 \qquad \hat{Z}_2 = \text{Im}\hat{Z} = \hat{a}_{\pi/2} - \hat{b}_{\pi/2} \,. \tag{2.1}$$

$\hat{Z}_1$ e $\hat{Z}_2$ are self-adjoint commuting operators, so that they can be jointly measured without the additional 3dB noise suffered by joint measurement of conjugated quadratures.
The eigenvectors of $\hat{Z}$ with complex eigenvalue $z = z_1 + iz_2$ are given by the following two equivalent expressions [4]

$$\begin{aligned} |z\rangle\rangle &= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi}} e^{2ixz_2} |x\rangle_0 \otimes |z_1 - x\rangle_0 \\ &= \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{\pi}} e^{-2iyz_1} |y + z_2\rangle_{\pi/2} \otimes |y\rangle_{\pi/2} \,. \end{aligned} \tag{2.2}$$

In Eq. (2.2) the tensor product $|\psi'\rangle \otimes |\psi''\rangle$ denotes kets in the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ pertaining the two modes a and b, whereas $|x\rangle_\phi$ denotes a quadrature eigenvector for phase ϕ. The notation $|\ \rangle\rangle$ stands for two-mode states. For a number state representation of Eq. (2.2), see Ref. [6]. Of course, the sharpest density probability of the output photocurrent is obtained for a field in a state $|z\rangle\rangle$. The states $\{|z\rangle\rangle\}$ are Dirac-normalized and have infinite total number of photons, hence they are not physically realizable. Nevertheless, we showed in Ref. [6] that the "twin-beams"

$$|0\rangle\rangle_\lambda = (1-\lambda^2)^{1/2} \sum_{n=0}^{\infty} (-\lambda)^n |n\rangle \otimes |n\rangle \tag{2.3}$$

at the output of a phase-insensitive amplifier (PIA) with gain $G = (1-\lambda^2)^{-1}$ approach the eigenstate $|0\rangle\rangle$ of $\hat{Z}$ (corresponding to zero eigenvalue) in the limit of infinite gain ($\lambda \to 1^-$). The eigenstate $|z\rangle\rangle$ for nonzero eigenvalue is approximated by the physical state

$$|z\rangle\rangle_\lambda = e^{za^\dagger - \bar{z}a}(1-\lambda^2)^{1/2}\sum_{n=0}^{\infty}(-\lambda)^n|n\rangle \otimes |n\rangle \,, \tag{2.4}$$

which is achieved by combining the twin-beams $|0\rangle\rangle_\lambda$ with a strong coherent local oscillator at the frequency of the signal mode a in a high-transmissivity beam splitter. The desired probability density of getting the value z for the output photocurrent $\hat{Z}$ with the field in the state $|w\rangle\rangle_\lambda$ is given by

$$|\langle\langle z|w\rangle\rangle_\lambda|^2 = \frac{1}{\pi\Delta_\lambda^2}\exp\left(-\frac{|z-w|^2}{\Delta_\lambda^2}\right) \,, \tag{2.5}$$

where

$$\Delta_\lambda^2 = \frac{1-\lambda}{1+\lambda} \,. \tag{2.6}$$

The state of the field $|w\rangle\rangle_\lambda$ has average number of photons

$$\bar{n} = {}_\lambda\langle\langle w|a^\dagger a + b^\dagger b|w\rangle\rangle_\lambda = |w|^2 + \frac{2\lambda^2}{1-\lambda^2} \tag{2.7}$$

and the marginal probability density

$$p(\phi) = \int_0^{+\infty} d|z|\,|z|\,|\langle\langle z|w\rangle\rangle_\lambda|^2 \tag{2.8}$$

for the phase $\hat{\Phi} = \arg(\hat{Z})$ approaches for $\Delta_\lambda \ll |w|$ the Gaussian form

$$p(\phi) \simeq \frac{|w|}{\sqrt{\pi}\Delta_\lambda}\exp\left[-\frac{|w|^2}{\Delta_\lambda^2}(\phi-\theta)^2\right] \,, \tag{2.9}$$

where $\theta = \arg(w)$. The corresponding r.m.s. phase sensitivity $\delta\phi = \langle\Delta\phi^2\rangle^{1/2}$ is optimized in the limit of infinite gain at the PIA, achieving the ideal limit

$$\delta\phi \simeq \frac{1}{\bar{n}} \tag{2.10}$$

for $|w|^2 = (1-\lambda)^{-1} = \bar{n}/2$.

The effect of nonunit quantum efficiency $\eta < 1$ of the heterodyne detector can be easily taken into account by changing Eq. (2.6) to

$$\Delta_\lambda^2 \to \Delta_\lambda^2(\eta) = \Delta_\lambda^2 + \frac{1-\eta}{\eta} \,. \tag{2.11}$$

and the result (2.10) still holds for $1-\eta \ll 2/\bar{n}$, otherwise sensitivity rapidly degrades towards shot noise.

3 Scheme for repeatable measurements

On the line of the main results for the two-mode phase heterodyne detection summarized in the previous section, here we present a scheme for a repeatable phase measurement, giving also some hints for its experimental realization. Our approach allows to perform a phase measurement on a generic two-mode field, without destroying it: in the following we compute the reduced state, depending on the outcome of the measurement.

We propose an interaction Hamiltonian bilinear in the four field modes a, b (for the system) and c, d (for the probe) as follows

$$\hat{H} = -K\frac{i}{2}\left[(a^\dagger c + bc + ad + b^\dagger d) - \text{h.c.}\right] , \tag{3.12}$$

where K is a coupling constant.
From definitions (2.1), and introducing the complex current for the probe field

$$\hat{A} \equiv c + d^\dagger = \hat{A}_1 + i\hat{A}_2 , \tag{3.13}$$

the Hamiltonian (3.12) rewrites

$$\hat{H} = K\left[\hat{Z}_1(\hat{c}_{\pi/2} + \hat{d}_{\pi/2}) - \hat{Z}_2(c_0 - d_0)\right] . \tag{3.14}$$

The Heisenberg evolution of a probe operator of the form $f(\hat{A})$ for an interaction time $\tau = \hbar/K$ is

$$\hat{U}^\dagger f(\hat{A})\hat{U} = f(\hat{A} + \hat{Z}) , \tag{3.15}$$

where

$$\hat{U} = \exp(-i\hat{H}\tau) = \exp\left\{-i\left[\hat{Z}_1(c_{\pi/2} + d_{\pi/2}) - \hat{Z}_2(c_0 - d_0)\right]\right\} \tag{3.16}$$

is the unitary evolution operator.
After the interaction, the probe modes c and d are heterodyne measured, with c as the signal mode and d as the image-band mode. As explained in the previous section, this corresponds to measure the photocurrent $\hat{A}$. Notice that here we no longer need a beam splitter for displacing the state, because the interaction Hamiltonian itself transfers signal to the twin-beams before heterodyne detection. This indirect measurement provides information about the probability density of the complex eigenvalue z pertaining the system operator $\hat{Z}$. The probability density is computed through the relation

$$\mathrm{Tr}_S[\hat{F}(z)\hat{\rho}_S] = \mathrm{Tr}_{S,P}[|z\rangle\!\rangle\langle\!\langle z|\,\hat{U}\,(\hat{\rho}_S \otimes \hat{\rho}_P)\,\hat{U}^\dagger] , \tag{3.17}$$

where $\hat{F}(z)$ is a probability operator-valued measure (POM) and $|z\rangle\!\rangle\langle\!\langle z|$ represents an orthogonal projector on the eigenspace of the two-mode probe Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_d$ relative to the eigenvalue z of $\hat{A}$. The tensor product $\hat{\rho}_S \otimes \hat{\rho}_P$ denotes the (disentangled) state of system and probe before the interaction. For a probe preparation in the twin-beams state (2.3), the POM $\hat{F}(z)$ writes

$$\hat{F}(z) = {}_\lambda\langle\langle 0|\hat{U}^\dagger|z\rangle\rangle\langle\langle z|\hat{U}|0\rangle\rangle_\lambda \doteq \hat{\Omega}^\dagger(z)\,\hat{\Omega}(z)\,. \tag{3.18}$$

From the Eqs. (2.2), (3.16) and (2.5) with $w = 0$, taking into account the additional phase ${}_\lambda\langle\langle 0|z\rangle\rangle = |{}_\lambda\langle\langle 0|z\rangle\rangle|e^{iz_1z_2}$ [8], the system "amplitude-operator" $\Omega^\dagger(z)$ is evaluated as follows

$$\begin{aligned}\hat{\Omega}^\dagger(z) &= \exp\left\{i\hat{Z}_1\left(i\frac{\partial}{\partial z_1} + z_2\right) + i\hat{Z}_2\left(i\frac{\partial}{\partial z_2} + z_1\right)\right\}{}_\lambda\langle\langle 0|z\rangle\rangle \\ &= \frac{e^{iz_1z_2}}{\sqrt{\pi}\Delta_\lambda}\exp\left(-\frac{\left|\hat{Z}-z\right|^2}{2\Delta_\lambda^2}\right)\,,\end{aligned} \tag{3.19}$$

where Δ_λ is given by Eq. (2.6). As a consequence, one gets the expression for the POM

$$\hat{F}(z) = \frac{1}{\pi\Delta_\lambda^2}\exp\left(-\frac{\left|\hat{Z}-z\right|^2}{\Delta_\lambda^2}\right)\,, \tag{3.20}$$

which provides the probability density

$$P(z\|\hat{\rho}_S) = \mathrm{Tr}[\hat{F}(z)\hat{\rho}_S] = \frac{1}{\pi\Delta_\lambda^2}\int_C d^2z'\; {}_S\langle\langle z'|\hat{\rho}_S|z'\rangle\rangle_S\,\exp\left(-\frac{|z'-z|^2}{\Delta_\lambda^2}\right)\,. \tag{3.21}$$

Here $\{|z'\rangle\rangle_S\}$ are the eigenstates of the system operator $\hat{Z}$ and the integral is over the complex plane. Eq. (3.21) is a convolution of the ideal probability with a Gaussian that narrows for increasing gain of the PIA. Notice that the result (3.20) can be derived more easily through Eqs. (3.18) and (3.15), upon defining formally the projector for $\hat{A}$ as a Dirac delta on the complex plane, namely

$$|z\rangle\rangle\langle\langle z| = \delta^{(2)}(\hat{A} - z)\,, \tag{3.22}$$

then obtaining from Eq. (3.18):

$$\hat{F}(z) = {}_\lambda\langle\langle 0|\delta^{(2)}(\hat{A} - \hat{Z} - z)|0\rangle\rangle_\lambda\,; \tag{3.23}$$

hence, using Eq. (2.5), one gets the result.
The operator $\Omega(z)$ has been explicitly computed because its adjoint action on the system state $\hat{\rho}_S$ provides the reduced state $\hat{\rho}_z$ after the measurement with outcome z. One has

$$\hat{\rho}_z = \frac{\hat{\Omega}(z)\,\hat{\rho}_S\,\hat{\Omega}^\dagger(z)}{\mathrm{Tr}[\hat{F}(z)\hat{\rho}_S]} = \frac{1}{\pi\Delta_\lambda^2}\,\frac{\exp\left(-\frac{|\hat{Z}-z|^2}{2\Delta_\lambda^2}\right)\hat{\rho}_S\,\exp\left(-\frac{|\hat{Z}-z|^2}{2\Delta_\lambda^2}\right)}{\mathrm{Tr}[\hat{F}(z)\hat{\rho}_S]}\,. \tag{3.24}$$

The POM that provides the probability density for the phase is the marginal one of $\hat{F}(z)$ in Eq. (3.20), namely

$$\begin{aligned} d\hat{\mu}(\phi) &= \int_0^{+\infty} d|z|\,|z|\hat{F}(z) \\ &= \frac{1}{2\pi}\exp\left(-\frac{|\hat{Z}|^2}{\Delta_\lambda^2}\right) + \frac{1}{\pi\Delta_\lambda}\mathrm{Re}\left(\hat{Z}e^{-i\phi}\right)\exp\left\{-\frac{1}{\Delta_\lambda^2}\left[\mathrm{Im}\left(\hat{Z}e^{-i\phi}\right)\right]^2\right\} \\ &\times \frac{\sqrt{\pi}}{2}\left\{1+\mathrm{erf}\left[\frac{\mathrm{Re}\left(\hat{Z}e^{-i\phi}\right)}{\Delta_\lambda}\right]\right\}, \end{aligned} \tag{3.25}$$

with $\phi = \arg(z)$. In the limit of infinite average number of photons at the twin-beams ($\Delta_\lambda \to 0$), Eq. (3.25) approaches the ideal POM

$$d\hat{\mu}(\phi) = \delta\left(\hat{\Phi} - \phi\right) \tag{3.26}$$

where $\hat{\Phi} = \arg(\hat{Z})$.

As regards the case of a heterodyne detection with quantum efficiency $\eta < 1$, the projector in the right side of Eq.(3.17) needs to be replaced by the POM [10]

$$\hat{A}_\eta(z) = \frac{\eta}{\pi(1-\eta)}\exp\left(-\frac{\eta}{1-\eta}|\hat{A}-z|^2\right). \tag{3.27}$$

This increases the variance (2.11) of the POM's (3.20) and (3.25). The corresponding reduced state $\hat{\rho}_z^{(\eta)}$ becomes

$$\hat{\rho}_z^{(\eta)} = \int_C d^2z' \, \frac{\eta}{\pi(1-\eta)} e^{-\frac{\eta}{1-\eta}|z'-z|^2} \hat{\Omega}(z')\,\hat{\rho}_S\,\hat{\Omega}^\dagger(z')\,\mathrm{Tr}[\hat{F}_\eta(z)\hat{\rho}_S]^{-1}, \tag{3.28}$$

where $\hat{\Omega}(z')$ is the same as in Eq. (3.19) and $\hat{F}_\eta(z)$ is the new POM after the substitution (2.11).
Eq. (3.28) displays a conceptually noteworthy difference from Eq. (3.24): the ideal measurement reduces a pure initial state into a pure state (it is a quasi-complete measurement, according to Ozawa's definition [9]), whereas a nonunit quantum efficiency leads to mixing.

Regarding the experimental feasibility of the repeatable measurement scheme here presented, we notice that Hamiltonian (3.12) can be achieved in the parametric approximation by means of two classical undepleted pumps. The interaction Hamiltonian in the Dirac picture is obtained in the rotating wave approximation from a non linear susceptibility $\chi^{(2)}$ (*three-wave mixing*). Indeed, the following frequency arrangement of the probe mode d and the pump modes γ, ξ in comparison with the probe mode c and the system modes a, b ($\omega_a < \omega_b$)

$$\begin{cases} \omega_d = \omega_c + \omega_b - \omega_a \\ \omega_\xi = \omega_c - \omega_a \\ \omega_\gamma = \omega_c + \omega_b \end{cases} \tag{3.29}$$

with the restrictions

$$\begin{aligned} &\omega_b \neq 2\omega_a\,; \qquad\qquad \omega_c > \omega_b \\ &\omega_c \neq \frac{3}{2}\omega_a\,, 2\omega_a\,, \omega_a + \frac{\omega_b}{2}\,, \omega_a + \omega_b\,, 2\omega_a + \omega_b \end{aligned} \tag{3.30}$$

insure that the only surviving terms in the rotating wave approximation are represented by the Hamiltonian

$$\hat{H} \propto \left[(a^\dagger c \xi^\dagger + b^\dagger d \xi^\dagger + a d \gamma^\dagger + b c \gamma^\dagger) + \text{h.c.} \right] \,. \tag{3.31}$$

The Hamiltonian (3.31) clearly coincides with the Hamiltonian in Eq. (3.12) in the parametric approximation of undepleted pumps. It is clear that for a suitable frequency arrangement, one could also use a *four-wave mixing* $\chi^{(3)}$ medium.

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