Modern Physics Letters B, Vol. 11, No. 29 (1997) 1263–1275

EQUIVALENCE BETWEEN SQUEEZED-STATE AND TWIN-BEAM COMMUNICATION CHANNELS

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Received 11 July 1997

We show the equivalence between two different communication schemes that employ a couple of modes of the electromagnetic field. One scheme uses unconventional heterodyne detection, with correlated signal and image band modes in a twin-beam state from parametric downconversion. The other scheme is realized through a complex-number coding over quadrature-squeezed states of two uncorrelated modes, each detected by ordinary homodyning. This equivalence concerns all the stages of the communication channel: the encoded state, the optimal amplifier for the channel, the master equation modeling the loss, and the output measurement scheme. The unitary transformation that connects the two communication schemes is realized by a frequency conversion device.

1. Introduction

The fundamental theorems of quantum communication theory[1,2] establish an upper bound on the mutual information for all possible coded inputs, and all kinds of output detection at the channel. These results, when applied to a single narrow-band linear bosonic channel,[3] specify the ultimate quantum capacity per use (i.e. the maximum mutual information *per mode* subjected to any constraint on the channel), which, for fixed average-power constraint, could be ideally achieved by direct detection of number-states with thermal *a priori* input probability. Any other communication scheme can transmit just less information than the number-state (NS) channel. At high power levels, for example, the ideal NS channel achieves 1.44 bits per use more than the ideal coherent-state (CS) channel, and 0.44 bits more than the quadrature-squeezed-state (QS) channel.[3] (The CS channel utilizes heterodyne detection of CS, with Gaussian *a priori* probability; the QS channel uses homodyne detection of QS having Gaussian probability and optimal fraction of squeezing photons).

Although the NS channel is the optimal one, several reasons relegate it to a purely theoretical plan, and lead to consider the QS channel as a more realistic

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(and not much less efficient) communication scheme. First of all, it is a challenge to achieve number eigenstates experimentally. Second, the optimal amplification for the QS communication channel is achieved by a phase-sensitive ideal amplifier, which can be realized experimentally, whereas a concrete realization of the ideal photon number amplifier for the NS channel is still unknown.[4,5] Finally, the detrimental effect of loss, widely neglected in the literature,[6] easily degrades the NS channel: for example, for ten photons of average power, a signal attenuation of 0.2 dB is enough to degrade the NS channel under the capacity of the CS one, and for higher power the effect is more dramatic.[7] The QS channel is more robust to losses than the NS channel, and at low powers and not too high losses it remains above the capacity level of the CS channel.[7]

In this paper we show the equivalence between two communication schemes that employ complex-number encoding carried by two-mode states. The potentiality of a quantum communication channel that uses two modes has already been noticed, in particular for phase-modulation-based digital communications.[9] The two equivalent schemes are the following:

(i) a complex number is encoded on a twin-beam state generated by parametric downconversion; decoding is achieved through unconventional heterodyne detection of both the signal and image-band modes which form the correlated twin-beam;
(ii) the real and the imaginary part of the complex number are independently encoded over two uncorrelated QS pertaining two different modes; the two QS are decoded through ordinary homodyne detection on each mode separately.

The equivalence between the above schemes involves all the stages of the communication channel: coding, decoding and optimal amplification. Moreover, it includes the master equation that models the loss along the line. The equivalence is realized by a unitary transformation that physically corresponds to 50–50 frequency conversion. The equivalence between the two channels is relevant for applications, because the heterodyne twin-beam communication channel is easier to achieve experimentally than the QS channel.

The paper is organized as follows. In Sec. 2 we describe the twin-beam communication scheme. We introduce the eigenstates of the heterodyne detector photocurrent, and give a scheme that realizes states that approach such eigenstates used to encode the transmitted information. In Sec. 3 we discuss the two-mode master equation that models the effect of both losses and linear phase-insensitive amplification, and then we derive a Fokker–Planck equation for the unconventional-heterodyne probability. In Sec. 4 the channel equivalence is analyzed. We show that the 50–50 frequency conversion disentangles a twin-beam state into a couple of QS, whereas, at the same time, it changes a phase-insensitive amplifier (PIA) into two independent phase-sensitive amplifiers (PSA). On the other hand, the master equation for the loss turns out to be invariant under frequency conversion. In Sec. 5

the Fokker–Planck equations for the two schemes are compared, and the mutual information that characterizes the communication channels is evaluated. Finally, maximization over input probabilities is carried out to get the capacity per use.

2. The Twin-Beam Communication Scheme

2.1. *Unconventional-heterodyne detection*

Under ideal conditions, the output photocurrent of a heterodyne detector corresponds[10,11] to the complex operator $\hat{Z} = a + b^\dagger$, a and b denoting the annihilator of the signal and the image-band mode, respectively. Ordinary heterodyne detection corresponds to the joint measurement of both quadratures of the field a, with b as the vacuum image-band mode responsible of the 3dB heterodyne added noise. The communication scheme here presented is based on unconventional heterodyne detection, namely with the signal and image-band modes both nonvacuum. The eigenvector of $\hat{Z}$ with complex eigenvalue z can be written in several equivalent forms. In terms of the eigenvectors $|x\rangle_\phi$ of the quadrature $\hat{X}_\phi = \frac{1}{2}(c^\dagger e^{i\phi} + \text{h.c.})$ of the mode $c = a, b$ one has[11,12]

$$|z\rangle\rangle = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi}} e^{2ix\text{Im}z} |x\rangle_0 \otimes |\text{Re}\, z - x\rangle_0$$
$$= \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{\pi}} e^{-2iy\text{Re}z} |y + \text{Im}\, z\rangle_{\pi/2} \otimes |y\rangle_{\pi/2}\, , \tag{1}$$

where $|\psi\rangle \otimes |\varphi\rangle$ denotes a vector in the two-mode Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ [the notation $|\,\rangle\rangle$ remembers that the state is a two-mode one]. A number representation of the state (1) can be found in Ref. 12. The eigenstate for zero eigenvalue reads

$$|0\rangle\rangle = \frac{1}{\sqrt{\pi}}\, e^{-a^\dagger b^\dagger} |0\rangle \otimes |0\rangle = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-)^n |n\rangle \otimes |n\rangle\, . \tag{2}$$

It is convenient to write the eigenstate of $\hat{Z}$ corresponding to complex eigenvalue z as follows

$$|z\rangle\rangle = \frac{e^{|z|^2/2}}{\sqrt{\pi}}\, e^{-a^\dagger b^\dagger} |z\rangle \otimes |\bar{z}\rangle\, , \tag{3}$$

where $|z\rangle$ denotes a customary coherent state. The states $\{|z\rangle\rangle\}$ are Dirac-normalized as $\langle\langle z|z'\rangle\rangle = \delta^{(2)}(z - z')$, and form a complete orthogonal set for $\mathcal{H}$.

In the following, we will introduce a set of physical (normalizable) two-mode states that approach the (infinite-energy) eigenstates of $\hat{Z}$.

2.2. *Limited power near-eigenstates of the photocurrent*

The physical realization of states that approach the eigenstates $|z\rangle\rangle$ in Eq. (3) is suggested by recognizing that the zero-eigenvalue state (2) is just a "twin-beam" at the output of a PIA in the limit of infinite gain.[13] All the other states with $z \neq 0$ can be approached by suitably displacing either a single mode a or b, or both of them, and this can be done by means of high-transmissivity beam-splitters with a strong coherent local oscillator. Such physical realizations $|z\rangle\!\rangle_\lambda$ can be written as follows

$$|z\rangle\!\rangle_\lambda = D_a(v) D_b(\bar{w}) e^{\tanh^{-1}\lambda\,(ab - a^\dagger b^\dagger)} |0\rangle \otimes |0\rangle \quad (z = v + w)\,, \tag{4}$$

where $D_c(u) = e^{uc^\dagger - \bar{u}c}$ denotes the displacement operator of the mode $c = a, b$, and $e^{\tanh^{-1}\lambda\,(ab-a^\dagger b^\dagger)}$ is the unitary transformation achieved by a PIA with gain $G = (1-\lambda^2)^{-1}$. [The PIA Hamiltonian $(a^\dagger b^\dagger + \text{h.c.})$ corresponds to a $\chi^{(2)}$ parametric downconversion in the rotating-wave approximation and for classical undepleted pump]. The average number N of photons *per mode* of the state (4) is given by

$$N = \frac{1}{2}\langle a^\dagger a + b^\dagger b\rangle = \frac{\lambda^2}{1-\lambda^2} + \frac{|v|^2}{2} + \frac{|w|^2}{2}\,. \tag{5}$$

The probability density of the photocurrent in the state $|u\rangle\!\rangle_\lambda$ has the Gaussian form

$$|\langle\!\langle z|u\rangle\!\rangle_\lambda|^2 = \frac{1}{\pi\Delta_\lambda^2}\exp\left(-\frac{|z-u|^2}{\Delta_\lambda^2}\right), \tag{6}$$

with variance

$$\Delta_\lambda^2 = \frac{1-\lambda}{1+\lambda}\,. \tag{7}$$

Using the identity

$$e^{-\xi(ab-a^\dagger b^\dagger)} D_a(v)\, D_b(\bar{w})\, e^{\xi(ab-a^\dagger b^\dagger)} = D_a(v\coth\xi + w\sinh\xi)\, D_b(\bar{v}\sinh\xi + \bar{w}\cosh\xi) \tag{8}$$

Eq. (4) rewrites as follows

$$|z\rangle\!\rangle_\lambda = e^{\tanh^{-1}\lambda\,(ab-a^\dagger b^\dagger)} \left|\frac{v+\lambda w}{\sqrt{1-\lambda^2}}\right\rangle \otimes \left|\frac{\lambda\bar{v}+\bar{w}}{\sqrt{1-\lambda^2}}\right\rangle \quad (z = v + w)\,. \tag{9}$$

Hence, the state $|z\rangle\!\rangle_\lambda$ can be also obtained through phase-insensitive amplification of input signal and idler which have the precise amplitude relation in Eq. (9). Notice that the probability density (6) does not depend explicitly on v and w (the share of displacement of the two modes), while the mean number of photons does. The constraint $v + w = z$ over states (4) and (9) implies that for each z there is a family of states approaching the eigenstate $|z\rangle\!\rangle$ according to Eq. (6). For $v = w = z/2$, the most symmetrical state

$$|z\rangle\!\rangle_\lambda = e^{\tanh^{-1}\lambda\,(ab-a^\dagger b^\dagger)} \left|\frac{z}{2\Delta_\lambda}\right\rangle \otimes \left|\frac{\bar{z}}{2\Delta_\lambda}\right\rangle \tag{10}$$

achieves the best phase sensitivity, as shown in Ref. 12. For high photon number, the marginal probability density for the phase $\hat{\phi} = \arg(\hat{Z})$ is Gaussian, and in the limit of infinite PIA gain ($\lambda \to 1^-$) the r.m.s. phase sensitivity is optimized versus the total photon number $\bar{n} \equiv 2N$ to the value[14]

$$\delta\phi = \langle \Delta\phi^2 \rangle^{1/2} \simeq \frac{1}{\sqrt{2}\,\bar{n}} \,. \tag{11}$$

For a repeatable phase measurement scheme on a two-mode field, see Ref. 15.

The couple of coherent states on the right of Eq. (10) can be generated by means of a single coherent state and a frequency conversion device with suitable pump strength and phase. Indeed, for any complex α, one has

$$\exp\left[\frac{\pi}{4}\left(e^{i\arg\alpha}a^\dagger b - e^{-i\arg\alpha}ab^\dagger\right)\right]|\sqrt{2}\alpha\rangle \otimes |0\rangle = |\alpha\rangle \otimes |\bar{\alpha}\rangle \,, \tag{12}$$

and the unitary operator on the left side of Eq. (12) describes a frequency conversion device [in the parametric approximation of classical undepleted pump, this can be realized through a three-wave (or degenerate four-wave) mixing in a nonlinear $\chi^{(2)}$ ($\chi^{(3)}$) medium]. The two equivalent experimental set-ups to generate the state (10) are sketched in Fig. 1.

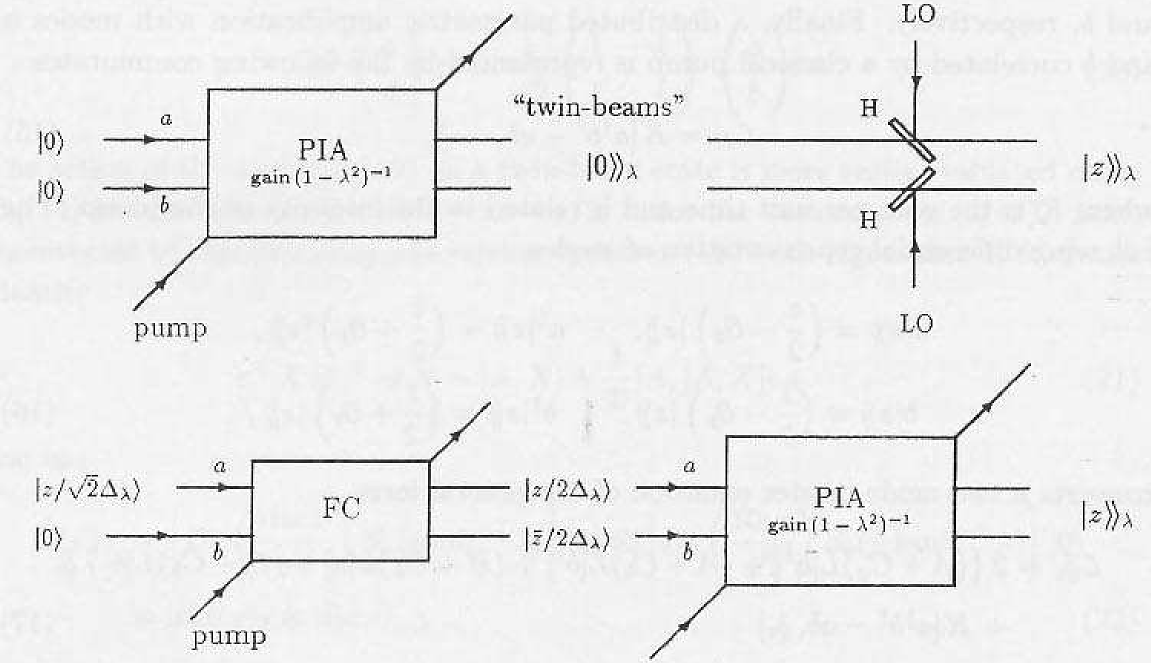

Fig. 1. Outline of the two alternative experimental set-ups to generate the two-mode states $|z\rangle\rangle_\lambda$ of Eq. (10). The labels H and LO denote high-transmissivity beam splitters and local oscillators respectively, which realize the displacement operators in Eq. (4). The input states at the parametric amplifier (PIA) are the vacuum state in the upper scheme, and two independent coherent states in the lower one. In the last case the couple of coherent states is generated by a single coherent state through frequency conversion (FC).

3. Losses and Distributed Amplification

The effect of losses on the communication channel can be modeled by the following master equation

$$\partial_t \hat{\varrho} = \mathcal{L}_\Gamma \hat{\varrho} \doteq \Gamma \left\{ (n_a+1)L[a] + (n_b+1)L[b] + n_a L[a^\dagger] + n_b L[b^\dagger] \right\} \hat{\varrho} \,. \tag{13}$$

In Eq. (13), the superoperator $\mathcal{L}_\Gamma$ gives the time derivative of the density matrix $\hat{\varrho}$ of the radiation state in the interaction picture, and acts on $\hat{\varrho}$ through the Lindblad superoperators $L[c]\hat{\varrho} = c\hat{\varrho}c^\dagger - \frac{1}{2}(c^\dagger c \hat{\varrho} + \hat{\varrho} c^\dagger c)$.[16] The damping rate Γ is supposed to be equal for both modes, whereas the mean number of thermal photons n_a and n_b at the frequency of modes a and b can be neglected at optical frequencies. The absence of cross-terms that correlate the two modes is a consequence of the rotating-wave approximation assumed in the ordinary derivation of the master equation.[17] In a similar fashion, an active medium amplifier in the linear regime can be described by the superoperator

$$\mathcal{L}_\Lambda = \Lambda \left\{ (m_a+1)L[a^\dagger] + (m_b+1)L[b^\dagger] + m_a L[a] + m_b L[b] \right\} \,, \tag{14}$$

where Λ denotes the gain per unit time — i.e. the amplifier length — and m_a and m_b are related to the population inversion of the lasing levels at resonance with a and b, respectively. Finally, a distributed parametric amplification with modes a and b correlated by a classical pump is represented by the following commutator

$$\mathcal{L}_K = K[a^\dagger b^\dagger - ab, \cdots] \,, \tag{15}$$

where K is the gain per unit time and is related to the intensity of the pump. The following differential representation of modes

$$\begin{aligned} a|z\rangle\rangle &= \left(\frac{z}{2} - \partial_{\bar{z}}\right)|z\rangle\rangle, \qquad a^\dagger|z\rangle\rangle = \left(\frac{\bar{z}}{2} + \partial_z\right)|z\rangle\rangle \,, \\ b|z\rangle\rangle &= \left(\frac{\bar{z}}{2} - \partial_z\right)|z\rangle\rangle, \qquad b^\dagger|z\rangle\rangle = \left(\frac{z}{2} + \partial_{\bar{z}}\right)|z\rangle\rangle \,, \end{aligned} \tag{16}$$

converts a two-mode master equation of the general form

$$\begin{aligned} \mathcal{L}\hat{\varrho}_t = 2\left\{ (A+C_a)L[a^\dagger] + (A+C_b)L[b^\dagger] + (B+C_a)L[a] + (B+C_b)L[b] \right\} \hat{\varrho}_t \\ + K[a^\dagger b^\dagger - ab, \hat{\varrho}_t] \end{aligned} \tag{17}$$

into the following Fokker–Planck equation

$$\partial_t P(z,\bar{z};t) = \left\{ Q\left(\partial_z z + \partial_{\bar{z}} \bar{z}\right) + 2D\partial^2_{z\bar{z}} \right\} P(z,\bar{z};t) \,, \tag{18}$$

where $P(z,\bar{z};t) \equiv \langle\langle z|\hat{\varrho}_t|z\rangle\rangle$ denotes the (unconventional) heterodyne probability density, and the drift and diffusion terms are given by $Q = B - A - K$ and $D = A + B + C_a + C_b$. Notice that the coefficients in the master equation (17) are not independent (the difference of the first two coefficients equals the difference of the last two in the curly brackets), and this is the condition under which all the derivatives in Eq. (16) gather to give the simple Fokker–Planck equation (18). Hence, when considering the effect of loss in Eq. (13), the assumption of equal damping for the two modes is crucial. Of course, the same argument holds true for the gain Λ in Eq. (14).

The solution of Eq. (18) will be given in Sec. 5, where also the mutual information transmitted by the channel will be evaluated. In the next section we will show the equivalence of the twin-beam channel with a communication scheme based on a couple of uncorrelated quadrature-squeezed states.

4. Equivalence Between Squeezed-State and Twin-Beam Channels

Let us consider the unitary transformation that describes a 50–50 frequency conversion device from mode a to mode b. Under suitable choice of phases, the corresponding unitary operator $\hat{U}$ writes

$$\hat{U} = \exp\left[\frac{\pi}{4}\left(ab^\dagger - a^\dagger b\right)\right] \tag{19}$$

so that the Heisenberg evolution of modes is given in matrix form as follows

$$\hat{U}^\dagger \begin{pmatrix} a \\ b \end{pmatrix} \hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{20}$$

The action of the operator (19) on a twin-beam state is more easily evaluated using Eq. (4) for the twin-beam with $v = w = z/2$. Since the vacuum state $|0\rangle \otimes |0\rangle$ is an eigenvector of the frequency conversion operator (19) with eigenvalue 1, from the identity

$$e^A\, X\, e^{-A} = X + [A, X] + \frac{1}{2!}[A, [A, X]] + \cdots\,, \tag{21}$$

one has

$$\begin{aligned} \hat{U}|z\rangle\!\rangle_\lambda &= \left[D_a\left(\frac{i\mathrm{Im}\, z}{\sqrt{2}}\right) S_a(\tanh^{-1}\lambda)\right]|0\rangle \otimes \left[D_b\left(\frac{\mathrm{Re}\, z}{\sqrt{2}}\right) S_b(-\tanh^{-1}\lambda)\right]|0\rangle \\ &\equiv |i\mathrm{Im}\, z\rangle_\lambda \otimes |\mathrm{Re}\, z\rangle_{-\lambda}\,, \end{aligned} \tag{22}$$

where $S_c(r)$ is the squeezing operator for the mode $c = a, b$, namely

$$S_c(r) = \exp\left[\frac{r}{2}\left(c^{\dagger 2} - c^2\right)\right], \tag{23}$$

and the squeezed state $|\alpha\rangle_\lambda$ is defined as follows

$$|\alpha\rangle_\lambda = D_c(\alpha/\sqrt{2})\, S_c(\tanh^{-1}\lambda)|0\rangle\,. \tag{24}$$

Hence, by means of frequency conversion the twin-beam state (4) disentangles into two squeezed states which are still related in intensity and phase. [For modes a and b at the same frequency and different wave vectors or polarization, this "disentanglement" can be also achieved by means of a 50–50 beam splitter,[18] i.e. by a passive device].

As regards the effect of frequency conversion at the output of a lossy/amplified channel, notice that a superoperator of the form

$$\mathcal{L} = \alpha L[a] + \beta L[b] + \gamma L[a^\dagger] + \delta L[b^\dagger] + \mathcal{L}_K \tag{25}$$

undergoes the following transformation

$$\begin{aligned} \hat{U}\mathcal{L}\hat{U}^\dagger = & \frac{\alpha+\beta}{2}\left(L[a]+L[b]\right) + \frac{\gamma+\delta}{2}\left(L[a^\dagger]+L[b^\dagger]\right) \\ & + \frac{K}{2}\left[a^2 - a^{\dagger 2}, \cdots\right] - \frac{K}{2}\left[b^2 - b^{\dagger 2}, \cdots\right] \\ & + \frac{\alpha-\beta}{2}(\text{cross-terms}) + \frac{\gamma-\delta}{2}(\text{cross-terms}) . \end{aligned} \tag{26}$$

We are not interested in the cross-terms in Eq. (26) that act jointly on $\mathcal{H}_a \otimes \mathcal{H}_b$, because they vanish when $n_a = n_b$ and $m_a = m_b$ in Eqs. (13) and (14), respectively. In this case one has

$$\hat{U}\mathcal{L}_\Gamma\hat{U}^\dagger = \mathcal{L}_\Gamma, \qquad \hat{U}\mathcal{L}_\Lambda\hat{U}^\dagger = \mathcal{L}_\Lambda . \tag{27}$$

Hence, the 50–50 conversion leaves the superoperators $\mathcal{L}_\Gamma$ and $\mathcal{L}_\Lambda$ (for both loss and PIA) invariant. This means that the disentanglement of the twin-beam occurs equivalently at whatever time during transmission. On the contrary, a distributed PIA with pump-correlated a and b is not invariant under the transformation (20). Indeed, a distributed PIA followed by frequency conversion is equivalent to a couple of independent PSA's, as shown by the commutator terms in Eq. (26), namely

$$\hat{U}\mathcal{L}_K\hat{U}^\dagger = \mathcal{L}'_K = \frac{K}{2}\left[a^2 - a^{\dagger 2}, \cdots\right] - \frac{K}{2}\left[b^2 - b^{\dagger 2}, \cdots\right] \tag{28}$$

Notice that for $K > 0$ the amplified quadrature components of the fields a and b through these commutators are $\hat{Y}_a \equiv \hat{a}_{\pi/2}$ and $\hat{X}_b \equiv \hat{b}_0$, respectively, which are the right quadratures in order to enhance the signal carried by the couple of squeezed states (22). Of course, the communication scheme that encodes the information on states (22) needs two independent homodyne measurements of the quadratures $\hat{Y}_a$ and $\hat{X}_b$.

We have shown the equivalence of the two channels as regards the input states and the evolution master equation. It remains to show that also the final detection stage is equivalent in the two schemes. The unconventional heterodyne detection is described by the following orthogonal resolution of the identity

$$d\hat{\mu}(z,\bar{z}) = d^2z\,\delta^{(2)}(\hat{Z} - z) \equiv d^2z\,|z\rangle\!\rangle\langle\!\langle z| . \tag{29}$$

The unitary operator (19) transforms the orthogonal resolution (29) as follows

$$\begin{aligned}\hat{U}\, d\hat{\mu}(z,\bar{z})\hat{U}^\dagger &= d^2z\, \delta(\sqrt{2}\hat{X}_b - \mathrm{Re}\, z)\, \delta(\sqrt{2}\hat{Y}_a - \mathrm{Im}\, z)\\ &= dx\, dy\, \delta(\hat{X}_b - x)\, \delta(\hat{Y}_a - y)\,, \qquad z = \sqrt{2}(x+iy)\,. \end{aligned} \tag{30}$$

The last orthogonal resolution in Eq. (30) is just the one corresponding to two independent homodyne measurements of quadratures $\hat{X}_b$ and $\hat{Y}_a$.

In conclusion of this section, we also show the equivalence between the QS scheme and the twin-beam scheme at the level of Fokker–Planck equations. Using the homodyne probability density

$$P_a(y;t)P_b(x;t) = \mathrm{Tr}\left[\hat{\varrho}_t |y\rangle_{\frac{\pi}{2}\,\frac{\pi}{2}}\langle y| \otimes |x\rangle_{00}\langle x|\right] \tag{31}$$

the master equation

$$\begin{aligned}\mathcal{L}'\hat{\varrho}_t &= 2\left\{A(L[a^\dagger] + L[b^\dagger]) + B(L[a] + L[b])\right\}\hat{\varrho}_t\\ &\quad + \frac{K}{2}\left[a^2 - a^{\dagger 2},\, \hat{\varrho}_t\right] - \frac{K}{2}\left[b^2 - b^{\dagger 2},\, \hat{\varrho}_t\right]\end{aligned} \tag{32}$$

can be written in the Fokker–Planck form

$$\partial_t\left\{P_a(y;t)P_b(x;t)\right\} = \left\{Q\left(\partial_x x + \partial_y y\right) + \frac{D}{4}\left(\partial^2_{xx} + \partial^2_{yy}\right)\right\} P_a(y;t)P_b(x;t) \tag{33}$$

with $Q = B - A - K$ and $D = A + B$. In obtaining Eq. (33) we used the Wigner representation of both a and b modes, and then evaluated the marginal integration over the quadratures $\hat{X}_a$ and $\hat{Y}_b$. The equivalence of the Fokker–Planck equation (33) with Eq. (18) is evident, after the coordinate transformation

$$z = \sqrt{2}(x+iy)\,, \qquad \bar{z} = \sqrt{2}(x-iy)\,, \tag{34}$$

and upon renaming the product of probabilities as follows

$$P'(z,\bar{z};t) = \frac{1}{2}P_a\left(\frac{\mathrm{Im}\, z}{\sqrt{2}};t\right)P_b\left(\frac{\mathrm{Re}\, z}{\sqrt{2}};t\right)\,. \tag{35}$$

The equivalence between the two communication channels is schematized in Fig. 2.

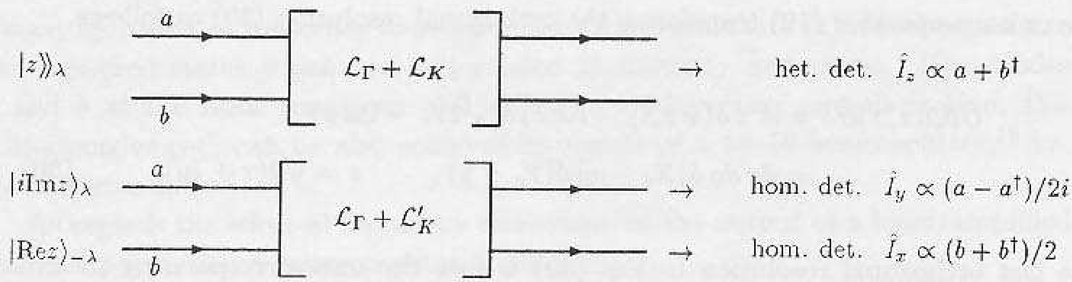

Fig. 2. Equivalence between the channels transmitting twin-beam states $|z\rangle\rangle_\lambda$ and couples of quadrature-squeezed states $|i\mathrm{Im}\, z\rangle_\lambda \otimes |\mathrm{Re}\, z\rangle_{-\lambda}$. The decoding measurements are the unconventional-heterodyne detection (complex photocurrent $\hat{I}_z$) and two independent ordinary homodyne detections (real photocurrents $\hat{I}_x$ and $\hat{I}_y$), respectively. The superoperators $\mathcal{L}_\Gamma$, $\mathcal{L}_K$ and $\mathcal{L}'_K$ model the loss in Eq. (13), the phase-insensitive amplification in Eq. (15) and the phase-sensitive amplification in Eq. (28), respectively. The equivalence between the two communication schemes is realized by the unitary transformation (20), namely a 50–50 frequency conversion.

5. Evaluation of the Channel Capacity

The solution of the Fokker–Planck equation (18) for a Gaussian initial probability density

$$P(z,\bar{z};0) = \frac{1}{\pi\Delta^2(0)} \exp\left(-\frac{|z-w|^2}{\Delta^2(0)}\right), \tag{36}$$

keeps the Gaussian form at all the time, and writes

$$P(z,\bar{z};t) = \frac{1}{\pi\Delta^2(t)} \exp\left(-\frac{|z-we^{-Qt}|^2}{\Delta^2(t)}\right), \tag{37}$$

where the "evolved" variance at time t is given by

$$\Delta^2(t) = \frac{D}{Q}(1-e^{-2Qt}) + \Delta^2(0)e^{-2Qt}. \tag{38}$$

For the twin-beam communication scheme one has $\Delta^2(0) \equiv \Delta^2_\lambda$, according to Eq. (6). For the scheme based on homodyne detection over the couple of QS states (22), one equivalently finds the Gaussian density

$$\left|{}_0\langle x| \otimes {}_{\frac{\pi}{2}}\langle y|\hat{U}|w\rangle\rangle_\lambda\right|^2 = \frac{2}{\pi\Delta^2_\lambda} \exp\left\{-\frac{2}{\Delta^2_\lambda}\left[\left(x-\frac{\mathrm{Re}\, w}{\sqrt{2}}\right)^2 - \left(y-\frac{\mathrm{Im}\, w}{\sqrt{2}}\right)^2\right]\right\}. \tag{39}$$

The mutual information of a communication channel with conditional probability density $p(w|z)$, encoded complex variable w, and *a priori* distribution $p(w)$ writes as follows

$$I = \int d^2w \int d^2z\, p(w)p(w|z) \ln \frac{p(w|z)}{\int d^2w'\, p(w')p(w'|z)}. \tag{40}$$

For *a priori* and conditional probability densities both Gaussian, i.e.

$$p(w) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|w|^2}{\sigma^2}\right) \tag{41}$$

$$p(w|z) = \frac{1}{\pi\Delta^2} \exp\left(-\frac{|z - gw|^2}{\Delta^2}\right) \tag{42}$$

one has

$$I = \ln\left(1 + \frac{g^2\sigma^2}{\Delta^2}\right) . \tag{43}$$

According to the Eq. (5) for the average number N of photons *per mode*, the variance of the prior distribution for the twin-beam scheme under the constraint of fixed average power reads[19]

$$\sigma^2 = 4\left(N - \frac{\lambda^2}{1-\lambda^2}\right) . \tag{44}$$

Hence, the mutual information transmitted at time t for a twin-beam channel modeled by the Fokker–Planck equation (18) is given by

$$I = \ln\left(1 + \frac{4\left[N - \lambda^2/(1-\lambda^2)\right]}{(D/Q)(e^{2Qt}-1) + \Delta_\lambda^2}\right) . \tag{45}$$

Equation (45) is valid also for the QS scheme, with parameter λ and variance Δ_λ^2 related to the squeezing parameter through Eqs. (22) and (7). In the lossless case, the mutual information is maximized by $\lambda = N/(N+1)$ to the value

$$I = 2\ln(1 + 2N) . \tag{46}$$

This information is always greater than the information of *two* lossless coherent-state channels, with one additional bit *per mode* for high average power ($N \gg 1$). For a lossy channel with damping rate Γ and $n_a = n_b = \bar{n}$ thermal photons one has $D = \Gamma(2\bar{n}+1)/2$ and $Q = \Gamma/2$. In this case, for $\lambda = N/(N+1)$, Eq. (45) rewrites

$$I = \ln\left(1 + \frac{4N(N+1)}{1 + (2N+1)(2\bar{n}+1)(e^{\Gamma t}-1)}\right) . \tag{47}$$

The exponential erasure of information in Eq. (47) becomes linear using the master equation (17), with $K = \Gamma/2$, namely

$$I = \ln\left(1 + \frac{4N(N+1)}{1 + (2N+1)(2\bar{n}+1)\Gamma t}\right) \tag{48}$$

This case represents an ideal distributed parametric amplification that works against the detrimental effect of loss. Notice that, due to the energy of the pump, the condition of fixed average power is not strictly satisfied. Indeed, after a transient, the increase of the mean number of photons is approximately linear in time as follows

$$N(t) = \frac{1}{2}\left[N(0) + \bar{n} - \frac{1}{2} + \left(N(0) - \bar{n} + \frac{1}{2}\right)e^{-2\Gamma t} + \Gamma(2\bar{n} + 1)\,t\right] . \qquad (49)$$

6. Conclusions

The maximum mutual information *per mode* for narrow-band linear bosonic channels under the constraint of fixed average power is achieved by the ideal NS channel with thermal input probabilities. Nevertheless, the experimental realization of number eigenstates and of an ideal photon number amplifier that can assist the channel is still unknown. Hence, our interest is turned into QS channels, which are feasible and still lead to a satisfactory efficiency. In this paper we have shown a new communication scheme that is based on unconventional-heterodyne detection on two-mode states ("twin-beam") and represents an alternative way to achieve the QS channel capacity. We have given two experimental set-ups to obtain the twin-beam states. We have shown that the twin-beam communication channel is equivalent to a channel that employs a couple of QS channels and ordinary homodyne detection as decoding. The correspondence between the two schemes holds also in the presence of loss and optimal amplification. The equivalence is realized by a unitary transformation that physically corresponds to the 50–50 frequency conversion between the two field modes. The twin-beam scheme is easier to obtain experimentally as compared to the QS scheme. Both the encoding and decoding stages are simpler for the twin-beam channel than for the QS one. As encoder one just needs parametric downconversion and coherent states, instead of squeezing. On the other hand, as decoder one has just one heterodyne detector versus a delicate balancing of two equal homodyne detectors. As compared to the single homodyne QS channel, the heterodyne one, by employing two field modes, carries a double amount of information.

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