



# Probability–fidelity tradeoffs for targeted quantum operations

G.M. D'Ariano<sup>a,b</sup>, S. Facchini<sup>a,b,\*</sup>, P. Perinotti<sup>a,b</sup>, M.F. Sacchi<sup>a,c</sup>

<sup>a</sup> *Quit group, Dipartimento di Fisica "A. Volta", via A. Bassi 6, 27100 Pavia, Italy*

<sup>b</sup> *Istituto Nazionale di Fisica Nucleare, Gruppo IV, via A. Bassi 6, 27100 Pavia, Italy*

<sup>c</sup> *CNR-INFN and CNISM, Unità di Pavia, via A. Bassi 6, 27100 Pavia, Italy*

## ARTICLE INFO

### Article history:

Received 8 May 2009

Accepted 25 June 2009

Available online 30 June 2009

Communicated by P.R. Holland

### PACS:

03.67.-a

## ABSTRACT

We present probability–fidelity tradeoffs for a varying quantum operation with fixed input–output states and for a varying inversion of a fixed quantum operation.

© 2009 Elsevier B.V. All rights reserved.

## 1. Introduction

Since the seminal work in Ref. [1], many information/disturbance tradeoffs have been derived in a wide range of frameworks [2–11]. Despite this variety, all tradeoffs were based on figures of merit defined as average over some ensemble, e.g. the uniform ensemble of all transformations.

In this Letter, following the suggestions contained in Ref. [12], we study the behavior of a *single* quantum operation in some simple cases, along the following lines. After reviewing the probability of transforming a pair of pure states to another given pair [13], we extend it to mixed target states, and then we provide a tradeoff between the probability and the fidelity of such a transformation. Finally, we present the probability–fidelity tradeoff in the inversion of an atomic (i.e. single–Kraus) quantum operation.

## 2. State transformations

We are given an ensemble  $E = \{q_{\pm}, |\psi_{\pm}\rangle\langle\psi_{\pm}|\}$  of two pure states  $|\psi_{\pm}\rangle$  with equal *a priori* probabilities  $q_{\pm} = 1/2$ , and a pair of (generally mixed) target states  $\rho_{\pm}$ . We want to find a quantum operation which realizes the transformation

$$|\psi_{\pm}\rangle \rightarrow \rho_{\pm} \quad (1)$$

maximizing the mean probability of success over the ensemble.

For pure final states  $\rho_{\pm} = |\phi_{\pm}\rangle\langle\phi_{\pm}|$  the problem has been solved in Ref. [13]:

\* Corresponding author at: Quit group, Dipartimento di Fisica "A. Volta", via A. Bassi 6, 27100 Pavia, Italy.

E-mail address: [facchini@fisicavolta.unipv.it](mailto:facchini@fisicavolta.unipv.it) (S. Facchini).

**Proposition 1.** *The maximum mean probability is*

$$p = \min \left\{ \frac{1 - |\langle\psi_+|\psi_-\rangle|}{1 - |\langle\phi_+|\phi_-\rangle|}, 1 \right\}. \quad (2)$$

Moreover, this probability is achieved with a balanced transformation, i.e. a transformation occurring with equal probability on both initial states.

Indeed, the above proposition can be extended also to final mixed states.

**Proposition 2.** *For generally mixed final states  $\rho_{\pm}$  the maximum mean probability is*

$$p = \min \left\{ \frac{1 - |\langle\psi_+|\psi_-\rangle|}{1 - F(\rho_+, \rho_-)}, 1 \right\}, \quad (3)$$

where  $F(\rho, \sigma) := \text{Tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$  is the Uhlmann fidelity [14]. Moreover, the probability is achieved with a balanced transformation.

**Proof.** Suppose we have a quantum operation  $\mathcal{E}$  realizing the transformation

$$|\psi_{\pm}\rangle \rightarrow \rho_{\pm}, \quad (4)$$

with certain probabilities  $p_{\pm}$ . Using the Ozawa dilation theorem [15] for quantum instruments we can realize the quantum operation in the following way

$$\mathcal{E}(\rho) = \text{Tr}_2[(I \otimes P)U(\rho \otimes |0\rangle\langle 0|)U^\dagger(I \otimes P)], \quad (5)$$

where  $|0\rangle$  is any pure state of an ancillary system,  $U$  is a unitary system–ancilla interaction,  $P$  is an orthogonal projector, and we

take the trace on the ancilla. Since unitaries and projectors cannot turn a pure state into a mixed one, the quantum operation  $\mathcal{E}$ , when applied to the our initial states  $|\psi_{\pm}\rangle$ , will have the form

$$\mathcal{E}(|\psi_{\pm}\rangle\langle\psi_{\pm}|) = p_{\pm} \text{Tr}_2(|\Phi_{\pm}\rangle\langle\Phi_{\pm}|), \quad (6)$$

where  $|\Phi_{\pm}\rangle$  are joint ancilla-system states and  $p_{\pm}$  are the success probabilities. Note that  $|\Phi_{\pm}\rangle$  are actually purifications of the final states  $\rho_{\pm}$ .

In this way we proved that every state transformation  $|\psi_{\pm}\rangle \rightarrow \rho_{\pm}$  can be realized with a transformation between pure states  $|\psi_{\pm}\rangle \rightarrow |\Phi_{\pm}\rangle$  followed by a partial trace. Thus, in order to maximize the probability of  $|\psi_{\pm}\rangle \rightarrow \rho_{\pm}$  it is not restrictive to search only among those transformations which take  $|\psi_{\pm}\rangle$  into purifications of the final states  $\rho_{\pm}$ .

From Uhlmann's theorem [14] we have that

$$|\langle\Phi_{+}|\Phi_{-}\rangle| \leq F(\rho_{+}, \rho_{-}), \quad (7)$$

for all the purifications of  $\rho_{\pm}$ , and thus

$$\frac{1 - |\langle\psi_{+}|\psi_{-}\rangle|}{1 - |\langle\Phi_{+}|\Phi_{-}\rangle|} \leq \frac{1 - |\langle\psi_{+}|\psi_{-}\rangle|}{1 - F(\rho_{+}, \rho_{-})}. \quad (8)$$

From the previous proposition we already know that the maximum probability for  $|\psi_{\pm}\rangle \rightarrow |\Phi_{\pm}\rangle$  is given by Eq. (2) and thus the upper bound holds

$$p \leq \min \left\{ \frac{1 - |\langle\psi_{+}|\psi_{-}\rangle|}{1 - F(\rho_{+}, \rho_{-})}, 1 \right\}. \quad (9)$$

This bound can be achieved by choosing the purifications  $|\Phi_{\pm}\rangle$  which give the equality in Eq. (7). The transformation is balanced by the previous proposition.  $\square$

### 3. Probability/fidelity tradeoff

Let us consider now the transformation

$$|\psi_{\pm}\rangle \rightarrow |\varphi_{\pm}\rangle, \quad |\langle\varphi_{+}|\varphi_{-}\rangle| \leq |\langle\psi_{+}|\psi_{-}\rangle|. \quad (10)$$

By Proposition 1 we know that it can be realized *exactly* only probabilistically. But if we allow also *approximate* transformations, realized by quantum operations which transform  $|\psi_{\pm}\rangle$  into states  $\rho_{\pm}$  close to  $|\varphi_{\pm}\rangle\langle\varphi_{\pm}|$

$$\begin{aligned} |\psi_{\pm}\rangle &\rightarrow \rho_{\pm} = \mathcal{E}(|\psi_{\pm}\rangle\langle\psi_{\pm}|)/p_{\pm}, \\ p_{\pm} &= \text{Tr}(\mathcal{E}(|\psi_{\pm}\rangle\langle\psi_{\pm}|)), \end{aligned} \quad (11)$$

we may be able to implement the transformation with greater probability, or even deterministically.

In general, there are two figures of merit characterizing the transformation: (1) the probability of success, (2) the fidelity between the target states and the states actually obtained. Intuitively, the more we try to tilt the pair  $|\psi_{\pm}\rangle$  towards the target states, the less the transformation is likely to happen.

The figures of merit are defined as follows

$$\begin{aligned} p &= \min\{p_{+}, p_{-}\}, \\ F &= \min\{F(|\varphi_{+}\rangle\langle\varphi_{+}|, \rho_{+}), F(|\varphi_{-}\rangle\langle\varphi_{-}|, \rho_{-})\}, \end{aligned} \quad (12)$$

where  $p$  is the minimum probability and  $F$  is the minimum fidelity over the two states (a *worst-case* criterion). Each transformation is characterized by a pair  $(p, F)$ , the set of all transformations thus being in correspondence with a subset of  $[0, 1] \times [0, 1]$ . Our task is to determine the frontier of this permitted subset, thus finding the transformations maximizing both figures of merit.

We can restrict our attention to approximated target states  $\rho_{\pm}$  having the same two-dimensional support, equal to the linear span

of the target states  $|\varphi_{\pm}\rangle$ . In fact, exploiting the Kraus representation for  $\mathcal{E}$  [16] (with Kraus operators  $K_j$ ) and defining the unnormalized states  $|\beta_{\pm}^{(j)}\rangle := K_j|\psi_{\pm}\rangle$ , we have

$$\rho_{\pm} = \frac{1}{p_{\pm}} \sum_j |\beta_{\pm}^{(j)}\rangle\langle\beta_{\pm}^{(j)}|. \quad (14)$$

We note that we can apply unitary operators  $U_j$  after the Kraus operators  $K_j$  without altering the probabilities, obtaining new states  $\rho'_{\pm}$

$$\rho'_{\pm} = \frac{1}{p_{\pm}} \sum_j U_j |\beta_{\pm}^{(j)}\rangle\langle\beta_{\pm}^{(j)}| U_j^{\dagger} \quad (15)$$

whose fidelity with the target states is

$$F(|\varphi_{\pm}\rangle\langle\varphi_{\pm}|, \rho'_{\pm}) = \sqrt{\frac{1}{p_{\pm}} \sum_j |\langle\varphi_{\pm}|U_j|\beta_{\pm}^{(j)}\rangle|^2}. \quad (16)$$

Thus, in order to have  $F(|\varphi_{\pm}\rangle\langle\varphi_{\pm}|, \rho'_{\pm}) \geq F(|\varphi_{\pm}\rangle\langle\varphi_{\pm}|, \rho_{\pm})$  for  $\rho'_{\pm}$  supported on the span of  $|\varphi_{\pm}\rangle$ , we only need to show that, given a pair of vectors  $|\beta_{\pm}\rangle$ , there is always a unitary transformation  $U$  moving  $|\beta_{\pm}\rangle$  in the span of  $|\varphi_{\pm}\rangle$  such that

$$|\langle\varphi_{\pm}|U|\beta_{\pm}\rangle| \geq |\langle\varphi_{\pm}|\beta_{\pm}\rangle|. \quad (17)$$

The operator  $U$  can be constructed in the following way.

Let us consider the component of  $|\beta_{+}\rangle$  orthogonal to  $\text{Span}\{|\varphi_{+}\rangle, |\beta_{-}\rangle\}$ . We rotate it into the one-dimensional subspace of  $\text{Span}\{|\varphi_{+}\rangle, |\varphi_{-}\rangle, |\beta_{-}\rangle\}$  orthogonal to  $\text{Span}\{|\varphi_{+}\rangle, |\beta_{-}\rangle\}$ . In this way, we have moved the four vectors in a three-dimensional space without changing the relevant scalar products  $|\langle\varphi_{\pm}|\beta_{\pm}\rangle|$ . The intersection  $V = \text{Span}\{|\varphi_{+}\rangle, |\varphi_{-}\rangle\} \cap \text{Span}\{|\beta_{+}\rangle, |\beta_{-}\rangle\}$  is one-dimensional, thus we can rotate the components of  $|\beta_{\pm}\rangle$  orthogonal to  $V$  into the one-dimensional subspace of  $\text{Span}\{|\varphi_{+}\rangle, |\varphi_{-}\rangle\}$  orthogonal to  $V$ . This rotation leaves all vectors in a two-dimensional space and increases the modulus of the scalar products  $|\langle\varphi_{\pm}|\beta_{\pm}\rangle|$ .

In the following we will then restrict to the span of  $|\varphi_{\pm}\rangle$ , and it is convenient to use the Bloch representation of states of bidimensional systems

$$\rho = (I + \mathbf{r} \cdot \boldsymbol{\sigma})/2, \quad (18)$$

where the Bloch vector  $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$  denotes a point in the unit ball  $|\mathbf{r}| \leq 1$  and  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  is the vector of Pauli matrices. In the Bloch representation the fidelity between the states  $\rho$  and  $\sigma$  (with Bloch vectors  $\mathbf{r}_{\rho}$  and  $\mathbf{r}_{\sigma}$ ) becomes [17]

$$F(\rho, \sigma) = \frac{(1 + \mathbf{r}_{\rho} \cdot \mathbf{r}_{\sigma} + \sqrt{(1 - |\mathbf{r}_{\rho}|^2)(1 - |\mathbf{r}_{\sigma}|^2)})^{\frac{1}{2}}}{\sqrt{2}}, \quad (19)$$

which, when one of the two states is pure simplifies as follows

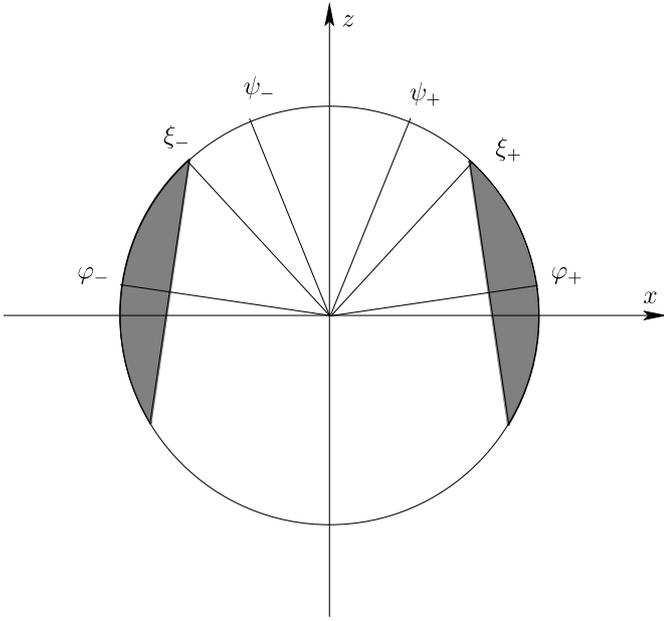
$$F(\rho, \sigma) = \sqrt{\frac{1 + \mathbf{r}_{\rho} \cdot \mathbf{r}_{\sigma}}{2}}. \quad (20)$$

The angle between vectors  $\mathbf{r}_{\rho_{+}}, \mathbf{r}_{|\varphi_{+}\rangle}$  and the angle between vectors  $\mathbf{r}_{\rho_{-}}, \mathbf{r}_{|\varphi_{-}\rangle}$  are both minimized for the pair  $\mathbf{r}_{\rho_{\pm}}$  coplanar with the pair  $\mathbf{r}_{|\varphi_{\pm}\rangle}$ , and with the same symmetry axis. This relative position of the couples of vectors can be achieved by a rotation of the couple  $\mathbf{r}_{\rho_{\pm}}$  in the Bloch sphere, corresponding to a unitary transformation which leaves the probabilities  $p_{\pm}$  invariant. Now, for each operation  $\mathcal{E}$  realizing a certain transformation

$$\mathcal{E}(|\psi_{\pm}\rangle\langle\psi_{\pm}|) = p_{\pm}\rho_{\pm}, \quad (21)$$

where  $\rho_{\pm}$  are coplanar with  $|\varphi_{\pm}\rangle$ , we can construct an operation  $\mathcal{E}'$  acting in the following way

$$\mathcal{E}'(|\psi_{\pm}\rangle\langle\psi_{\pm}|) = \frac{1}{2}(p_{\pm}\rho_{\pm} + p_{\mp}\sigma_z\rho_{\mp}\sigma_z), \quad (22)$$



**Fig. 1.** The section of the Bloch ball containing the initial pair  $|\psi_{\pm}\rangle$  and the target pair  $|\varphi_{\pm}\rangle$ . The shadowed area contains all the states  $\rho_{\pm}$  with fidelity  $F(\rho_{\pm}, |\varphi_{\pm}\rangle) \geq f$ .

where we have chosen the basis of the representation such that  $\sigma_z$  is the  $\pi$ -rotation around the symmetry axis of the pair  $|\varphi_{\pm}\rangle$ , i.e.  $\sigma_z|\varphi_{-}\rangle = |\varphi_{+}\rangle$ . The second term in r.h.s. is simply the “mirror image” of  $\mathcal{E}$ . This new quantum operation is symmetric since  $\sigma_z \mathcal{E}'(|\psi_{\pm}\rangle\langle\psi_{\pm}|)\sigma_z = \mathcal{E}'(|\psi_{\mp}\rangle\langle\psi_{\mp}|) = \mathcal{E}'(\sigma_z|\psi_{\pm}\rangle\langle\psi_{\pm}|\sigma_z)$  and behaves better than the original one w.r.t. both figures of merit in Eqs. (12) and (13), since

$$\text{Tr}(\mathcal{E}'(|\psi_{\pm}\rangle\langle\psi_{\pm}|)) = \frac{1}{2}(p_{+} + p_{-}) \geq \min\{p_{+}, p_{-}\},$$

$$F(|\varphi_{\pm}\rangle, \mathcal{E}'(|\psi_{\pm}\rangle\langle\psi_{\pm}|)) \geq \min\{F(|\varphi_{+}\rangle, \rho_{+}), F(|\varphi_{-}\rangle, \rho_{-})\}. \quad (23)$$

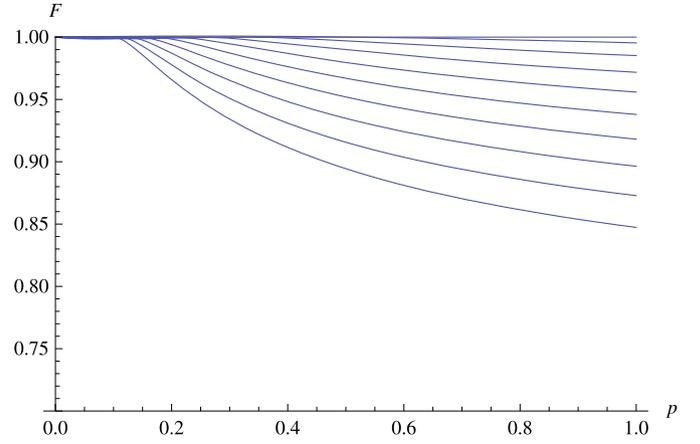
Thus, the frontier of the set of permitted couples  $(p, F)$  can be determined considering only symmetric transformations. Notice that w.l.o.g. we can assume the initial states  $|\psi_{\pm}\rangle$  to be in the symmetric configuration coplanar with  $|\varphi_{\pm}\rangle$  (with  $|\psi_{+}\rangle$  close to  $|\varphi_{+}\rangle$  and  $|\psi_{-}\rangle$  close to  $|\varphi_{-}\rangle$ ), since this can be always achieved by a rotation of the pair, corresponding to an additional unitary transformation, which doesn't change probabilities. In the  $(p, F)$  plane this configuration corresponds to the point  $(1, f_0)$ , where  $f_0 := |\langle\psi_{+}|\varphi_{+}\rangle| \equiv |\langle\psi_{-}|\varphi_{-}\rangle|$ .

Now, let  $f$  be the fidelity we want to achieve, with  $f_0 \leq f \leq 1$ . Then, the set of possible final states  $\rho_{\pm}$  compatible with the constraint  $F(\rho_{\pm}, |\varphi_{\pm}\rangle) \geq f$  is the shadowed area depicted in Fig. 1, where  $|\xi_{\pm}\rangle$  are pure states such that  $|\langle\xi_{\pm}|\varphi_{\pm}\rangle| := f$ . We claim that among such states, the most probably attainable final configuration is the pair  $|\xi_{\pm}\rangle$ . In order to prove the claim, we need to prove that the probability

$$p = \frac{1 - |\langle\psi_{+}|\psi_{-}\rangle|}{1 - F(\rho_{+}, \rho_{-})} \quad (24)$$

reaches the maximum at the pair  $|\xi_{\pm}\rangle$ , over any symmetric pair  $\rho_{\pm}$  inside the area. The fidelity  $F(\rho_{+}, \rho_{-})$  for states compatible with the constraints  $F(\rho_{\pm}, |\varphi_{\pm}\rangle) \geq f$  is maximized by the pair  $|\xi_{\pm}\rangle$ . Indeed, the fidelity  $F(\rho_{+}, \rho_{-})$  for states of the form  $\rho_{\pm} = \frac{1}{2}(I \pm \beta\sigma_x + \gamma\sigma_z)$  can be obtained from Eq. (19)

$$F(\rho_{+}, \rho_{-}) = \sqrt{1 - \beta^2}. \quad (25)$$



**Fig. 2.** Tradeoff curves  $F(p)$  for  $|\langle\psi_{+}|\psi_{-}\rangle| = 0.9$  and for  $|\langle\varphi_{+}|\varphi_{-}\rangle|$  ranging from 0 to 0.9, at intervals of 0.1.

This clearly shows that the optimal states maximizing probability (24) are those minimizing  $\beta$ . The pair  $|\xi_{\pm}\rangle$  satisfies this request, whence it is the most probable.

The remaining part of the optimal tradeoff curve can now be completed quite easily; we only need to sweep the pure states in the arc between  $|\psi_{\pm}\rangle$  and  $|\varphi_{\pm}\rangle$  to obtain the points in the  $(p, F)$ -plane connecting  $(1, f_0)$  and  $(p_0, 1)$ , where  $p_0 = (1 - |\langle\psi_{+}|\psi_{-}\rangle|)/(1 - |\langle\varphi_{+}|\varphi_{-}\rangle|)$ . After a little trigonometry, we obtain the explicit expression for this part of the curve

$$F(p) = \cos \left[ \frac{\arccos |\langle\varphi_{+}|\varphi_{-}\rangle| - \arccos \left( 1 - \frac{1 - |\langle\psi_{+}|\psi_{-}\rangle|}{p} \right)}{2} \right]. \quad (26)$$

In Figs. 2–3 we plot these curves for different values of the fidelities  $|\langle\psi_{+}|\psi_{-}\rangle|$  and  $|\langle\varphi_{+}|\varphi_{-}\rangle|$ .

#### 4. Tradeoff for the inversion of a quantum operation

Suppose we want to know whether a given quantum operation  $\mathcal{E}$  can be inverted deterministically on some subspace  $\mathcal{L} \subseteq \mathcal{H}$ , in other words whether there is a quantum channel  $\mathcal{R}$  such that

$$\rho \rightarrow \rho' = \frac{\mathcal{E}(\rho)}{\text{Tr}(\mathcal{E}(\rho))} \rightarrow \mathcal{R}(\rho') = \rho \quad (27)$$

for every  $\rho$  with  $\text{supp}(\rho) \subseteq \mathcal{L}$ . Necessary and sufficient conditions for this inversion have been proved in Ref. [18], while in Ref. [19] an equivalent condition based on information-theoretical quantities such as entropy and coherent information is provided. If the quantum operation cannot be inverted by a channel, or the inversion is not required to be perfect, it is still possible to achieve an approximate inversion which brings  $\rho'$  close to  $\rho$ . Such “closeness” has been quantified in Ref. [20], whenever  $\mathcal{E}$  is a channel, and in Ref. [21] for general quantum operations.

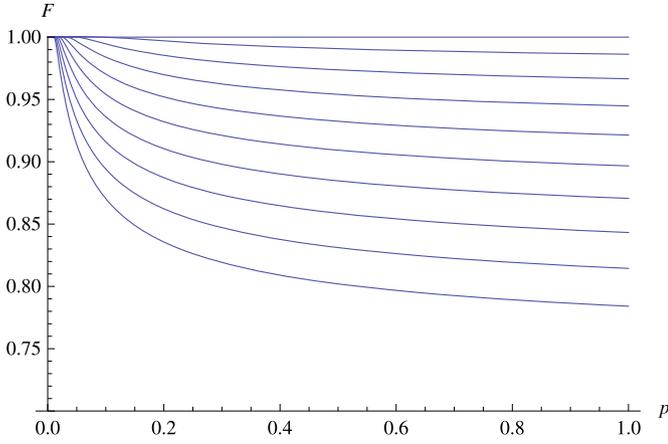
In the present Letter we explore the possibility of *probabilistic* inversions, including *exact* inversions as a particular case. In the following we will focus on a two-dimensional system undergoing an atomic quantum operation

$$\mathcal{E}(\rho) = M\rho M^{\dagger}, \quad (28)$$

where  $M$  is a *contraction*, i.e. satisfying  $\|M\| \leq 1$ . Using the polar decomposition  $M = UP$  with unitary  $U$  and  $P \geq 0$ , w.l.o.g. we can take  $M = M_{\beta} \geq 0$  with the following matrix representation

$$M_{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \quad (29)$$

where  $\beta$ ,  $0 \leq \beta \leq 1$ , is the smallest singular value. The largest singular value can be fixed at 1 up to an overall probability rescaling



**Fig. 3.** Tradeoff curves  $F(p)$  for  $|\langle\psi_+|\psi_-\rangle| = 0.99$  and for  $|\langle\varphi_+|\varphi_-\rangle|$  ranging from 0.09 to 0.99, at intervals of 0.1.

independent of the state (we assume that the quantum operation has happened).

We will consider two case studies with a given set  $\mathcal{D}$  of initial states, and a given set  $\mathcal{Q}$  of quantum operations inverting  $M_\beta$  approximately. After the transformation on the state  $\rho \in \mathcal{D}$

$$\rho' = \frac{M_\beta \rho M_\beta}{\text{Tr}(\rho M_\beta^2)}, \quad (30)$$

a following “inverting” quantum operation  $\mathcal{R} \in \mathcal{Q}$  leaves the system in the state

$$\rho'' = \frac{\mathcal{R}(\rho')}{\text{Tr}(\mathcal{R}(\rho'))}. \quad (31)$$

The quality of the inversion is assessed by two figures of merit: (i) the probability of success

$$p(\mathcal{R}; \rho) = \text{Tr}(\mathcal{R}(\rho')), \quad (32)$$

(ii) the fidelity with the initial state

$$f(\mathcal{R}; \rho) = F(\rho, \rho''). \quad (33)$$

In order to keep the probability of success above some threshold  $\bar{p}$  we consider only the subset  $\mathcal{Q}_{\bar{p}} \subseteq \mathcal{Q}$  whose elements  $\mathcal{R}$  satisfy the constraint:

$$p(\mathcal{R}; \rho) \geq \bar{p}, \quad \forall \rho \in \mathcal{D}. \quad (34)$$

In a worst-case criterion we have to choose the inversion  $\bar{\mathcal{R}} \in \mathcal{Q}_{\bar{p}}$  maximizing the minimum fidelity over  $\mathcal{D}$

$$\bar{\mathcal{R}} = \arg \max_{\mathcal{R} \in \mathcal{Q}_{\bar{p}}} \min_{\rho \in \mathcal{D}} f(\mathcal{R}; \rho). \quad (35)$$

This gives the point  $(\bar{p}, \bar{F})$ , with  $\bar{F} = \min_{\rho \in \mathcal{D}} f(\bar{\mathcal{R}}; \rho)$ , in the  $(p, F)$  plane. The tradeoff curve is obtained varying  $\bar{p}$  in the interval  $[0, 1]$ . In this way we obtain a curve  $F = F(p)$  giving the minimum fidelity over  $\mathcal{D}$  achievable with probability of success at least  $p$ .

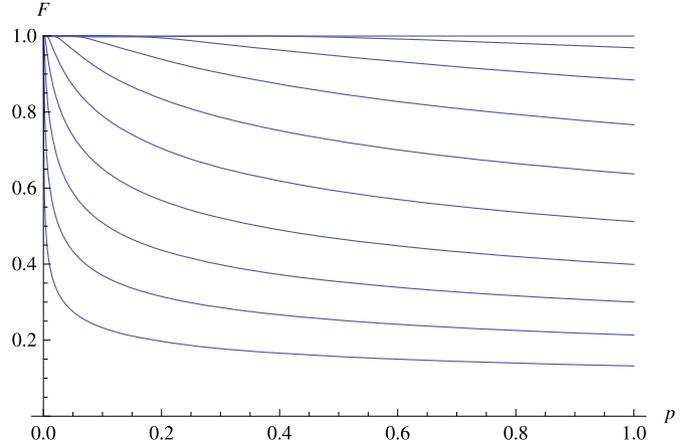
#### 4.1. Semiclassical case

The set of input states  $\mathcal{D} = \{\rho_x\}$  consists of all density operators jointly diagonal with the contraction

$$\rho_x = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}, \quad 0 \leq x \leq 1, \quad (36)$$

while the set of possible inversions  $\mathcal{Q} = \{N_\gamma\}$  consists of the diagonal contractions

$$N_\gamma = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta \leq \gamma \leq 1. \quad (37)$$



**Fig. 4.** Tradeoff curves for the semiclassical case, for various  $\beta$ . Each curve gives the minimum guaranteed fidelity in the inversion of  $M_\beta$ , as a function of the minimum probability  $p$  of success over all the initial states. Curves from the bottom to the top correspond to increasing values of  $\beta$ , ranging from 0.1 to 1.

The unit-fidelity case is the matrix inverse (rescaled in order to keep it a contraction)  $N_\beta = M_\beta^{-1} / \|M_\beta^{-1}\|$ .

The states  $\rho'_x$  and  $\rho''_x$  are easily computed

$$\rho'_x = \frac{1}{x + \beta^2(1-x)} \begin{pmatrix} x & 0 \\ 0 & \beta^2(1-x) \end{pmatrix}, \quad (38)$$

$$\rho''_x = \frac{1}{\gamma^2 x + \beta^2(1-x)} \begin{pmatrix} \gamma^2 x & 0 \\ 0 & \beta^2(1-x) \end{pmatrix}, \quad (39)$$

and so are the probability and the fidelity

$$p(N_\gamma; \rho_x) = \frac{\gamma^2 x + \beta^2(1-x)}{x + \beta^2(1-x)}, \quad (40)$$

$$f(N_\gamma; \rho_x) = \frac{\gamma x + \beta(1-x)}{\sqrt{\gamma^2 x + \beta^2(1-x)}}. \quad (41)$$

By inspection of these expressions one can see that the set  $\mathcal{Q}_{\bar{p}}$  is

$$\mathcal{Q}_{\bar{p}} = \{N_\gamma, \gamma^2 \geq \bar{p}\} \quad (42)$$

and that

$$\arg \max_{N_\gamma \in \mathcal{Q}_{\bar{p}}} \min_{\rho_x} f(N_\gamma; \rho_x) = N_{\sqrt{\bar{p}}}. \quad (43)$$

The corresponding tradeoff curves are plotted in Fig. 4 for various  $\beta$ . The uppermost curves are obtained when  $\beta$  approaches 1, i.e. when  $M_\beta$  is near to the identity (clearly, in this case there is almost no need of inversion). On the other hand, as  $\beta$  goes to zero  $M_\beta$  approaches an orthogonal projector which, in our worst-case criterion, cannot be inverted with nonvanishing minimum fidelity.

#### 4.2. Quantum case

We consider a set of two non-orthogonal states  $\mathcal{D} = \{|\psi_\pm\rangle\}$ , and we let  $\mathcal{Q}$  to be the set of all quantum operations. The states after the first transformation are

$$|\psi'_\pm\rangle = \frac{M_\beta |\psi_\pm\rangle}{\|M_\beta |\psi_\pm\rangle\|}. \quad (44)$$

The required inversion is then

$$|\psi'_\pm\rangle \rightarrow |\psi_\pm\rangle \quad (45)$$

which we already studied in Section 3.

<sup>1</sup> The “flipping” contractions  $P_\delta = \begin{pmatrix} 0 & \delta \\ \delta & 1 \end{pmatrix}$ ,  $0 \leq \delta \leq 1$  maintain the diagonal form of the states. However, they do not improve both the fidelity and the probability of inversion.

## 5. Conclusions

After generalizing the state-transformation probability formula of Ref. [13] to mixed target states, we derived a probability-fidelity tradeoff for a varying quantum operation with fixed input-output states. We have then presented the first tradeoff between the probability and the fidelity in the inversion of a quantum operation in a semiclassical and in a quantum case.

## Acknowledgements

This work has been supported by the EC through the project CORNER.

## References

- [1] C.A. Fuchs, A. Peres, Phys. Rev. A 53 (1996) 2038.
- [2] C.A. Fuchs, Fortschr. Phys. 46 (1998) 535.
- [3] H. Barnum, Report University of Bristol, 2000, quant-ph/0205155.
- [4] K. Banaszek, Phys. Rev. Lett. 86 (2001) 1366.
- [5] K. Banaszek, I. Devetak, Phys. Rev. A 64 (2001) 052307.
- [6] L. Mišta Jr., J. Fiurášek, R. Filip, Phys. Rev. A 72 (2005) 012311.
- [7] M.F. Sacchi, Phys. Rev. Lett. 96 (2006) 220502.
- [8] U.L. Andersen, M. Sabuncu, R. Filip, G. Leuchs, Phys. Rev. Lett. 96 (2006) 020409.
- [9] M.G. Genoni, M.G.A. Paris, Phys. Rev. A 74 (2006) 012301.
- [10] F. Buscemi, M.F. Sacchi, Phys. Rev. A 74 (2006) 052320.
- [11] M.F. Sacchi, Phys. Rev. A 75 (2007) 012306.
- [12] G.M. D'Ariano, Fortschr. Phys. 51 (2003) 318.
- [13] A. Chefles, S.M. Barnett, J. Phys. A 31 (1998) 10097.
- [14] A. Uhlmann, Rep. Math. Phys. 9 (1976) 273.
- [15] M. Ozawa, J. Math. Phys. 25 (1984) 79.
- [16] K. Kraus, States, Effects and Operations. Fundamental Notions of Quantum Theory, Springer-Verlag, Berlin, Heidelberg, 1983.
- [17] M. Hübner, Phys. Lett. A 163 (1992) 239.
- [18] E. Knill, R. Laflamme, arXiv:quant-ph/9604034.
- [19] B. Schumacher, M.A. Nielsen, Phys. Rev. A 54 (1996) 2629.
- [20] B. Schumacher, M. Westmoreland, Quantum Inf. Proc. 1 (2002) 5.
- [21] F. Buscemi, M. Hayashi, M. Horodecki, Phys. Rev. Lett. 100 (2008) 210504.