

On the parametric approximation in quantum optics^(*)

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(ricevuto il 16 Settembre 1998; approvato il 15 Ottobre 1998)

Summary. — We perform the exact numerical diagonalization of the Hamiltonians that describe both degenerate and nondegenerate parametric amplifiers, by exploiting the conservation laws pertaining each device. We clarify the conditions under which the parametric approximation holds, showing that the most relevant requirement is the coherence of the pump after the interaction, rather than its undepletion.

PACS 03.65 – Quantum mechanics.

PACS 42.50 – Quantum optics.

PACS 42.65.Yj – Optical parametric oscillators and amplifiers.

PACS 42.50.Ar – Photon statistics and coherence theory.

1. – Introduction

The interactions of different radiation modes through nonlinear crystals allow the generation of interesting states of light, which exhibit a rich variety of phenomena [1-15]. Most of the theoretical analysis usually refers to situations where one mode—the so-called “pump” mode—is placed in a high-amplitude coherent state. In such cases the parametric approximation is widely used to compute the dynamical evolution [16]. In the parametric approximation the pump mode is classically treated as a c -number, thus neglecting both the depletion mechanism and quantum fluctuations. As a result, bilinear and trilinear Hamiltonians are reduced to linear and quadratic forms in the field operators, respectively, and hence some useful mathematical tools—typically decomposition formulas for Lie algebras—can be exploited for calculations [3, 17, 18].

In the validity regime of the parametric approximation different optical devices experimentally realize different fundamental unitary operators in quantum optics. For example, a beam splitter, by suitably mixing the signal state with a strong local oscillator at the same frequency, realizes the displacement operator, which generates coherent states from

(*) The authors of this paper have agreed to not receive the proofs for correction.

the vacuum. Similarly, a degenerate parametric amplifier realizes the squeezing operator, which is the generator of the squeezed vacuum. Finally, a nondegenerate parametric amplifier realizes the two-mode squeezing operator, *i.e.* the generator of twin-beam. In the above scenario, it is matter of great interest to study the conditions under which the parametric approximation holds. This issue has been considered by a number of authors [19-22], however without giving a general validity criterion, which is the main concern of this paper. Quantum effects in two-mode optical amplifiers have been extensively analyzed [20,21,23-30]. Frequency couplers with intensity-dependent coupling have been studied [25,26,31], whereas the case of degenerate parametric amplifier has been considered by many authors [20,21,24,27-30]. Phase correlations [28] and the signal-pump degree of entanglement [29] have been examined. The effect of pump squeezing has been also considered [30]. On the other hand, though trilinear processes have been thoroughly analyzed in refs. [15,24,32], only little attention has been devoted to the parametric approximation in nondegenerate amplifiers [22].

The most explicit conditions for the validity of the parametric approximation have been derived in refs. [19] and [20], for the beam splitter and the degenerate parametric amplifier, respectively. In both references sufficient conditions have been derived, which however can be widely breached in relevant cases of interest, as we will show in the following.

In this paper we perform the exact numerical diagonalization of the full Hamiltonians pertaining the three above-mentioned devices. As was already noted by other authors [23,24,32], such a numerical treatment is made amenable by the presence of constants of motion that characterizes each Hamiltonian. In fact, the Hilbert space can be decomposed into the direct sum of subspaces that are invariant under the action of the unitary evolution. Therefore, one needs to diagonalize the Hamiltonian just inside each invariant subspace, thus considerably reducing the dimension of the diagonalization space [23,24,32].

We analyze in different sections the cases of the beam splitter, the degenerate parametric amplifier and the nondegenerate parametric amplifier. The case of the beam splitter can be treated analytically, but we also present some numerical results in order to introduce the general approach that will be used for the parametric amplifiers. For each device, we look for the conditions under which the parametric approximation holds, for both vacuum and non-vacuum input signal states. The comparison between the exact evolution and the theoretical predictions from the parametric approximation is made in terms of the overlap $\mathcal{O} = \sqrt{\text{Tr}(\hat{\rho}_{\text{out}} \hat{\rho}_{\text{th}})}$ between the state $\hat{\rho}_{\text{out}}$ that exits the device and the theoretical state $\hat{\rho}_{\text{th}}$ obtained within the approximation. An explicit comparison in terms of photon number distributions and Wigner functions is also given for some interesting and representative cases.

The main result of the paper is to show that the usual requirements for the validity of the parametric approximation, namely short interaction time and strong classical undepleted pump, are too strict. Indeed, we show that the only relevant requirement is the coherence of the pump after the interaction, rather than its undepletion. In fact, we will show typical examples in which the pump at the input is weak (one photon in average), after the interaction it is highly depleted, and notwithstanding the parametric approximation still holds. On the other hand, there are cases in which the pump after the interaction is only slightly depleted, however is no longer coherent, and the approximation fails. Finally, we show some interesting features such as pump squeezing and Schrödinger-cat-like state generation that arise when the parametric approximation breaks down.

2. – Displacer

The beam splitter is a passive device that couples two different modes of radiation at the same frequency through a first-order susceptibility-tensor $\chi^{(1)}$ medium. Such device is widely used in quantum optics [1, 33], from homodyne/heterodyne detection [34], to directional couplers [35] and cavity QED [36]. In the rotating wave approximation and under phase-matching conditions, the beam splitter Hamiltonian writes in terms of the two mode operators a and b as follows:

$$(1) \quad \hat{H}_{\text{BS}} = \kappa (ab^\dagger + a^\dagger b) ,$$

where κ is the coupling constant proportional to the $\chi^{(1)}$ of the medium. The unitary evolution operator of the device in the interaction picture writes

$$(2) \quad \hat{U}_{\text{BS}} = \exp [-i\tau (ab^\dagger + a^\dagger b)] = e^{-i \tan \tau ab^\dagger} |\cos \tau|^{a^\dagger a - b^\dagger b} e^{-i \tan \tau a^\dagger b} ,$$

where τ is the interaction time rescaled by the coupling κ . The factorization of the operator \hat{U}_{BS} in eq. (2) has been derived by applying the Baker-Campbell-Hausdorff formula for the $SU(2)$ algebra [3, 17, 18]. The Heisenberg evolution of the field modes reads

$$(3) \quad \hat{U}_{\text{BS}}^\dagger \begin{pmatrix} a \\ b \end{pmatrix} \hat{U}_{\text{BS}} = \begin{pmatrix} a \cos \tau - ib \sin \tau \\ -ia \sin \tau + b \cos \tau \end{pmatrix} .$$

From eq. (3) it turns out that the transmissivity θ at the beam splitter is given by the relation $\theta = \cos^2 \tau$.

The parametric approximation refers to situations in which one mode—say mode b —is excited in a strong coherent state. In this case in the first line of eq. (2) the operator b might be replaced by a c -number, namely the complex amplitude β of the coherent state. Under this assumption, the evolution operator (2) would rewrite as the following displacement operator:

$$(4) \quad \hat{D}(-i\beta\tau) \equiv \exp [-i\tau (\beta a^\dagger + \bar{\beta}a)] .$$

A more refined approximation that takes into account the 2π -periodicity in the exact Heisenberg equations (3) is

$$(5) \quad \hat{D}(-i\beta \sin \tau) \equiv \exp [-i \sin \tau (\beta a^\dagger + \bar{\beta}a)] .$$

Indeed, the more precise result in eq. (5) can be obtained by recasting the factorized expression in eq. (2) in normal order with respect to mode a , after taking the expectation over mode b [37, 38]. The simple form of the bilinear Hamiltonian in eq. (1) allows to clarify the conditions under which the parametric approximation (5) holds [19]. A set of sufficient requirements are given by

$$(6) \quad \begin{aligned} |\beta| &\rightarrow \infty , & \sin \tau &\rightarrow 0 \\ |\beta| \sin \tau &= \text{const} , \end{aligned}$$

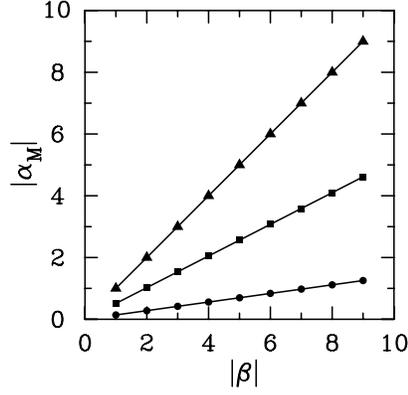


Fig. 1. – Performances of a beam splitter in achieving the displacement operator. We report the maximum displacing amplitude $|\alpha_M|$ achievable by a beam splitter as a function of the pump amplitude $|\beta|$. In the picture triangles refer to vacuum input, squares to coherent state input $|\alpha \equiv 1\rangle$, and circles to number state input $|n \equiv 1\rangle$.

without any assumption on the “signal” state for mode a . Hence, by combining a signal input state $\hat{\rho}_{\text{in}}$ with a strong coherent local oscillator $|\beta\rangle$ in a beam splitter with very high transmissivity, one can achieve the displacement operator in eq. (5). The theoretically expected state $\hat{\rho}_{\text{th}}$ then writes

$$(7) \quad \hat{\rho}_{\text{th}} = \hat{D}(-i\beta \sin \tau) \hat{\rho}_{\text{in}} \hat{D}^\dagger(-i\beta \sin \tau).$$

Here we present some numerical results concerning the exact unitary evolution of eq. (2). The dynamics generated by the Hamiltonian (1) preserves the total number of photons involved in the process, in agreement with the following commutation relation:

$$(8) \quad \left[\hat{H}_{\text{BS}}, a^\dagger a + b^\dagger b \right] = 0.$$

Therefore, it is convenient to decompose the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ as a direct sum of subspaces with a fixed number N of photons, since these are invariant under the action of the unitary evolution operator (2). Such a decomposition can be written as follows:

$$(9) \quad \mathcal{H}_a \otimes \mathcal{H}_b = \bigoplus_{N=0}^{+\infty} \mathcal{H}_N,$$

where

$$(10) \quad \mathcal{H}_N = \text{Span} \{ |m\rangle \otimes |N-m\rangle, m \in [0, N] \},$$

$\text{Span}\{\cdot\}$ denoting the Hilbert subspace linearly spanned by the orthogonal vectors within the brackets, and $|n\rangle \otimes |m\rangle \equiv |n, m\rangle$ representing the common eigenvector of the number operator of the two modes. The decomposition in eq. (9) makes the Hamiltonian (1) block-diagonal, namely

$$(11) \quad \hat{H}_{\text{BS}} = \sum_{N=0}^{+\infty} \hat{h}_N,$$

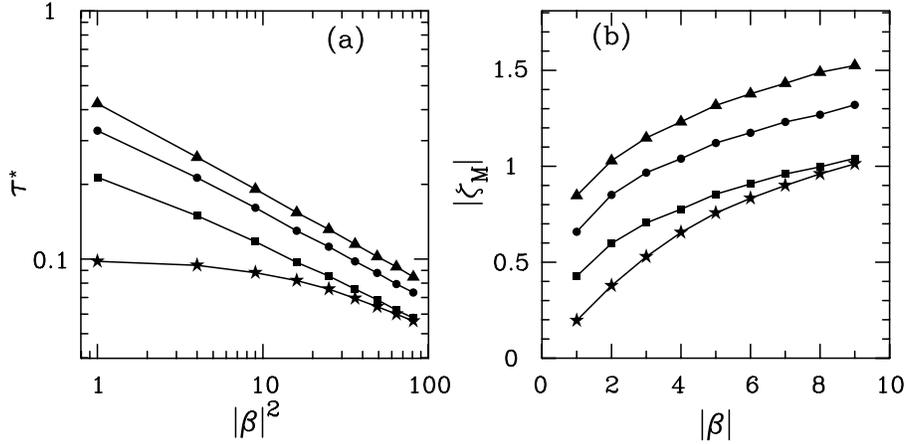


Fig. 2. – Performances of a degenerate parametric amplifier in providing the squeezing operator $\hat{S}(\zeta)$. In both pictures triangles refer to vacuum input, squares to coherent state input $|\alpha \equiv 1\rangle$, circles to number state input $|n \equiv 1\rangle$, and stars to number state input $|n \equiv 2\rangle$. In (a) we report the quantity τ^* , namely the maximum interaction time that leads to an output signal whose overlap with the theoretical squeezed state is larger than 99%, as a function of the pump intensity $|\beta|^2$. In (b) we show the maximum squeezing parameter $|\zeta_M|$ achievable by the degenerate parametric amplifier, as a function of the pump amplitude $|\beta|$.

where \hat{h}_N acts just inside the subspace \mathcal{H}_N . Correspondingly, a generic two-mode state $|\psi_0\rangle$ can be written in the orthogonal basis (10) as follows:

$$(12) \quad |\psi_0\rangle = \sum_{N=0}^{+\infty} \sum_{m=0}^N c_{m,N-m} |m, N-m\rangle.$$

The diagonalization is performed inside each invariant subspace, and the truncation of the series in eqs. (11) and (12) corresponds to fix the maximum eigenvalue of the constant of motion $a^\dagger a + b^\dagger b$.

The state $\hat{\rho}_{\text{out}}$ evaluated by the exact evolution operator (2) is given by

$$(13) \quad \hat{\rho}_{\text{out}} = \text{Tr}_b[\hat{U}_{\text{BS}}(\hat{\rho}_{\text{in}} \otimes |\beta\rangle\langle\beta|)\hat{U}_{\text{BS}}^\dagger],$$

where Tr_b denotes the partial trace on \mathcal{H}_b . The comparison between the theoretical state $\hat{\rho}_{\text{th}}$ of eq. (7) within the parametric approximation and the actual state $\hat{\rho}_{\text{out}}$ is made in terms of the relative overlap

$$(14) \quad \mathcal{O} \equiv \sqrt{\text{Tr}[\hat{\rho}_{\text{th}}\hat{\rho}_{\text{out}}]}.$$

In the case of coherent input signal the overlap is evaluated analytically. One has

$$(15) \quad \hat{U}_{\text{BS}} |\alpha\rangle \otimes |\beta\rangle = |\alpha \cos \tau - i\beta \sin \tau\rangle \otimes |\beta \cos \tau - i\alpha \sin \tau\rangle,$$

and thus

$$(16) \quad \mathcal{O} = |\langle\alpha - i\beta \sin \tau | \alpha \cos \tau - i\beta \sin \tau\rangle| = \exp\left[-4|\alpha|^2 \sin^4 \frac{\tau}{2}\right].$$

From eq. (16) it is apparent that the parametric approximation gives always exact results for vacuum input state ($\alpha \equiv 0$), whereas it is justified for coherent state as long as $4|\alpha|^2 \sin^4(\tau/2) \ll 1$, independently of the pump intensity.

We introduce the quantity τ^* which represents, for a fixed value of the pump amplitude $|\beta|$, the maximum interaction time leading to an overlap larger than 99%. The value of τ^* clearly depends on the input signal: in agreement with eq. (16) it is not defined for the vacuum (parametric approximation is exact), whereas for a coherent input signal $|\alpha\rangle$ it is given by

$$\tau^* = 2 \arcsin \sqrt{\frac{C}{|\alpha|}} \quad C = \frac{1}{2} (-\ln 0.99)^{1/2} \simeq 0.05.$$

The quantity τ^* also determines the maximum displacing amplitude $|z_M| \equiv |\beta| \sin \tau^*$ that can be achieved by a beam splitter with a coherent pump $|\beta\rangle$. In fig. 1 we have reported $|z_M|$ for the vacuum, a coherent state and a number state as a function of the pump amplitude $|\beta|$. The linear behavior of the plots indicates that τ^* is independent of the pump intensity. In the case of vacuum input we have complete energy transfer from the pump to the signal (slope of $|z_M|$ vs. $|\beta|$ equal to unit). Although they have the same energy, the coherent and number input states show different slopes, the coherent being more similar to the vacuum. Actually the set of coherent states is closed under the action of the displacement operator, so that the parametric approximation can fail only in predicting the exact amplitude of the output coherent state.

We conclude that the first of requirements (7) is too tight. At least for coherent and number states, as long as the signal average photon number is less than the pump one, the beam splitter can “displace” the signal also for very weak pump.

In the next sections we will deal with the problem of parametric approximation in nonlinear amplifiers.

3. – Squeezer

The degenerate parametric amplifier couples a signal mode a at frequency ω_a with a pump mode c at double frequency $\omega_c = 2\omega_a$. The interaction is mediated by the second-order susceptibility tensor $\chi^{(2)}$ of the medium. Each photon in the pump mode produces a photon pair in the signal mode, giving rise to light with a number of interesting properties, such as phase-sensitive amplification, squeezing and antibunching [4-7, 15]. In the rotating wave approximation and under phase-matching conditions the Hamiltonian writes

$$(17) \quad \hat{H}_{\text{DP}} = \kappa (a^2 c^\dagger + a^{\dagger 2} c),$$

with $\kappa \propto \chi^{(2)}$. The corresponding unitary evolution operator in the interaction picture reads

$$(18) \quad \hat{U}_{\text{DP}} = \exp [-i\tau (a^2 c^\dagger + a^{\dagger 2} c)],$$

where τ represents a rescaled interaction time. The parametric approximation replaces the pump mode c by the complex amplitude β of the corresponding coherent state, such

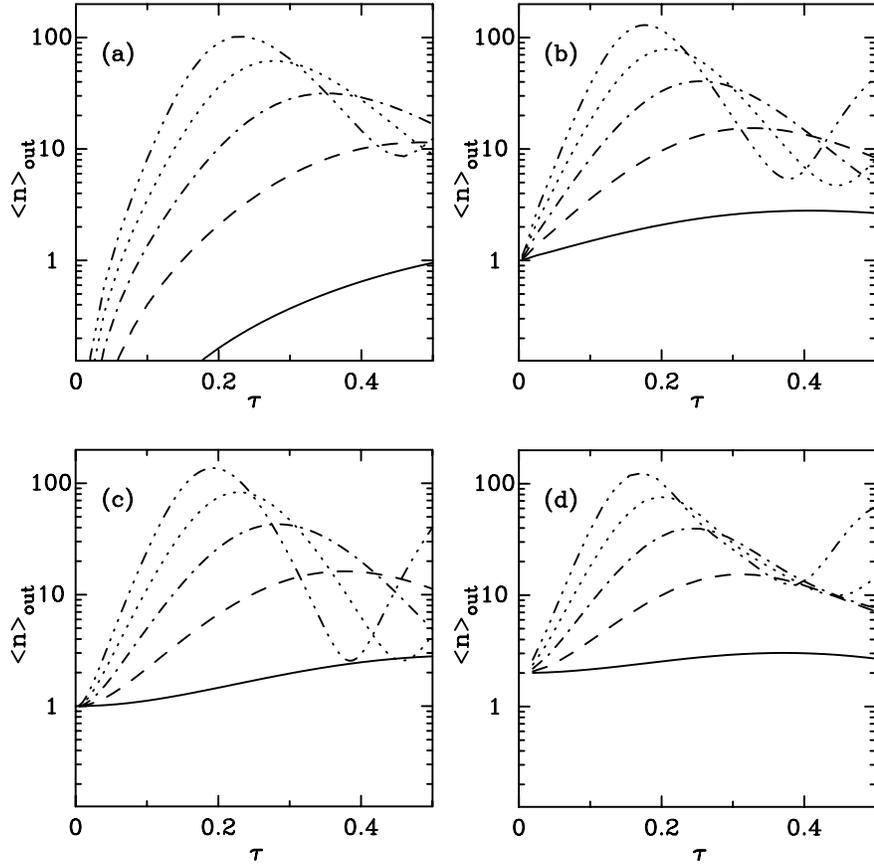


Fig. 3. – Average photon number $\langle \hat{n} \rangle_{\text{out}}$ of the signal at the output of a degenerate parametric amplifier, as a function of the interaction time τ . In (a) the case of vacuum input, in (b) coherent input $|\alpha \equiv 1\rangle$, in (c) number input $|n \equiv 1\rangle$, and in (d) number input $|n \equiv 2\rangle$. Different line-styles refer to different pump amplitudes: $\beta = 9$ (dot-dot-dashed), $\beta = 7$ (dotted), $\beta = 5$ (dot-dashed), $\beta = 3$ (dashed), $\beta = 1$ (solid).

that the operator (18) rewrites as

$$(19) \quad \hat{S}(-2i\tau\beta) \equiv \exp[-i\tau(\beta a^{\dagger 2} + \bar{\beta} a^2)],$$

$\hat{S}(\zeta)$ being the squeezing operator [4]. In the case of coherent input signal $|\alpha\rangle$, the predicted state at the output is the squeezed state

$$(20) \quad \hat{S}(-2i\tau\beta)|\alpha\rangle = \hat{S}(-2i\tau\beta)\hat{D}(\alpha)|0\rangle = \hat{D}(\tilde{\alpha})\hat{S}(-2i\tau\beta)|0\rangle \equiv |\tilde{\alpha}, -2i\tau\beta\rangle,$$

with $\tilde{\alpha} = \alpha \cosh(-2i\tau\beta) + \bar{\alpha} \sinh(-2i\tau\beta)$. Notice that, differently from the beam splitter operator of eq. (2), we have no method available to order eq. (18) normally with respect to mode c (as in eq. (2) for b) and then replace such mode by the c -number $|\beta|$. Hence, we have no analogous nonperturbative method to estimate the validity of the parametric approximation. Hillery and Zubairy have been approached the question [20] in terms

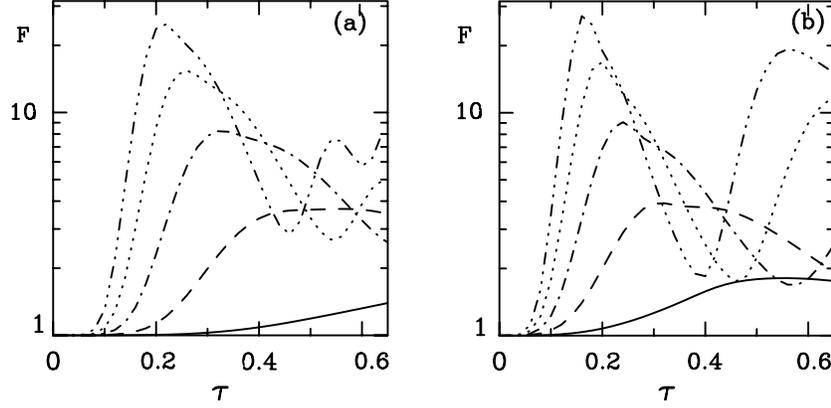


Fig. 4. – Fano factor F of the pump at the output of a degenerate parametric amplifier, as a function of the interaction time τ . In (a) the case of vacuum input; in (b) of coherent input $|\alpha \equiv 1\rangle$. Different line-styles refer to different pump amplitude: $\beta = 9$ (dot-dot-dashed), $\beta = 7$ (dotted), $\beta = 5$ (dot-dashed), $\beta = 3$ (dashed), $\beta = 1$ (solid).

of a perturbation series for the propagator of the Hamiltonian (17). For initial vacuum state at mode a , they write the following conditions:

$$(21) \quad \begin{aligned} 1/|\beta| \ll 1, \quad \tau \ll 1, \\ \tau e^{4|\beta|\tau} \ll 1, \quad e^{4|\beta|\tau} \ll |\beta|. \end{aligned}$$

Here we evaluate the exact evolution generated by the operator (18) through numerical diagonalization of the Hamiltonian (17), using the method based on the constant of motion. In this case one has

$$(22) \quad \left[\hat{H}_{\text{DP}}, a^\dagger a + 2c^\dagger c \right] = 0,$$

and the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_c$ is decomposed in terms of invariant subspaces corresponding to the eigenvalues of the constant of motion $a^\dagger a + 2c^\dagger c$, namely

$$(23) \quad \mathcal{H}_a \otimes \mathcal{H}_c = \bigoplus_{N=0}^{+\infty} \mathcal{H}_N,$$

with

$$(24) \quad \mathcal{H}_N = \text{Span} \{ |N - 2m\rangle \otimes |m\rangle, m \in [0, \lfloor N/2 \rfloor] \},$$

$\lfloor \cdot \rfloor$ denoting the integer part of x . Hence the Hamiltonian in eq. (17) rewrites in the same fashion as in eq. (11) and the block-diagonalization is performed for each \hat{h}_N , with N from 0 to the maximum allowed value of the constant of motion. Similarly to eq. (12), a generic two-mode state $|\psi_0\rangle$ is written as follows:

$$(25) \quad |\psi_0\rangle = \sum_{N=0}^{+\infty} \sum_{m=0}^{\lfloor N/2 \rfloor} c_{N-2m,m} |N - 2m, m\rangle.$$

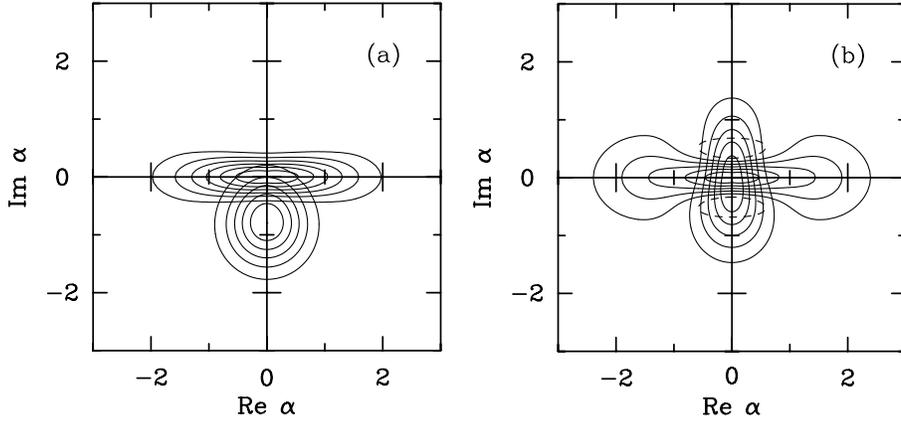


Fig. 5. – Contour plot Wigner functions of both the signal and the pump modes at the output of a degenerate parametric amplifier. The input signal is the vacuum, whereas the pump is initially in a coherent state $\beta = -i$. The time interaction is equal to $\tau = \tau^* = 0.42$ in (a) and to $\tau = 2\tau^* = 0.84$ in (b).

The performances of a degenerate parametric amplifier in realizing the squeezing operator $\hat{S}(\zeta) = \exp[1/2(\zeta a^{\dagger 2} - \bar{\zeta} a^2)]$ of eq. (19) are depicted in fig. 2. In fig. 2a) we have reported the maximum interaction time τ^* that leads to an output signal whose overlap with the theoretical squeezed state is larger than 99%, as a function of the pump intensity $|\beta|^2$. In 2b) we have shown the maximum squeezing parameter $|\zeta_M|$ achievable by the amplifier, as a function of the pump amplitude $|\beta|$. According to eq. (19) one has $|\zeta_M| = 2|\beta|\tau^*$. In both pictures we have considered the vacuum, a coherent state and two different number states at the input of the amplifier. For the same set of input states, we have also shown in fig. 3 the average signal photon number as a function of the interaction time τ , for five different values of the pump amplitude.

From fig. 2 it turns out that the requirements for vacuum input signal in eq. (22) are too strict. In particular, the two conditions in the second line are not satisfied for $1 < |\beta| < 9$ (see the line with triangles in fig. 2a)). Moreover, fig. 2 shows that one can realize a squeezing operator even through a weak pump with just one photon.

By definition the validity of the parametric approximation is guaranteed for $\tau < \tau^*$, τ^* depending on $|\beta|$ and on the input state. However, we want to provide a general criterion that can be easily checked experimentally. As we will show in the following, the undepletion of the pump is not a valid criterion. We argue that the relevant parameter, in order to confirm whether the parametric approximation is justified or not, is the degree of coherence of the pump *after* the nonlinear interaction. These statements are supported by the following numerical results.

Let us consider the case of a number input state $|n = 1\rangle$ with pump amplitude $|\beta| = 9$. From fig. 2 one can extract the maximum interaction time $\tau^* \simeq 0.073$ for the validity of the parametric approximation, and the corresponding maximum squeezing parameter $|\zeta_M| \simeq 1.314$. The average photon number of the output state can be drawn from fig. 3c) as $\langle n \rangle_{\text{out}} \simeq 9.68$, corresponding to a pump depletion of about 5.4%. One might consider such a small depletion as the sign of the goodness of the parametric approximation. On the other hand, from fig. 3c) one recognizes the region $0.33 \lesssim \tau \lesssim 0.44$, in which the output signal is even less excited than that in the above example, and consequently the

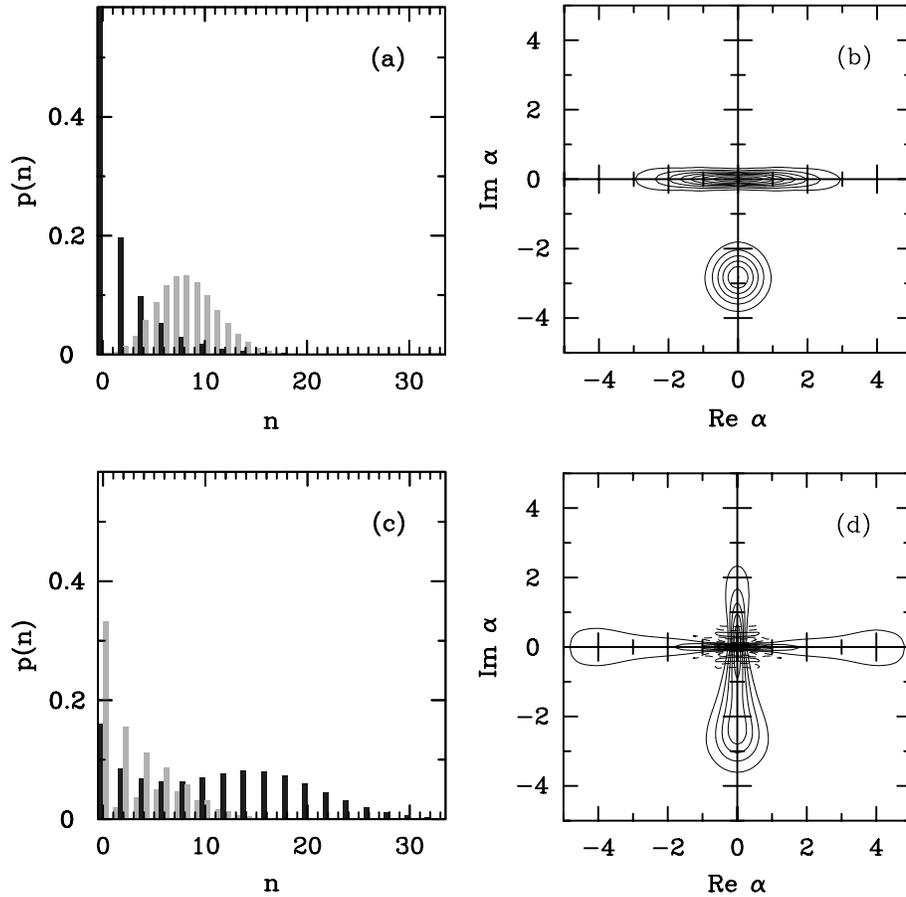


Fig. 6. – Photon number probability and contour plot Wigner function for both the signal and the pump mode at the output of a degenerate parametric amplifier. The plots refer to a situation in which the signal mode is initially in the vacuum and the pump mode is excited to a coherent state with amplitude $\beta = -3i$. In (a) and (b) the interaction time is equal to $\tau \equiv \tau^* = 0.19$, whereas (c) and (d) refer to an interaction time $\tau = 0.43 > 2\tau^*$. In the first case the parametric approximation well describes the real interaction, which produces a squeezed vacuum state with squeezing parameter $r = 1.146$ corresponding to about 2 squeezing photons. On the other hand, parametric approximation does not hold in the second case. Notice that the break-down of parametric approximation is connected with the appearance of negative values in the Wigner function, which is a signature of quantum interference in the phase space.

pump is less depleted. Nevertheless the parametric approximation does not hold in such range of interaction time, since τ is larger than τ^* . Let us now consider the Fano factor $F = \langle \Delta \hat{n}^2 \rangle / \langle \hat{n} \rangle$ of the pump at the output. One finds that in the region $0.33 \lesssim \tau \lesssim 0.44$ the Fano factor is always larger than $F = 1.13$, whereas for $\tau < \tau^*$ it never exceeds $F = 1.10$.

More generally, in all situations in which the parametric approximation is satisfied we found that the Fano factor of the pump at the output never exceeds $F = 1.10$. This holds also when the pump is weak ($|\beta|^2 = 1-10$). Indeed, in this case the depletion of the

pump can be strong, nevertheless the parametric approximation does not break down. In fig. 4 we show the Fano factor of the output pump as a function of the interaction time τ , for different values of the pump amplitude. Plots refer to vacuum input and to coherent state input ($|\alpha \equiv 1\rangle$): similar plots can be obtained for other input states.

As the condition of pump undepletion does not guarantee the validity of the parametric approximation, so pump depletion by itself does not sign its failure: rather we have to consider the Fano factor of the pump. In order to stress this point, let us consider the extreme case of a pump with only one photon, and the input signal in the vacuum. The exact numerical solution indicates that the parametric approximation holds for interaction time up to $\tau^* \simeq 0.42$, the squeezing parameter and the output signal photon number increasing up to $|\zeta_M| \simeq 0.84$ and $\langle n \rangle_{\text{out}} \simeq 0.74$, respectively (see figs. 2 and 3a)). Correspondingly, the pump depletion grows up to 37% at $\tau = \tau^*$. In spite of the strong depletion, the pump preserves a good degree of coherence: the Fano factor achieves at most the value $F = 1.10$ at $\tau = \tau^*$.

In summary, the validity regime $\tau < \tau^*$ for the parametric approximation does not identify with the condition of pump undepletion, rather it corresponds to a Fano factor not exceeding the initial coherent level more than 10%.

What happens beyond the parametric regime? For interaction time larger than τ^* new quantum effects arise at the output. In fig. 5 we show the Wigner functions of both the signal and the pump modes at the output of the amplifier for $\tau = \tau^*$ and $\tau = 2\tau^*$, with vacuum input and weak pump $|\beta| = 1$. In fig. 6 the case of a stronger pump is given. As τ increases, the pump first empties, then it starts refilling, preferably for even photon numbers, leading to oscillations in the photon number distribution. Remarkably, the corresponding Wigner functions of the pump and the signal exhibit interference in the phase space [39], the signal resembling a Schrödinger-cat.

4. – Two-mode squeezer

The nondegenerate parametric amplifier involves three different modes of the radiation field—say the signal a , the idler b and the pump c —which are coupled by a $\chi^{(2)}$ nonlinear medium. The relation between the frequencies of the field modes is given by $\omega_c = \omega_a + \omega_b$. The Hamiltonian of the amplifier under phase-matched conditions can be written in the rotating wave approximation as follows:

$$(26) \quad \hat{H}_{\text{NP}} = \kappa (abc^\dagger + a^\dagger b^\dagger c) ,$$

with $\kappa \propto \chi^{(2)}$. The Hamiltonian in eq. (26) describes also the case in which the frequencies pertaining modes a and b are the same, provided that the respective wave vectors and/or polarizations are different. The dynamics induced by the Hamiltonian (26) leads to a considerably rich variety of phenomena, such as generation of strongly correlated photon pairs by parametric downconversion [8-10], phase insensitive amplification [3,16], generation of heterodyne eigenstates that are suitable for optimal phase detection [11], polarization entanglement [12] and realization of Bell states [10,12-14]. The unitary evolution operator in the interaction picture reads

$$(27) \quad \hat{U}_{\text{NP}} = \exp [-i\tau (abc^\dagger + a^\dagger b^\dagger c)] ,$$

where τ represents a rescaled interaction time. The parametric approximation replaces in eq. (27) the pump mode c with the complex amplitude β of the corresponding coherent

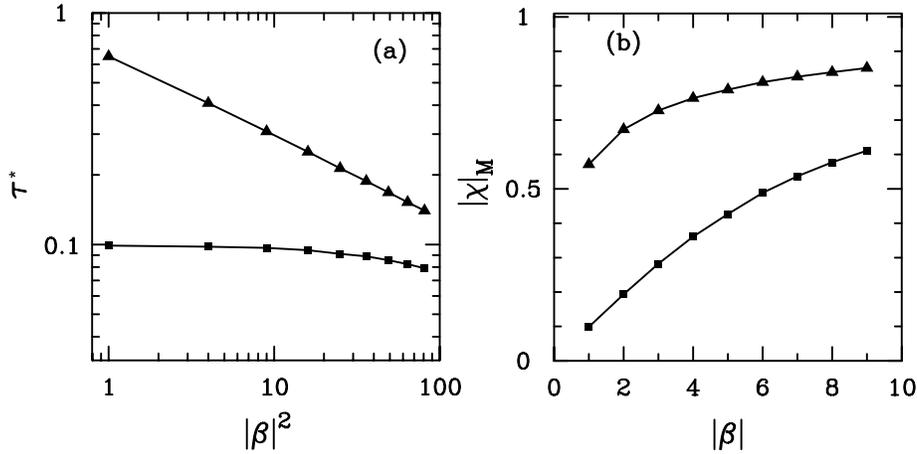


Fig. 7. – Performances of a nondegenerate parametric amplifier in achieving the two-mode squeezing operator. In both pictures triangles refer to vacuum input and circles to photon number state $|1,1\rangle$ input. In (a) we report the quantity τ^* , namely the maximum interaction time that leads to an output signal whose overlap with the theoretical state is larger than 99%, as a function of the pump intensity $|\beta|^2$. In (b) we show the corresponding maximum two-mode squeezing parameter $|\chi_M|$ achievable by the nondegenerate parametric amplifier, as a function of the pump amplitude $|\beta|$.

state, thus achieving the two-mode squeezing operator

$$(28) \quad \hat{S}_2(-i\tau\beta) \equiv \exp[-i\tau(\beta a^\dagger b^\dagger + \bar{\beta}ab)].$$

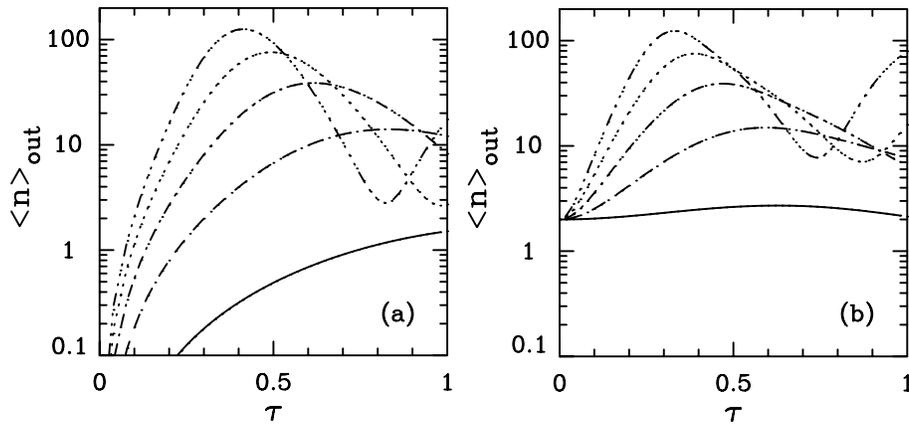


Fig. 8. – Average photon number $\langle \hat{n} \rangle_{out}$ of the signal at the output of a nondegenerate parametric amplifier, as a function of the interaction time τ . In (a) the case of vacuum input, and in (b) the case of photon number input $|1,1\rangle$. Different line-styles refer to different pump amplitudes: $\beta = 9$ (dot-dot-dashed), $\beta = 7$ (dotted), $\beta = 5$ (dot-dashed), $\beta = 3$ (dashed), $\beta = 1$ (solid).

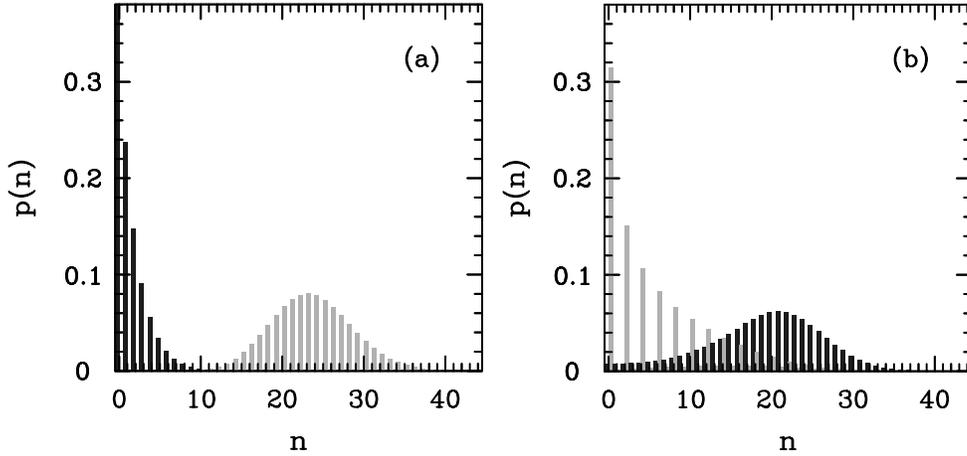


Fig. 9. – Photon number probabilities for both the signal and the pump modes at the output of a nondegenerate parametric amplifier. The plots refer to a situation in which the signal mode is initially in the (two-mode) vacuum and the pump mode is excited to a coherent state with amplitude $\beta = -5i$. The interaction time is equal to $\tau \equiv \tau^* = 0.214$ in (a), and to $\tau = 3\tau^*$ in (b). In the first case the parametric approximation well describes the real interaction, which produces a twin-beam state with two-mode squeezing parameter $\chi = 0.789$ corresponding to about 3.3 output photons. On the other hand, parametric approximation does not hold in the second case, as can be easily recognized from the pump squeezing. The pump Fano factor in (b) is about $F = 7.4$.

The two-mode squeezing operator yields a suppression of the quantum fluctuations in one quadrature of the sum and difference of modes $a \pm b$ [40]. When applied to vacuum input, the unitary operator in eq. (28) generates the so-called twin-beam

$$(29) \quad \hat{S}_2(\chi)|0,0\rangle = (1 - |\lambda|^2)^{1/2} \sum_{n=0}^{\infty} \lambda^n |n,n\rangle,$$

where $\lambda = \arg(\chi) \tanh |\chi|$. The expression in eq. (29) can be easily derived by factorizing the \hat{S}_2 operator through the decomposition formulas for the $SU(1,1)$ Lie algebra [3,17,18].

The dynamics of the nondegenerate parametric amplifier admits two independent constants of motion. We choose them as follows:

$$(30) \quad \hat{N} = \frac{1}{2} [a^\dagger a + b^\dagger b + 2c^\dagger c], \quad \hat{K} = a^\dagger a + c^\dagger c.$$

Correspondingly, we decompose the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ in the direct sum

$$(31) \quad \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c = \oplus_{N=0}^{\infty} \oplus_{K=0}^N \mathcal{H}_{NK},$$

where the invariant subspaces \mathcal{H}_{NK} are given by

$$(32) \quad \mathcal{H}_{NK} = \text{Span} \{ |K-m\rangle \otimes |N-K-m\rangle \otimes |m\rangle, \\ m \in [0, \min(K, N-K)] \}.$$

The Hamiltonian \hat{H}_{NP} and a generic three-mode state $|\psi_0\rangle$ will be consistently written as follows:

$$(33) \quad \hat{H}_{NP} = \sum_{N=0}^{+\infty} \sum_{K=0}^N \hat{h}_{NK},$$

$$(34) \quad |\psi_0\rangle = \sum_{N=0}^{+\infty} \sum_{K=0}^N \sum_{m=0}^{\min(K, N-K)} c_{K-m, N-K-m, m} |K-m, N-K-m, m\rangle.$$

To compute the exact dynamical evolution, one then diagonalizes each block \hat{h}_{NK} in eq. (33) up to a fixed maximum value of N and makes the input state evolve in the representation of eq. (34).

As in the previous section, we have evaluated the maximum interaction time τ^* that provides an output state—in the signal and idler modes—whose overlap with the state predicted by the parametric approximation is larger than 99%. The time τ^* for vacuum input and number input $|n \equiv 1, n \equiv 1\rangle$ is plotted as a function of the pump intensity in fig. 7a). The corresponding achievable two-mode squeezing parameter—the maximum argument $|\chi_M|$ in the operator (29)—is represented in fig. 7b). In fig. 8 we show the average photon number $\langle n \rangle_{\text{out}}$ of the output signal mode as a function of the interaction time, and for different values of the pump amplitude. Notice that the quantity $a^\dagger a - b^\dagger b$ is conserved, so that, for the considered input states, the idler mode has the same average photon number as the signal one.

As shown for the degenerate case, here also the requirement of a strong pump is not peremptory, whereas the undepletion of the pump does not guarantee the validity of the parametric approximation. Again, it is the Fano factor F of the pump after the interaction that well discriminates the working regimes of the amplifier. As long as $F \leq 1.10$, the overlap between the states at the output and those predicted by the parametric approximation is larger than 99%. For interaction time longer than τ^* , the pump mode reveals its quantum character, by showing oscillations in the number probability. This is illustrated in fig. 9, where we report the photon number probabilities for both the signal and the pump modes in the case of vacuum input and pump amplitude equal to $|\beta| = 5$, for different values of the interaction time.

5. – Conclusions

The quantum description of many optical devices is based on interaction Hamiltonians that couple different modes of radiation through the susceptibility tensor of the medium that supports the interaction. The theoretical predictions about such interactions are usually drawn in the so-called parametric approximation, *i.e.* by treating the pump mode classically as a fixed c -number. Owing to such approximation, an analytical treatment is possible with the help of the factorization formulas for Lie algebras.

In this paper we have investigated the conditions under which the parametric approximation holds in the treatment of $\chi^{(2)}$ nonlinear amplifiers, by resorting to the exact diagonalization of their full Hamiltonians. We have explicitly compared the states evaluated by the exact evolution with those predicted by the parametric approximation, in terms of the overlap between such states. On the one hand, we have shown that the

regime of validity of the parametric approximation is very large, including also the case of weak pump with 1–10 mean photon number. On the other hand, we have found that neither the condition of pump undepletion guarantees the goodness of the approximation, nor the condition of pump depletion signs its failure. We found that the degree of coherence of the pump *after* the interaction is a univocal parameter that discriminates the working regimes of the amplifiers. In terms of the pump Fano factor F we found that a deviation smaller than 10% guarantees an overlap larger than 99% between the states predicted within the parametric approximation and those evaluated by the exact Hamiltonian.

For long interaction times the approximation breaks down, and the quantum character of the pump mode is revealed. Oscillations in the pump number probability appear and, correspondingly, the Wigner function of the signal mode assumes negative values and resembles a Schrödinger-cat state.

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M. G. A. PARIS would like to acknowledge the “Francesco Somaini” foundation for partial support.

REFERENCES

- [1] MANDEL L. and WOLF E., *Optical Coherence and Quantum Optics* (Cambridge University Press) 1995.
- [2] PEŘINA J., *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Dordrecht, Kluwer) 1991.
- [3] D’ARIANO G. M., *Int. J. Mod. Phys. B*, **6** (1992) 1292.
- [4] YUEN H. P., *Phys. Rev. A*, **13** (1976) 2226.
- [5] YAMAMOTO Y. and HAUS H. A., *Rev. Mod. Phys.*, **58** (1986) 1001.
- [6] Special issues on Squeezed states: KIMBLE H. J. and WALLS D. F. (Editors), *J. Opt. Soc. Am. B*, **4** (1987); LOUDON R. and KNIGHT P. L. (Editors), *J. Mod. Opt.*, **34** (1987).
- [7] DE MARTINI F. *et al.* (Editors), *Quantum Interferometry* (VCH, Weinheim) 1996, pp. 95-224.
- [8] ALLEY C. O. and SHIH Y. H., *Phys. Rev. Lett.*, **61** (1988) 2921.
- [9] OU Z. Y. and MANDEL L., *Phys. Rev. Lett.*, **61** (1988) 50.
- [10] RARITY J. G. and TAPSTER P. B., *Phys. Rev. Lett.*, **64** (1990) 2495.
- [11] D’ARIANO G. M. and SACCHI M. F., *Phys. Rev. A*, **52** (1995) R4309; in ref. [7], pp. 307-313.
- [12] BOSCHI D., DE MARTINI F. and DI GIUSEPPE G., in ref. [7], pp. 135-143.
- [13] KLYSHKO D. N., *Phys. Lett. A*, **132** (1988) 299.
- [14] HORNE M. A., SHIMONY A. and ZEILINGER A., *Phys. Rev. Lett.*, **62** (1989) 2209.
- [15] BANDILLA A., DROBNÝ G. and JEX I., *Phys. Rev. Lett.*, **75** (1995) 4019; *Phys. Rev. A*, **53** (1996) 507; *Opt. Commun.*, **128** (1996) 353.
- [16] MOLLOW B. R. and GLAUBER R. J., *Phys. Rev.*, **160** (1967) 1076, 1097.
- [17] TRUAX D. R., *Phys. Rev. D*, **31** (1985) 1988.
- [18] BAN M., *J. Opt. Soc. Am. B*, **10** (1993) 1347.
- [19] PARIS M. G. A., *Phys. Lett. A*, **217** (1996) 78.
- [20] HILLERY M. and ZUBAIRY M. S., *Phys. Rev. A*, **29** (1984) 1275.
- [21] CROUCH D. D. and BRAUNSTEIN S. L., *Phys. Rev. A*, **38** (1988) 4696.
- [22] SCHARF G. and WALLS D. F., *Opt. Commun.*, **50** (1984) 245.
- [23] WALLS D. F. and BARAKAT R., *Phys. Rev. A*, **1** (1970) 446.
- [24] WALLS D. F. and TINDLE C. T., *J. Phys. A*, **5** (1972) 534.
- [25] BUCK B. and SUKUMAR C. V., *Phys. Lett. A*, **81** (1981) 132; SUKUMAR C. V. and BUCK B., *J. Phys. A*, **17** (1984) 885.

- [26] BUŽEK V., *Quantum Opt.*, **1** (1989) 53; *J. Mod. Opt.*, **37** (1990) 303.
- [27] MOSTOWSKI J. and RZAZEWSKI K., *Phys. Lett. A*, **66** (1978) 275.
- [28] TANAŚ R. and GANTSOG TS., *Phys. Rev. A*, **45** (1992) 5031.
- [29] BUŽEK V. and DROBNÝ G., *Phys. Rev. A*, **47** (1993) 1237.
- [30] HILLERY M., YU D. and BERGOU J., *Phys. Rev. A*, **49** (1994) 1288.
- [31] DATTOLI G., GALLARDO J. C. and TORRE A., *Riv. Nuovo Cimento*, **11** (1988) 1.
- [32] DROBNÝ G. and JEX I., *Phys. Rev. A*, **46** (1992) 499.
- [33] CAMPOS R. A., SALEH B. E. A. and TEICH M. C., *Phys. Rev. A*, **40** (1989) 1371.
- [34] YUEN H. P. and CHAN V. W. S., *Opt. Lett.*, **8** (1983) 177.
- [35] JANSZKY J., SIBILIA C., BERTOLOTTI M. and YUSHIN Y., *J. Mod. Opt.*, **35** (1988) 1757;
JANSZKY J., ADAM P., BERTOLOTTI M. and SIBILIA C., *Quantum Opt.*, **4** (1992) 163.
- [36] FEARN H., *Quantum Opt.*, **2** (1990) 103; FEARN H. and LOUDON R., *Opt. Commun.*, **64** (1987) 485.
- [37] D'ARIANO G. M., *Nuovo Cimento B*, **107** (1992) 643.
- [38] D'ARIANO G. M., *Quantum estimation theory and optical detection*, in *Quantum Optics and the Spectroscopy of Solids*, edited by HAKIOĞLU T. and SHUMOVSKY A. S. (Kluwer, Dordrecht) 1997, pp. 139-174.
- [39] SCHLEICH W. and WHEELER J. A., *Nature*, **326** (1987) 574.
- [40] CAVES C. M. and SCHUMAKER B. L., *Phys. Rev. A*, **31** (1985) 3068.